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New Jersey's Science & Technology University

THE EDGE IN KNOWLEDGE

Digital Communication – Part II

- Using the law of total probability, the BEP can be written as:

$$P_B(E) = \int_{-\infty}^{+\infty} f(v_I)\Pr\left(M \neq \widehat{M} \middle| V_I(T_P) = v_I\right) dv_I$$

probability density function of $V_I(T_P)$

BEP conditioned on $V_I(T_P) = v_I$

- For each value v_I of $V_I(T_P)$, $\widehat{M}$ should then be selected so that $\Pr(M \neq \widehat{M}|V_I(T_P) = v_I)$ is minimized.

- If $\widehat{M}(v_I) = 0$, the BEP conditioned on $V_I(T_P) = v_I$ is:

$$\Pr(M \neq 0|V_I(T_P) = v_I) = \Pr(M = 1|V_I(T_P) = v_I)$$

and similarly if $\widehat{M}(v_I) = 1$.

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⇒ in order to minimize the BEP:

$$\Pr(M = 1|V_I(T_P) = v_I) \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \Pr(M = 0|V_I(T_P) = v_I)$$

a posterior probability (APP) of $M = 1$ given $V_I(T_P) = v_I$

a posterior probability (APP) of $M = 0$ given $V_I(T_P) = v_I$

This rule is known as

MAXIMUM A POSTERIOR BIT DEMODULATION (MAPBD)

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- How do we choose γ in order to obtain the MAPBD?

- Using Bayes rule, we can write:

$$\Pr(M = 1|V_I(T_P) = v_I) = \frac{f(v_I|I = 1)\Pr(M = 1)}{f(v_I)}$$

π_1

probability density function
of $V_I(T_P)$ given $M = 1$

- Therefore, the MAPBD can be expressed as:

$$f(v_I|M = 1)\pi_1 \quad \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} \quad f(v_I|M = 0)\pi_0$$

or equivalently

$$\frac{f(v_I|M = 1)}{f(v_I|M = 0)} \gtrless \frac{\pi_0}{\pi_1}$$ (likelihood ratio test)

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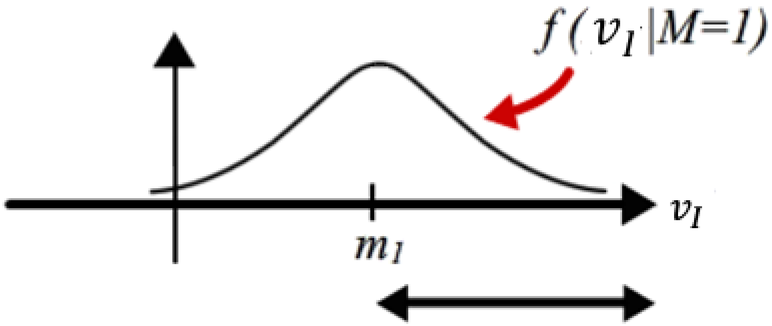
- What is the probability density $f(v_I|M=1)$?
- Recall that if $M=1$, then:

$$V_I(T_P) = V_{I,1}(T_P) = m_1 + N_I \text{ with } N_I \sim N\left(0, \sigma_{N_I}^2\right)$$

$$\Rightarrow \; f(v_I|M=1) = \frac{1}{\sqrt{2\pi}\sigma_{N_I}} e^{\frac{-(v_I - m_1)^2}{2\sigma_{N_I}^2}}$$



spread depends on σ_{N_I}

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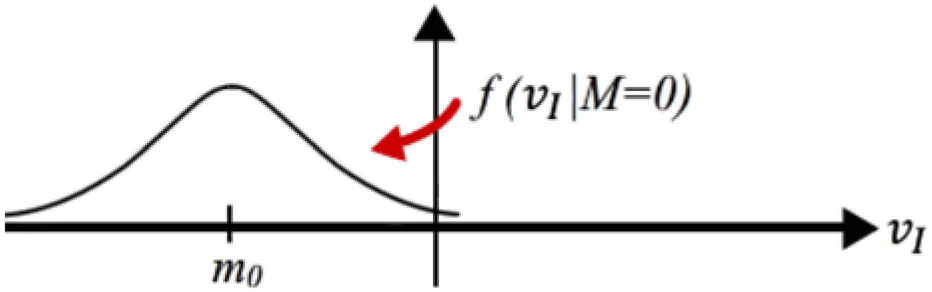
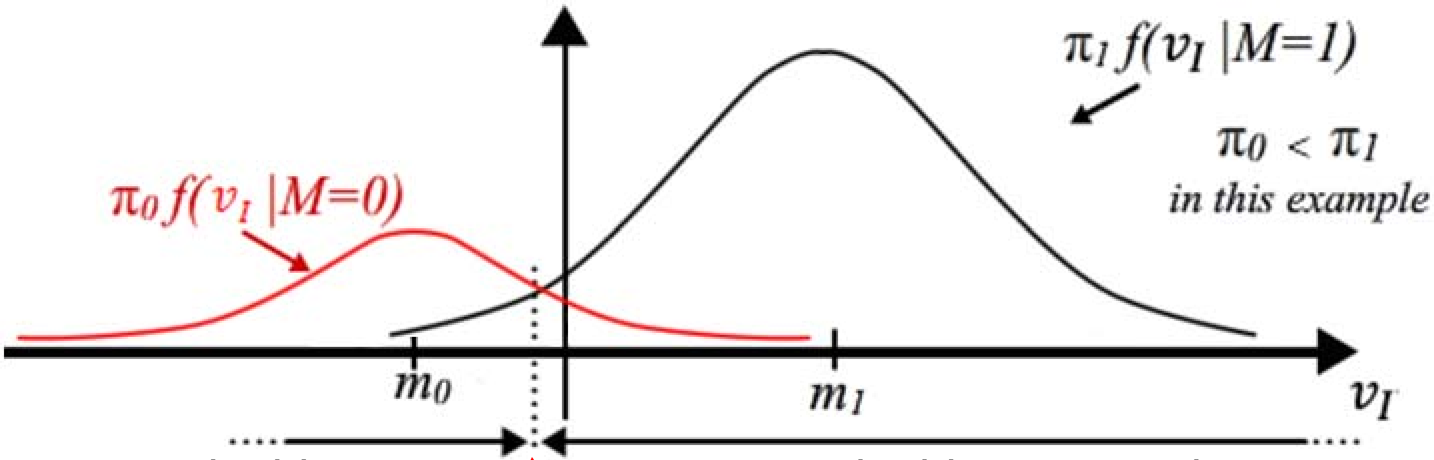
- Similarly, we have that:

$$f(v_I|M=0) = \frac{1}{\sqrt{2\pi}\sigma_{N_I}} e^{\frac{-(v_I - m_0)^2}{2\sigma_{N_I}^2}}$$



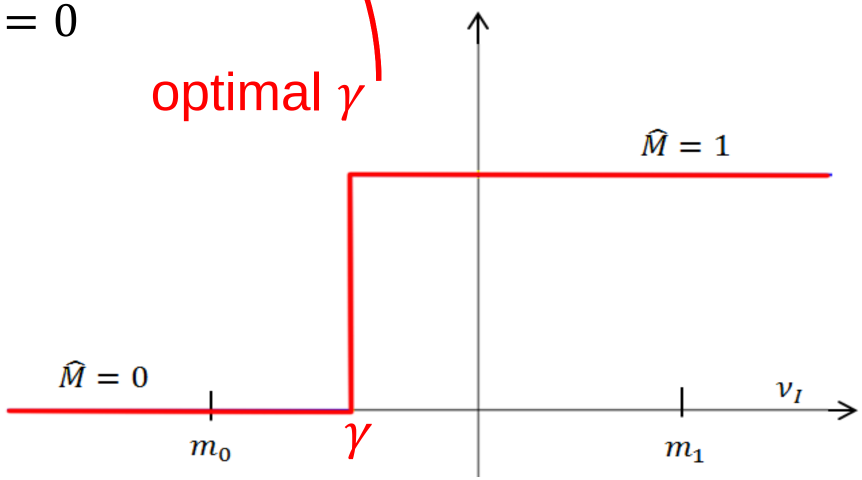
- Illustration of the MAPBD



in this area,
we have
$$\pi_1 f(v_1|M=1) < \pi_0 f(v_I|M=0)$$
$$\Rightarrow \widehat{M}=0$$

in this area, we have
$$\pi_1 f(v_1|M=1) > \pi_0 f(v_I|M=0)$$
$$\Rightarrow \widehat{M}=1$$

optimal γ



$$v_I \underset{\widehat{M}=0}{\overset{\widehat{M}=1}{\gtrless}} \gamma$$

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- To find the optimal γ, one must calculate the value of v_I for which

$$\pi_1 f(v_I|M = 1) = \pi_0 f(v_I|M = 0)$$

- The calculation of the optimal γ is simpler in the special case

$$\boxed{\pi_0 = \pi_1 = \frac{1}{2}}$$

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- In this case, the MAPBD rule is

$$f(v_I|M=1) \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} f(v_I|M=0),$$

which is equivalent to:

$$-\log f(v_I|M=1) \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} -\log f(v_I|M=0),$$

which in turn is equivalent to

$$\boxed{(v_I-m_0)^2 \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} (v_I-m_1)^2}$$

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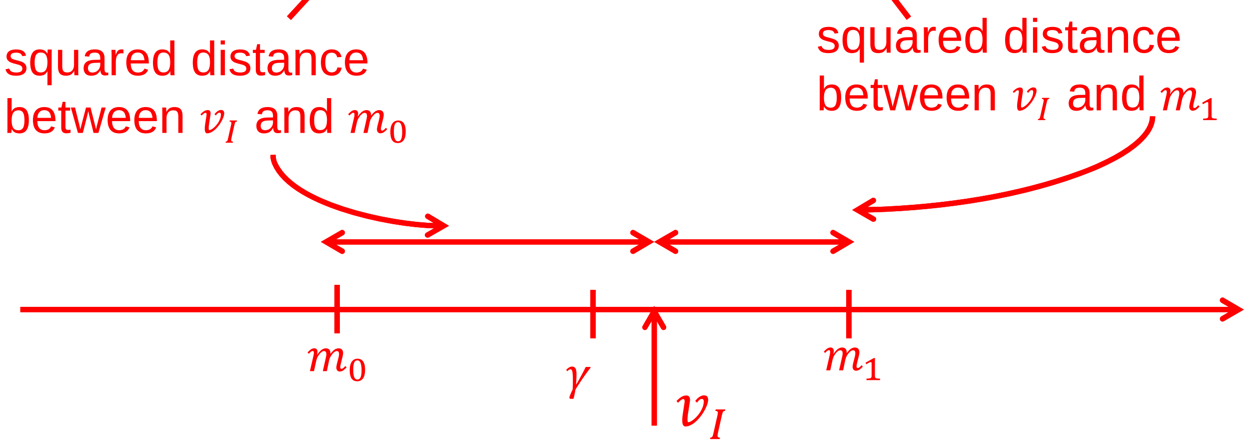
$$\boxed{(v_I - m_0)^2 \underset{\hat{M}=0}{\overset{\hat{M}=1}{\gtrless}} (v_I - m_1)^2}$$

ex:



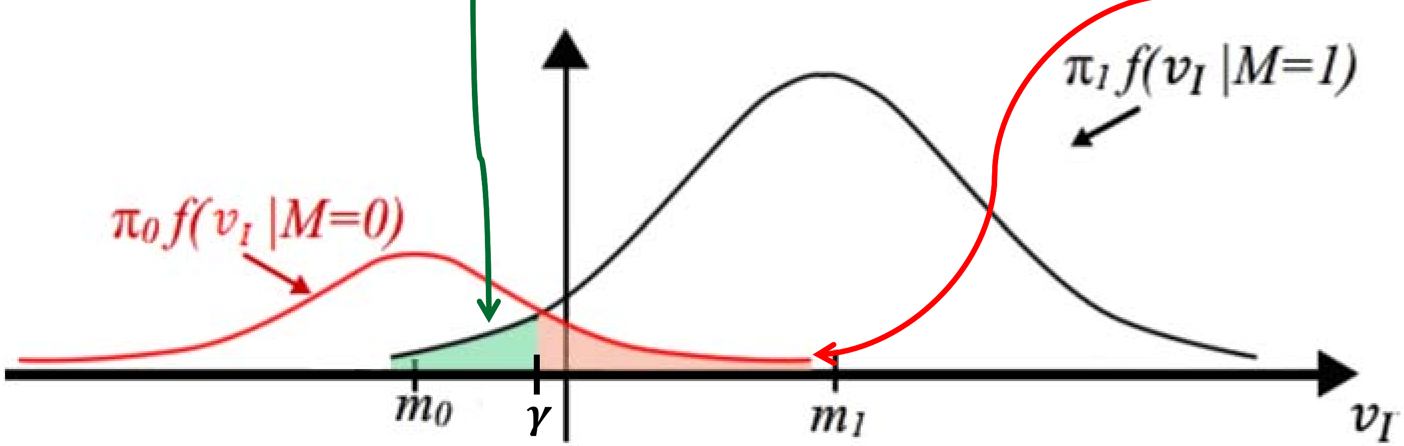
- The optimal threshold is hence the midpoint between m_0 and m_1:

$$\gamma = \frac{m_0 + m_1}{2}$$

∎

- Given the optimal threshold, we now evaluate the BEP.
- Using the law of total probability, we have

$$
\begin{aligned}
&P_B(E) = \Pr\left(\widehat{M} \neq M\right) \\
&= \pi_1 \Pr\left(\widehat{M} = 0 \middle| M = 1\right) + \pi_0 \Pr\left(\widehat{M} = 1 \middle| M = 0\right) \\
&= \pi_1 \Pr(V_I(T_P) < \gamma | M = 1) + \pi_0 \Pr(V_I(T_P) > \gamma | M = 0)
\end{aligned}
$$



$$\pi_1 \int_{-\infty}^{\gamma} f(v_I | M = 1) dv_I$$
$$= \pi_1 \Pr(v_I\ (T_P) < \gamma | M = 1)$$

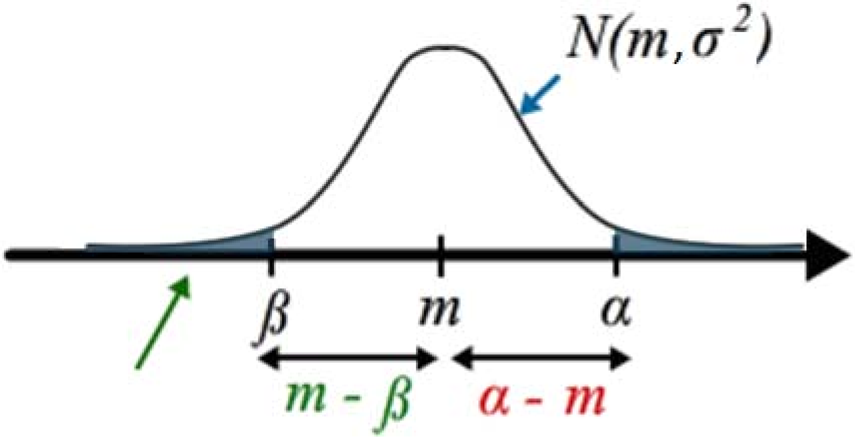
$$\pi_0 \int_{\gamma}^{+\infty} f(v_I | M = 0) dv_I$$
$$= \pi_0 \Pr(v_I\ (T_P) > \gamma | M = 0)$$

Digital Communication – Part II

- As seen above, in order to calculate the BEP, we need to evaluate the integral of the tails of a Gaussian distribution.
- While a simple closed-form expression is lacking, the values of such integrals can be obtained from the so called ERFC function or from the Q function, which are available in MATLAB.
- $\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$
- $\mathrm{Q}(z) = 1 - \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2}\, dt$

Digital Communication – Part II

- Using this function, we can calculate the tails of a Gaussian distribution as follows:



$$\int_{a}^{+\infty} f(x)dx = \frac{1}{2}\operatorname{erfc}\left(\frac{a-m}{\sqrt{2}\sigma}\right)$$
$$= \mathrm{Q}\left(\frac{a-m}{\sigma}\right)$$

$$\int_{-\infty}^{\beta} f(x)dx = \frac{1}{2}\operatorname{erfc}\left(\frac{m-\beta}{\sqrt{2}\sigma}\right) = Q\left(\frac{m-\beta}{\sigma}\right)$$

Digital Communication – Part II

- Using the Q function, the BEP is given as

$$P_B(E) = \pi_0 Q\left(\frac{\gamma - m_0}{\sigma_{N_I}}\right) + \pi_1 Q\left(\frac{m_1 - \gamma}{\sigma_{N_I}}\right)$$

which can be directly calculated in MATLAB.

- If $\boxed{\pi_0 = \pi_1 = \frac{1}{2}}$, we have $\gamma - m_0 = m_1 - \gamma = \frac{m_1 - m_0}{2}$ since $\gamma = \frac{m_0 + m_1}{2}$

$$\Rightarrow \boxed{P_B(E) = Q\left(\frac{m_1 - m_0}{2\sigma_{N_I}}\right)}$$

Digital Communication – Part II

⇒ The BEP depends on the ratio

$$\boxed{\eta = \frac{(m_1 - m_0)^2}{4\sigma_{N_I}^2} \quad \text{effective SNR}}$$

since

$$P_B(E) = Q(\sqrt{\eta})$$

Optimum Filter $H(f)$

- Given $\{x_{z,0}(t), x_{z,1}(t)\}$ and the optimal γ, we wish to design $H(f)$, or $h(t)$, that minimizes the BEP $P_B(E)$
- Focusing on the case $\pi_0 = \pi_1 = \frac{1}{2}$, we have

$$P_B(E) = Q\left(\sqrt{\eta}\right)$$

with $\eta = \frac{(m_1 - m_0)^2}{4\sigma_{N_I}^2}$ effective SNR

where $m_i = Re\left\{x_{z,i}(t) * h(t)\right\}|_{t=T_P}\ i = 0,1$

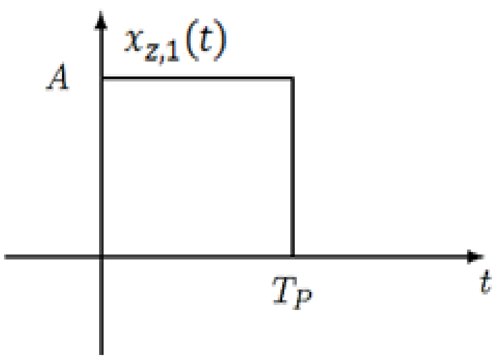
and $\sigma_{N_I}^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\, df$

→ the filter affects both the numerator and the denominator of η.

Optimum Filter $H(f)$

- Let us first try to build an intuition about the optimal filter.
- For this purpose, consider first the transmission of $M = 1$ and try to guess the optimal filter under the assumption that $M = 1$.

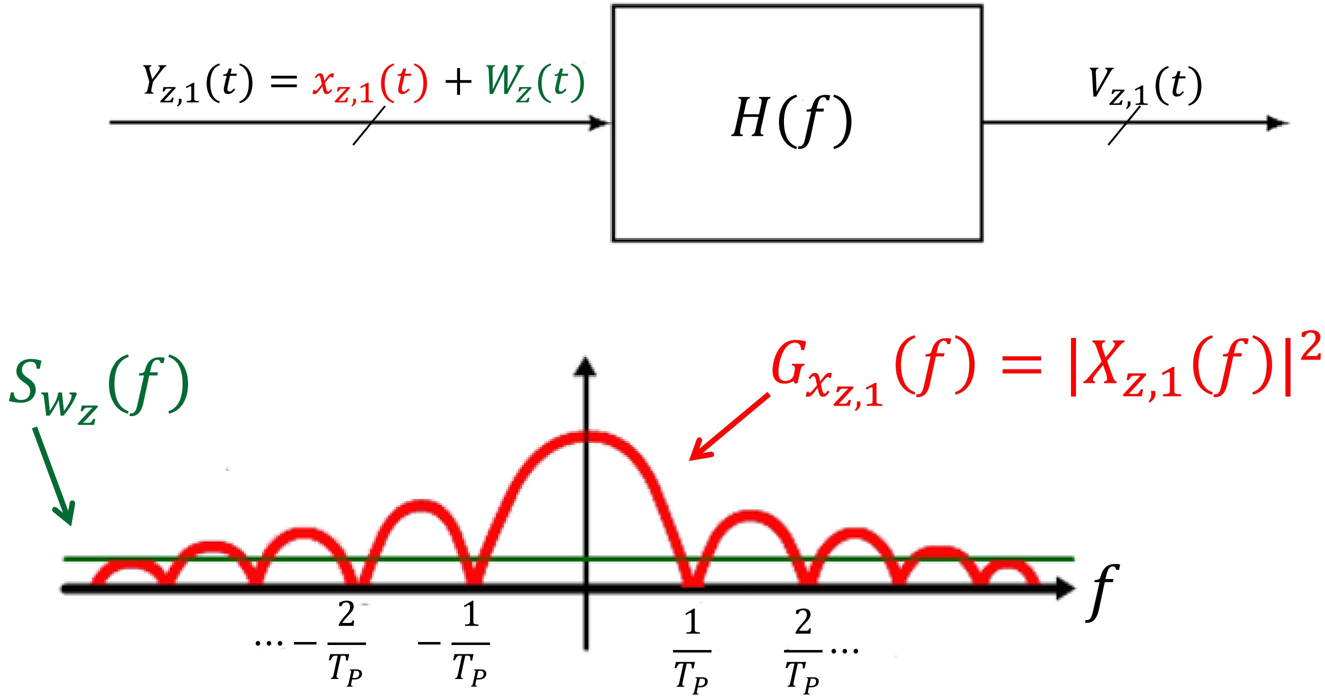
Ex.:



$E_1 = A^2 T_P$ waveform $x_{z,1}(t)$

Optimum Filter $H(f)$

- The filter $H(f)$ should emphasize the signal $x_{z,1}(t)$ over the noise $W_z(t)$

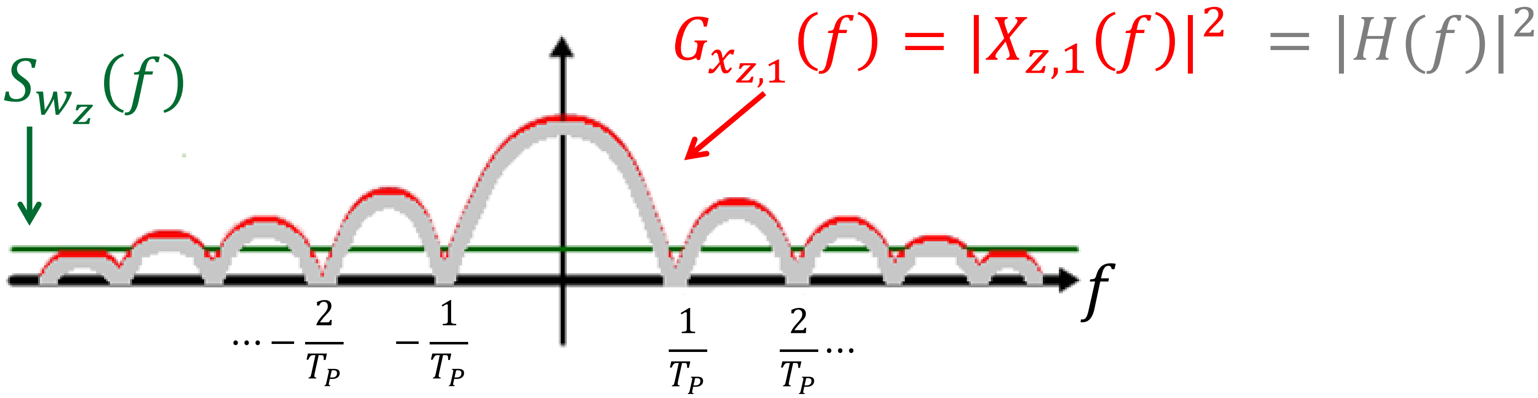


Optimum Filter $H(f)$

$\Rightarrow$ it can be proved that the best such filter is

$$H(f) = X^*_{z,1}(f)$$

filter matched to the signal $x_{z,1}(t)$

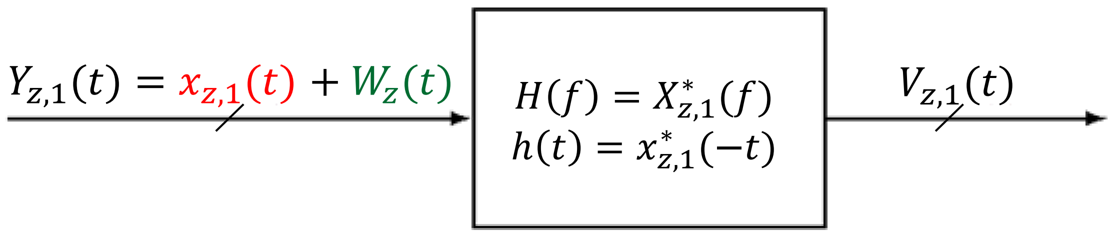


Optimum Filter $H(f)$

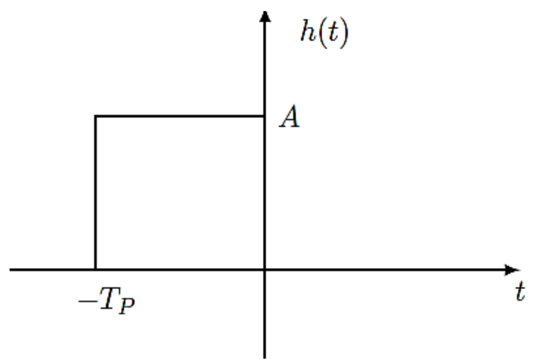
- The filter has the impulse response:

$$h(t) = x^*_{z,1}(-t) = \mathcal{F}^{-1}\{X^*_{z,1}(f)\}$$

and hence we have,



Ex.:

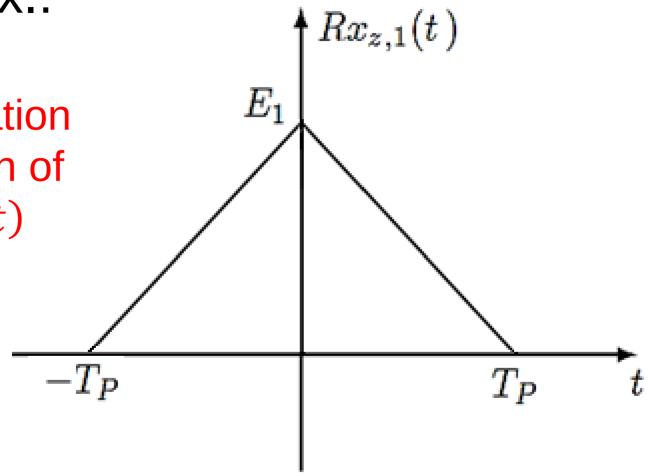


Optimum Filter $H(f)$

where $V_{z,1}(t) = \underbrace{x_{z,1}(t) * x^*_{z,1}(-t)}_{m_1(t) = R_{x_{z,1}}(t)} + \underbrace{W_z(t) * x^*_{z,1}(-t)}_{N_z(t)}$

Ex.:

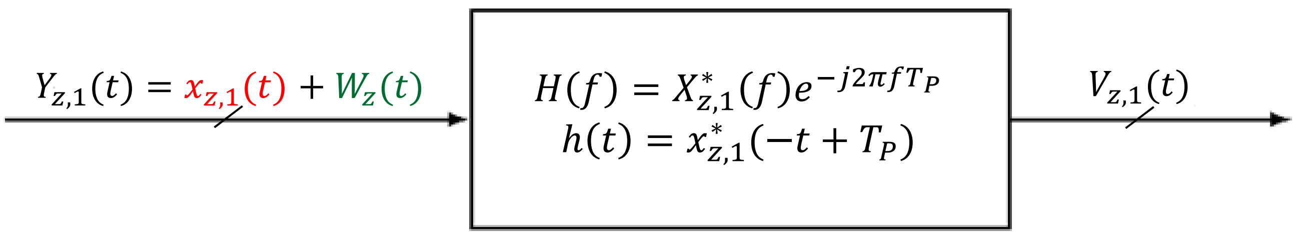
correlation function of $x_{z,1}(t)$



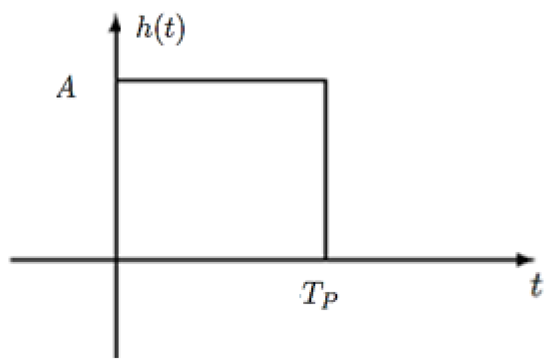
maximum of the signal at time zero

Optimum Filter $H(f)$

- The filter $h(t) = x^*_{z,1}(-t)$ is anti-causal and hence not realizable.
- We hence counter the delayed version $h(t) = x^*_{z,1}(-t + T_P)$, which is causal:

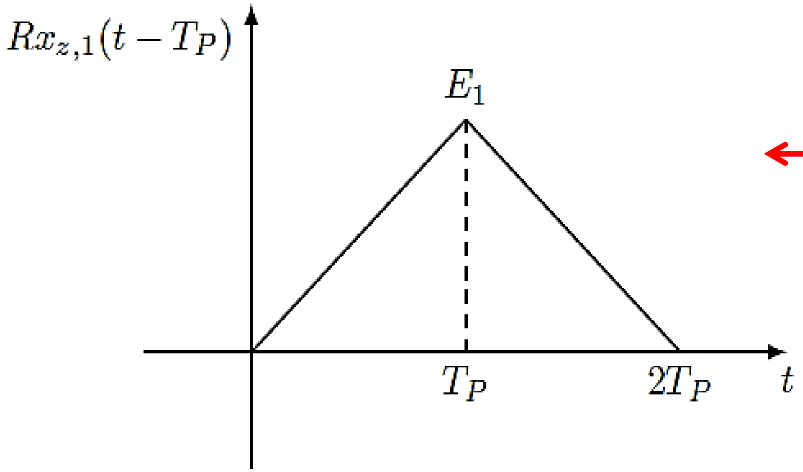


Ex.:



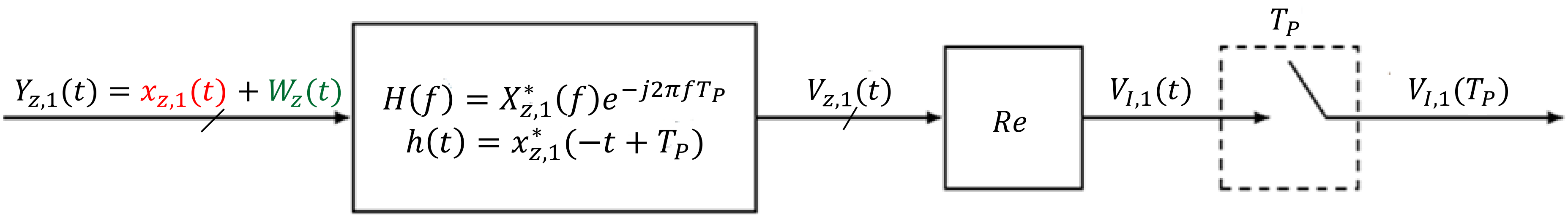
Optimum Filter $H(f)$

where $V_{z,1}(t) = \underbrace{x_{z,1}(t) * x^*_{z,1}(-t+TP)}_{m_1(t) = R_{x_{z,1}}(t-T_P)} + \underbrace{W_z(t) * x^*_{z,1}(-t+TP)}_{N_z(t)}$



maximum of the signal at time T_P

- Consider now the effect of the filter matched to $x_{z,1}(t)$ on sufficient statistics $V_I(T_P)$:



$$V_{I,1}(T_P) = \underbrace{R_{x_{z,1}}(t - T_P)|_{t=T_P}}_{= E_1} + \underbrace{N_{I,1}(T_P)}_{\sim N(a, \sigma^2_{N_{I,1}})}$$

with

$$\sigma^2_{N_{I,1}} = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2 \, df$$

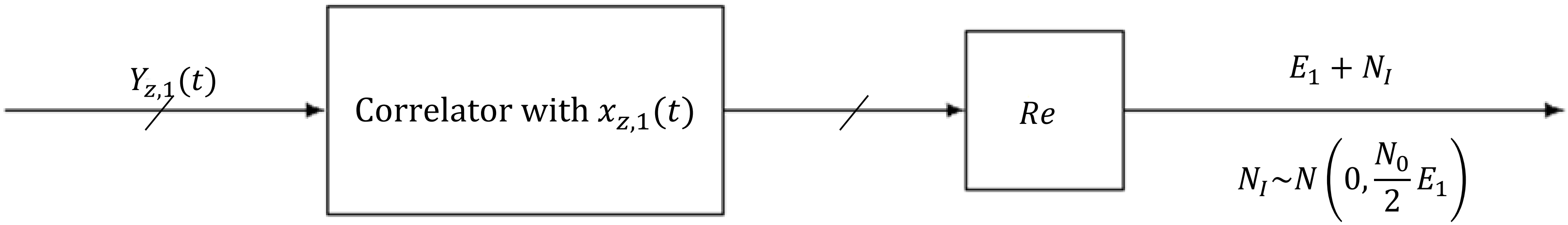
$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} G_{x_{z,1}}(f) \, df$$

$$= \frac{N_0}{2} E_1$$

Optimum Filter $H(f)$

- Matched filter + Re + Sampler = correlator + Re

- Correlator + Re



$$\int_{-\infty}^{+\infty} Y_{z,1}(t)\, x_{z,1}^{*}(t) dt$$

Optimum Filter $H(f)$

- To see the equivalence:

$$\text{matched filter} + \text{Re} + \text{sampler} = Re\{Y_{z,1}(t) * x_{z,1}^{*}(-t + T_P)\}\Big|_{t=T_P}$$

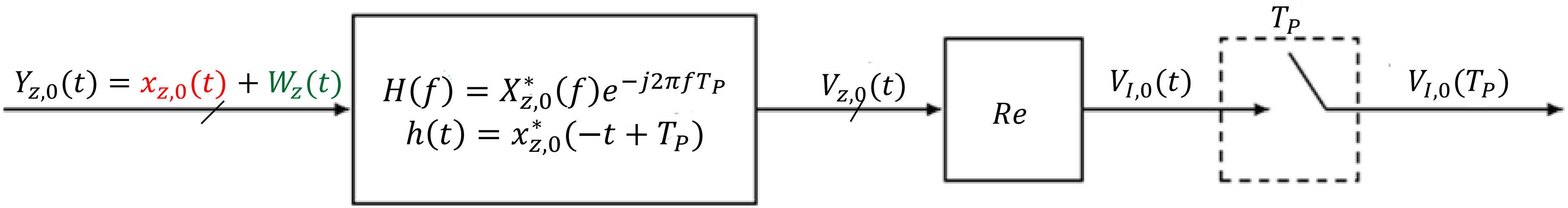
$$= Re\left\{\int_{-\infty}^{+\infty} Y_{z,1}(\tau) x_{z,1}^{*}\,(-t + T_P + \tau)\, d\tau\right\}\Bigg|_{t=T_P}$$

$$= Re\left\{\int_{-\infty}^{+\infty} Y_{z,1}(\tau) x_{z,1}^{*}\,(\tau)\, d\tau\right\}$$

$$= \text{ correlator } + \text{Re}$$

- Remark: Correlator + Re may be easer to implement especially using digital signal processing (e.g., in MATLAB)

- So far, we have built an intuition for the optimal filter if it is known that $x_{z,1}(t)$ has been transmitted
- Similarly, if $x_{z,0}(t)$ is known to have been transmitted, then the optimal filter is:



$$V_{I,0}(T_P) = R_{x_{z,0}}(t - T_P)\Big|_{t=T_P} + N_{I,0}(T_P)$$

$$\text{with } \sigma^2_{N_{I,0}} = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H(f)|^2\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} G_{x_{z,0}}(f)\, df$$

$$= \frac{N_0}{2} E_0$$

- But the receiver, of course, does not know whether $x_{z,0}(t)$ or $x_{z,1}(t)$ is transmitted.
- In this case, we have seen that the filter that minimizes the BEP is such that the effective SNR η is maximized.
- It can be proved that in this case, the optimal filter is as follows:

$$\boxed{\begin{aligned} H(f) &= \left(X_{z,1}^*(f) - X_{z,0}^*(f)\right)e^{-j2\pi f T_P} \\ h(t) &= x_{z,1}^*(-t+T_P) - x_{z,0}^*(-t+T_P) \end{aligned}}$$

matched filter (to $\{x_{z,0}(t), x_{z1}(t)\}$)

- Remark: The matched filter (to $\{x_{z,0}(t), x_{z,1}(t)\}$) is the difference between the filters matched to $x_{z,0}(t)$ and $x_{z,1}(t)$.
- The formal proof that the matched filter maximizes η can be found in the textbook, and is based on the so called Cauchy-Schwartz inequality.

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