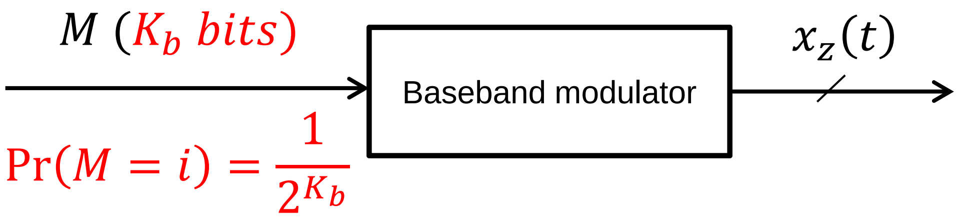
NJIT

New Jersey's Science & Technology University

THE EDGE IN KNOWLEDGE

Transmitting more than one bit (Ch. 14-15)

- Baseband modulator:



$$x_{z,i}(t) \quad if\ M=i$$

with $E_i = \int |x_{z,i}(t)|^2 dt$

for $i = 0,1,\dots,2^{K_b}-1$

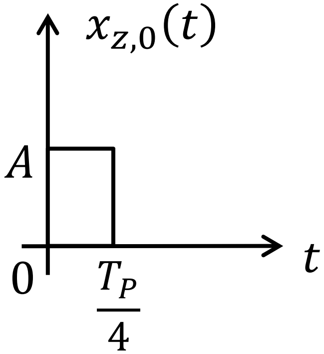
- Average energy:

$$E_s = \frac{1}{2^{K_b}} \sum_{i=0}^{2^{K_b}-1} E_i = K_b E_b$$

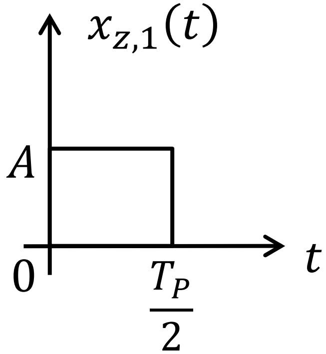
energy per bit

- If $x_z(t)$ is of duration T_P,
  - Average power $= P_s = \frac{E_s}{T_P}$
  - Bit rate $= \frac{K_b}{T_P}$

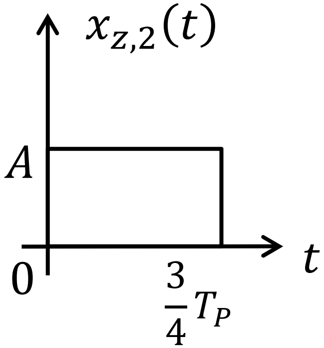
Ex.: a) Pulse Width Modulation (PWM) with $K_b = 2$ (4-PWM)



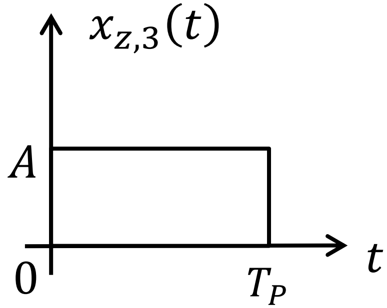
$$E_0 = A^2 \frac{T_P}{4}$$



$$E_1 = A^2 \frac{T_P}{2}$$



$$E_2 = A^2 \frac{3}{4} T_P$$



$$E_3 = A^2 T_P$$

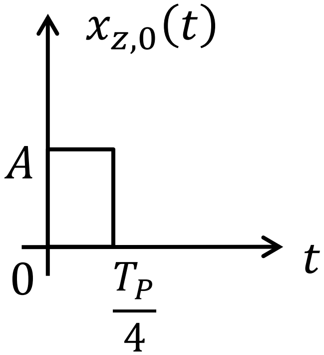
- Average energy:

$$E_s = \frac{1}{4} A^2 T_P \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 \right)$$
$$= A^2 T_P \frac{10}{16}$$

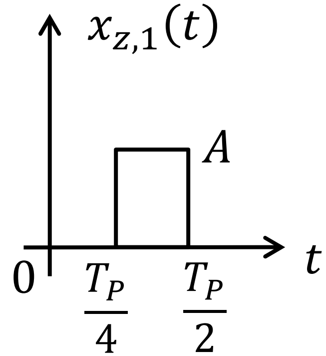
- We now impose that $E_s = 2E_b$ since $K_b = 2$ in order to find A:

$$A^2 T_P \frac{10}{16} = 2E_b \Rightarrow A = \sqrt{\frac{16}{5} \frac{E_b}{T_P}}$$

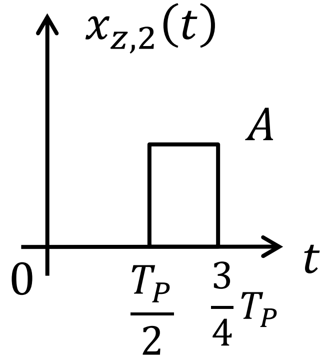
b) Pulse Position Modulation (PPM) with $K_b = 2$ (4-PPM)



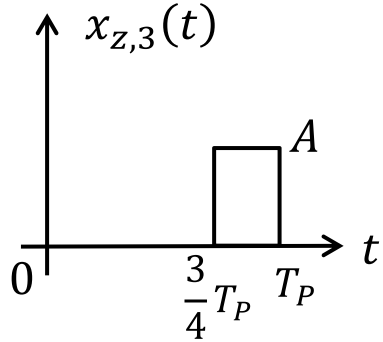
$$E_0 = A^2 \frac{T_P}{4}$$



$$E_1 = A^2 \frac{T_P}{2}$$



$$E_2 = A^2 \frac{T_P}{4}$$



$$E_3 = A^2 \frac{T_P}{4}$$

- Average energy:

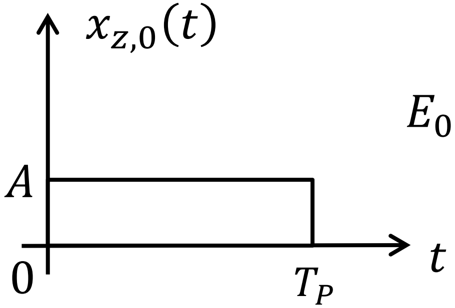
$$E_s = A^2 \frac{T_P}{4}$$

- We now impose that $E_s = 2E_b$ since $K_b = 2$ in order to find A:

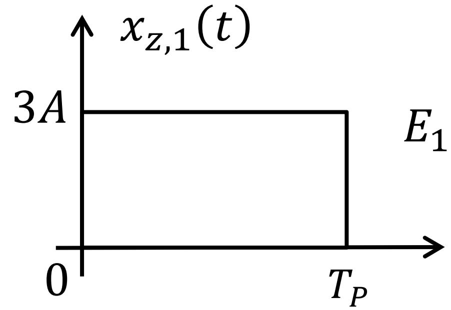
$$A^2 \frac{T_P}{4} = 2E_b \Rightarrow A = \sqrt{\frac{8\, E_b}{T_P}}$$

c) Pulse Amplitude Modulation (2^{K_b}-PAM)

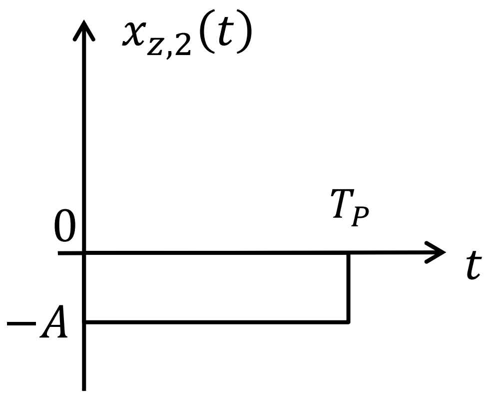
- As an example, 4-PAM with $K_b = \log_2 4 = 2$, we have



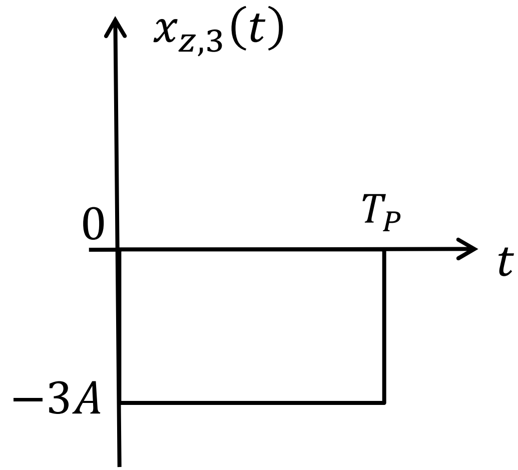
$$E_0 = A^2 T_P$$



$$E_1 = 9A^2 T_P$$



$$E_2 = A^2 T_P$$



$$E_3 = 9A^2 T_P$$

- Average energy:

$$E_s = \frac{A^2 T_P}{4}(2 \times 1 + 2 \times 9) = \frac{20}{4} A^2 T_P = 5A^2 T_P$$

- We now impose that $E_s = 2E_b$ since $K_b = 2$ in order to find A:

$$5A^2 T_P = 2E_b \Rightarrow A = \sqrt{\frac{2\ E_b}{5T_P}}$$

d) Phase Shift Keying (2^{K_b}-PAM)

- $$x_{z,i}(t) = \begin{cases} Ae^{j\theta(i)} & 0 \le t \le T_P \\ 0 & elsewhere \end{cases} \qquad i = 0,1 \ldots,\ 2^{K_b}-1$$

  with $\theta(i) = \pi \frac{2i+1}{2^{K_b}}$

- As an example, if $K_b = \log_2 4 = 2$, we have

$$\theta(0) = \frac{\pi}{4} \Rightarrow x_{z,0}(t) = \begin{cases} \frac{1}{\sqrt{2}}(A + jA) & 0 \le t \le T_P \\ 0 & elsewhere \end{cases}$$

$$\theta(1) = \frac{3\pi}{4} \Rightarrow x_{z,1}(t) = \begin{cases} \frac{1}{\sqrt{2}}(-A + jA) & 0 \le t \le T_P \\ 0 & elsewhere \end{cases}$$

$$\theta(2) = \frac{5\pi}{4} \Rightarrow x_{z,2}(t) = \begin{cases} \frac{1}{\sqrt{2}}(-A - jA) & 0 \le t \le T_P \\ 0 & elsewhere \end{cases}$$

$$\theta(3) = \frac{7\pi}{4} \Rightarrow x_{z,3}(t) = \begin{cases} \frac{1}{\sqrt{2}}(A - jA) & 0 \le t \le T_P \\ 0 & elsewhere \end{cases}$$

- Imposing $E_s = K_b E_b$, we easily get

$$A = \sqrt{\frac{K_b E_b}{T_P}}$$

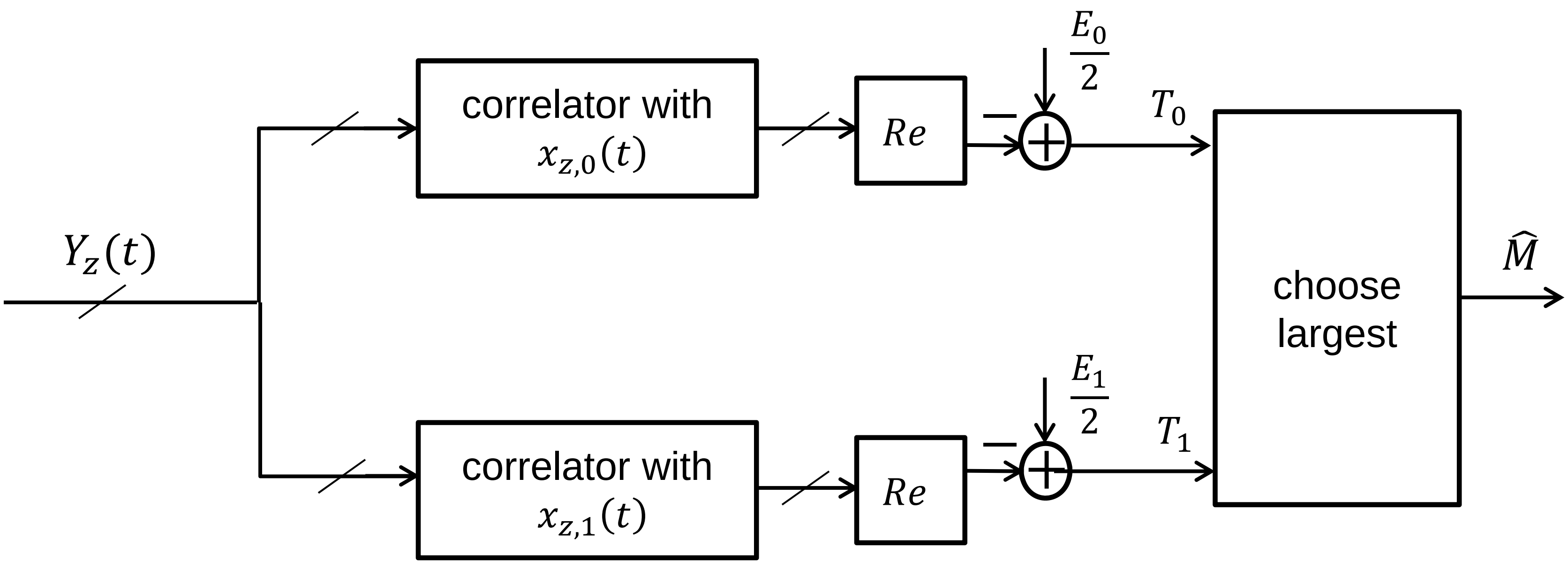
$\blacksquare$

Quick Quiz

Find $x_{I,i}(t)$ and $x_{Q,i}(t)$ for 4-PSK $(i = 0,1,2,3)$.

- Recall the optimal demodulator for $K_b = 1$ (and $\pi_i = \frac{1}{2}$):

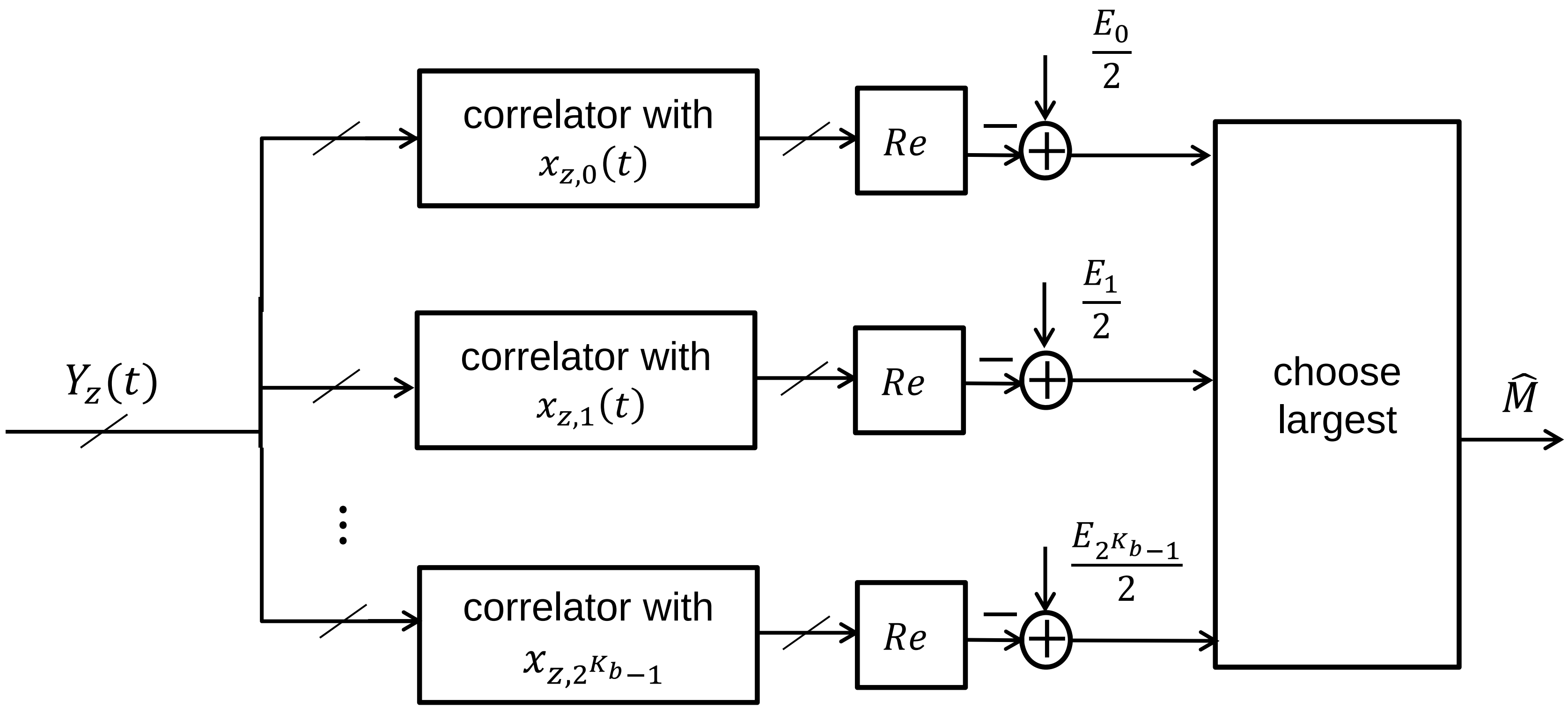
MAXIMUM LIKELIHOOD BIT DEMODULATOR (MLBD)



where T_j is the likelihood metric for message $M = j$

- Generalizing to $K_b \geq 1$ (and $\pi_i = \frac{1}{2^{K_b}}$):

MAXIMUM LIKELIHOOD WORD DEMODULATOR (MLWD)

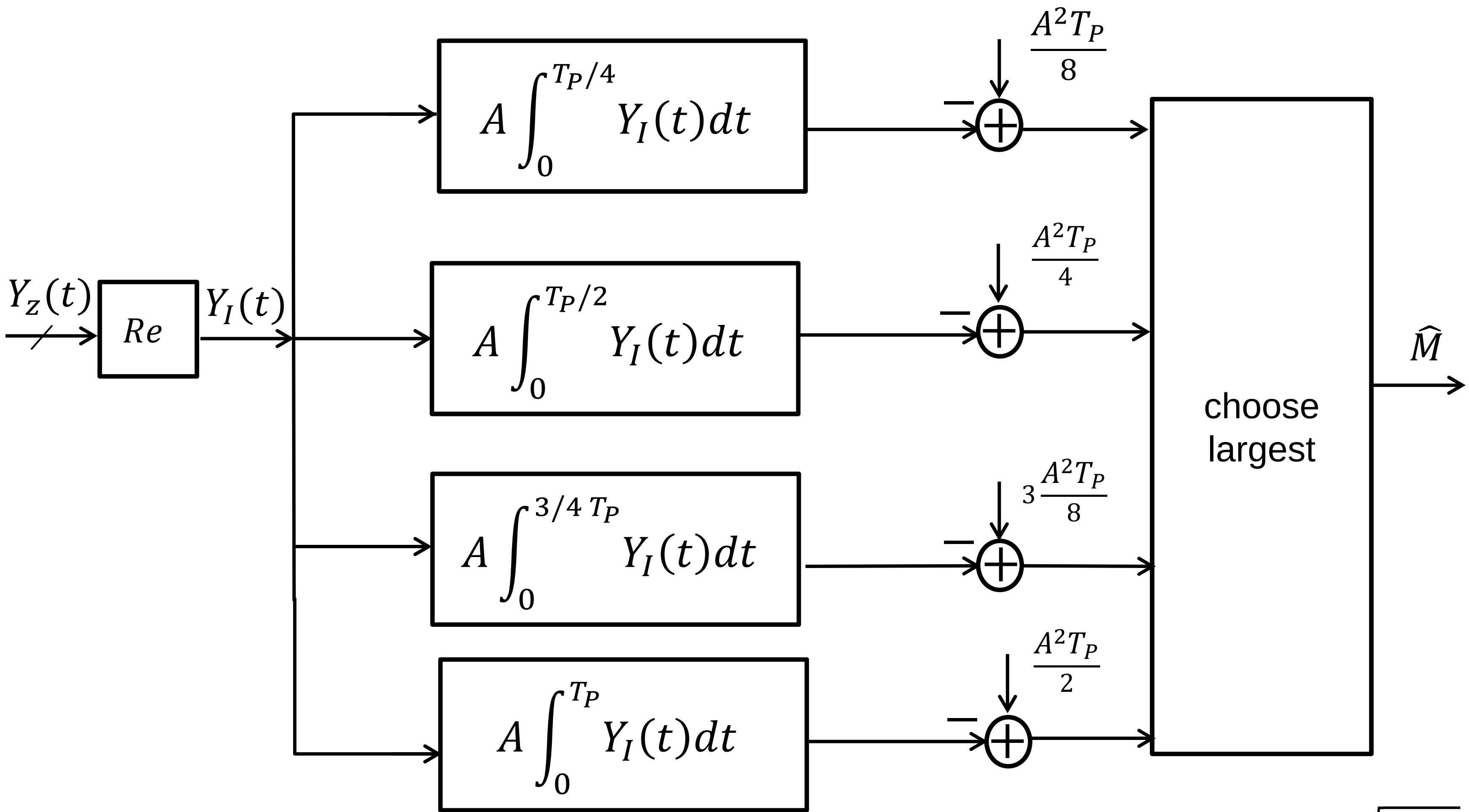


where T_j is the likelihood metric for message $M = j$

Remarks:

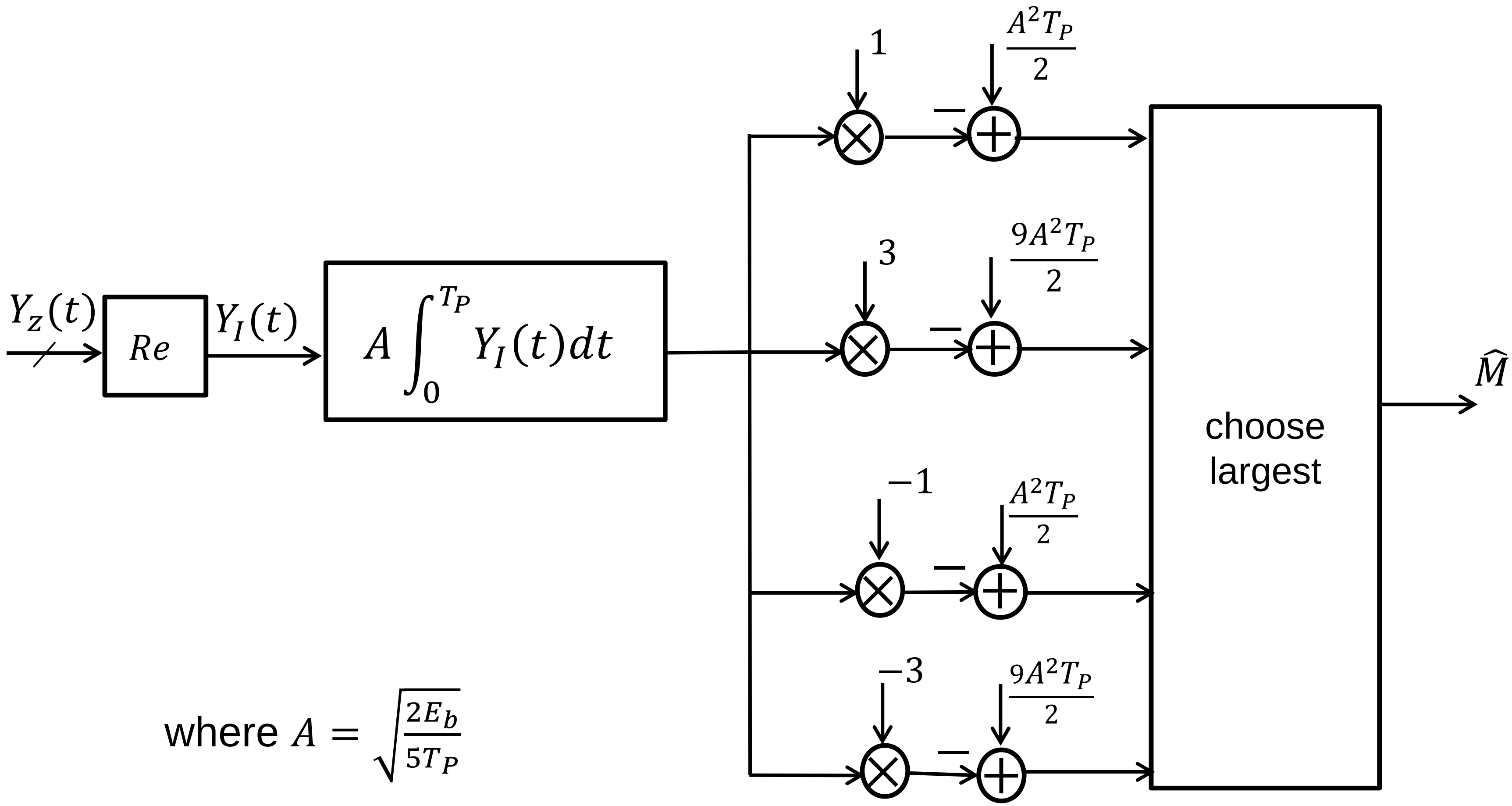
- 2^{K_b} branches ⇒ the complexity grows exponentially with K_b
- This complexity can be mitigated if the waveforms have special properties, e.g., real or proportional to one another.

Ex.: a) 4-PWM ($K_b = 2$) – The MLWD can be simplified as:



$$A = \sqrt{\frac{16}{5}\frac{E_b}{T_P}}$$

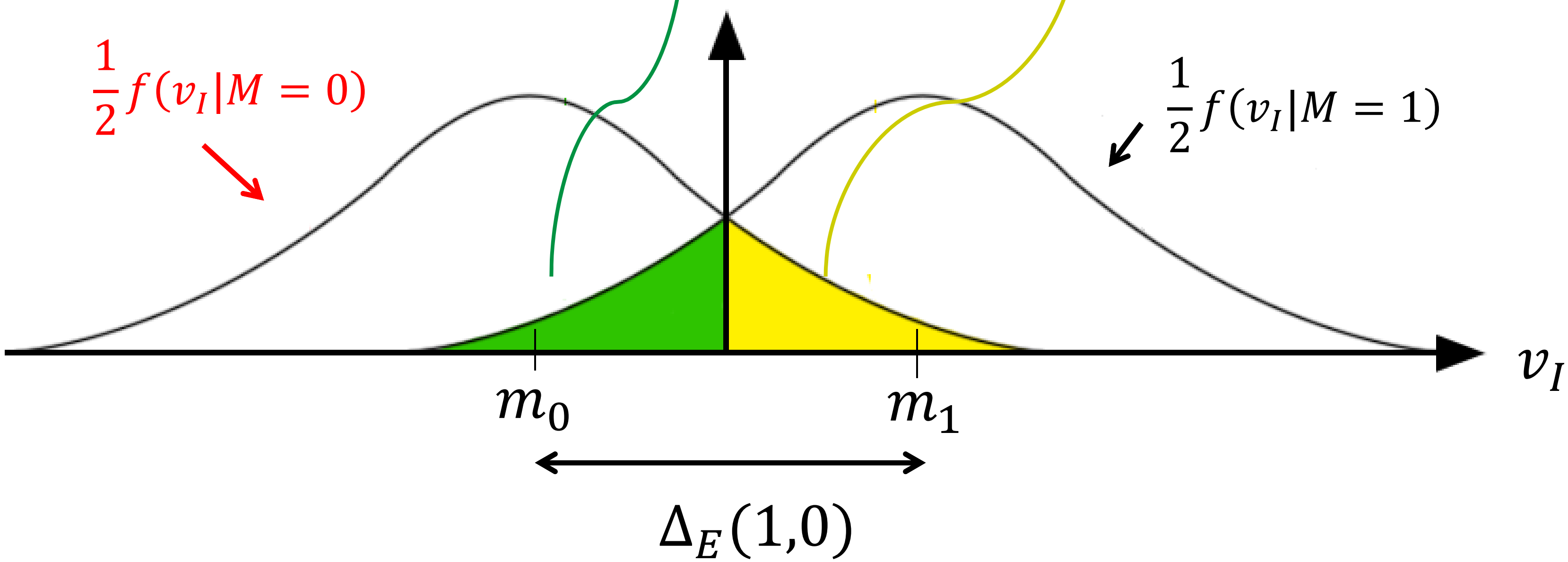
b) 4-PAM – The MLWD can be simplified as:



where $A = \sqrt{\frac{2E_b}{5T_P}}$

- We are now interested in calculating the word error probability (WEP): $P_W(E) = \Pr(\widehat{M} \neq M)$
- Recall that with $K_b = 1$:

$$P_W(E) = P_B(E) = \frac{1}{2}\Pr\left(\widehat{M} \neq 1 | M = 1\right) + \frac{1}{2}\Pr\left(\widehat{M} \neq 0 | M = 0\right) = Q\left(\sqrt{\frac{\Delta_E(1,0)}{2N_0}}\right)$$



- Generalizing to $K_b \geq 1$, we get

$$P_W(E) = \frac{1}{2^{K_b}} \sum_{j=0}^{2^{K_b-1}} \Pr\left(\widehat{M} \neq j | M = j\right)$$

- Calculating $\Pr\left(\widehat{M} \neq j | M = j\right)$ is more difficult than for $K_b = 1$:

$$\Pr\left(\widehat{M} \neq j | M = j\right) = \Pr\left(\underbrace{T_0 > T_j \text{ or } T_2 > T_j \text{or} \dots \text{or } T_{2^{Kb}-1} > T_j}_{\text{excluding } M = j} | M = j\right)$$

$$= \Pr\left(\bigcup_{i \neq j} \{T_i > T_j\} | M = j\right)$$

- We have:

$$\Pr\left(\hat{M} \neq j | M = j\right) = \Pr\left(\bigcup_{i \neq j} \{T_i > T_j\} | M = j\right)$$

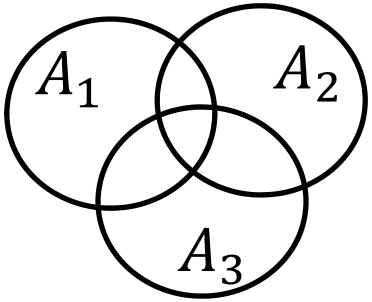
$$\leq \sum_{\substack{i=0 \\ i \neq j}}^{2^{K_b}-1} \Pr\left(T_i > T_j | M = j\right)$$

$$= \sum_{\substack{i=0 \\ i \neq j}}^{2^{K_b}-1} Q\left(\sqrt{\frac{\Delta_E(i,j)}{2N_0}}\right)$$

pairwise error probability

The union bound generally states that for any events A_i, the following holds:

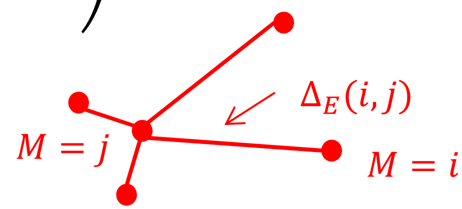
$$\Pr\left(\bigcup_i A_i\right) \leq \sum_i \Pr(A_i)$$



with equality if and only if the events A_i are disjoint.

- This leads to the so called union bound on $P_W(E)$:

$$P_W(E) \leq P_{WUB}(E) = \frac{1}{2^{K_b}} \sum_{j=0}^{2^{K_b}-1} \sum_{i \neq j} \mathrm{Q}\left(\sqrt{\frac{\Delta_E(i,j)}{2N_0}}\right)$$



different signals i may have the same distance from a signal j

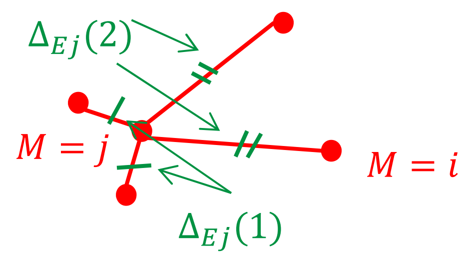
N_j = number of different distances from node j

$$= \frac{1}{2^{K_b}} \sum_{j=0}^{2^{K_b}-1} \sum_{k=1}^{N_j} A_{dj}(k)\, \mathrm{Q}\left(\sqrt{\frac{\Delta_{Ej}(k)}{2N_0}}\right)$$

number of signals at distance $\Delta_{Ej}(k)$ from j: <u>conditional distance spectrum</u>

grouping pairs of signals (i,j) with the same distance

$$= \frac{1}{2^{K_b}} \sum_{k=1}^{N} A_d(k)\, \mathrm{erfc}\left(\sqrt{\frac{\Delta_E(k)}{2N_0}}\right)$$



N = number of different distances

number of pairs of signals at distance $\Delta_E(k)$: <u>distance spectrum</u>

- Remark: We have

$$\sum_{k=1}^{N_j} A_{dj}(k) = 2^{K_b} - 1$$

$$\sum_{k=1}^{N} A_d(k) = 2^{K_b}\,(2^{K_b} - 1)$$

(Why?)

Ex.: 4-PWM $(K_b = 2)$

- Conditional distance spectrum:
  - for $M = 0$

$$\Delta_E(0,1) = \int_{T_P/4}^{T_P/2} \frac{16}{5} \frac{E_b}{T_P} dt = \frac{4E_b}{5}$$

$$\Delta_E(0,2) = \frac{8E_b}{5}$$

$$\Delta_E(0,3) = \frac{12E_b}{5}$$

$$\Rightarrow \left[\left\{ \underset{\Delta_{E0}(1)}{\frac{4E_b}{5}}, \underset{A_{d0}(1)}{1} \right\}, \ \left\{ \frac{8E_b}{5}, 1 \right\}, \left\{ \frac{12E_b}{5}, 1 \right\} \right] \qquad (N_0 = 3)$$

– for $M = 1$ and $M = 2$

$$\Rightarrow \left[\left\{\frac{4E_b}{5}, 2\right\},\ \left\{\frac{8E_b}{5}, 1\right\}\right] \qquad (N_1 = N_2 = 2)$$

– for $M = 3$

$$\Rightarrow \left[\left\{\frac{4E_b}{5}, 1\right\},\ \left\{\frac{8E_b}{5}, 1\right\}, \left\{\frac{12E_b}{5}, 1\right\}\right] \qquad (N_3 = 3)$$

- Distance spectrum

$$\left[\left\{\frac{4E_b}{5}, 6\right\},\ \left\{\frac{8E_b}{5}, 4\right\}, \left\{\frac{12E_b}{5}, 2\right\}\right] \qquad (N = 3)$$

$\Delta_E(1)$ $\qquad$ $A_d(1)$

- Union bound:

$$P_W(E) \le P_{WUB}(E) = \frac{1}{4}\left(6Q\left(\sqrt{\frac{4E_b}{10N_0}}\right) + 4\mathrm{Q}\left(\sqrt{\frac{8E_b}{10N_0}}\right) + 2\mathrm{Q}\left(\sqrt{\frac{12E_b}{10N_0}}\right)\right)$$

$$= \frac{3}{2}\,\mathrm{Q}\left(\sqrt{\frac{2E_b}{5N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{4E_b}{5N_0}}\right) + \frac{1}{2}\mathrm{Q}\left(\sqrt{\frac{6E_b}{5N_0}}\right)$$

$\blacksquare$

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