Source Coding with in-Block Memory and Causally Controllable Side Information

O. Simeone

NJIT

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Motivation

\[ X_i = (\text{pollution level, congestion, noise level, ...}) \]
Block correlation (e.g., neighborhood, blocks, ...)
Motivation

Compression of the observations

\[ X^n \rightarrow W \ (nR \text{ bits}) \]
Motivation
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Prior Work

- Causal side information [Weissman and El Gamal ’06]
- Controllable side information [Permuter and Weissman ’11]
  [Choudhuri and Mitra ’12]
Causal Side Information
Weissman and El Gamal ’06

- The receiver filters the side information

\[ \hat{X}_i = u_i(W, Y^i) \]

- Function \( u^n(w) = (u_1(w, y_1), u_2(w, y^2), u_3(w, y^3), \ldots) \) can be thought of as a tree
Causal Side Information
Weissman and El Gamal '06

- The receiver filters the side information

\[ X^n \xrightarrow{\text{Enc}} W \xrightarrow{\hat{X}_i = u_i(W, Y^i)} \text{Dec} \]

- Function \( u^n(w) = (u_1(w, y_1), u_2(w, y^2), u_3(w, y^3), \ldots) \) can be thought of as a tree
Causal Side Information

Weissman and El Gamal '06

- Function $u^n(w) = (u_1(w, y_1), u_2(w, y^2), u_3(w, y^3), ...,)$
The side information can be controlled via cost-constrained actions

\[
\begin{align*}
X^n \xrightarrow{\text{Enc}} W & \xrightarrow{\hat{X}_i = u_i(W, Y^i)} \\
& \xrightarrow{A_i = v_i(W, Y^{i-1})} \\
X_i \xrightarrow{Y_i} & \xrightarrow{P(y_i \mid a_i, x_i)}
\end{align*}
\]

Action \( A^n \) is cost constrained as

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\Lambda(A_i)] \leq \Gamma
\]

Ex.: If \( A = 1 \), \( Y = X + Z \); and if \( A = 0 \), \( Y = \phi \). With \( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[A_i] \leq \Gamma \)
The side information can be controlled via cost-constrained actions.

Action $A^n$ is cost constrained as $\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\Lambda(A_i)] \leq \Gamma$

Ex.: If $A = 1$, $Y = X + Z$; and if $A = 0$, $Y = \phi$. With $\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[A_i] \leq \Gamma$
Function $v^n(w) = (v_1(w), v_2(w, y_1), v_3(w, y^2), \ldots)$ can be thought of as a tree.
For memoryless sources, rate-distortion-cost function

\[ R(D, \Gamma) = \min_{p(a,u|x), f(U,Y)} I(X; A, U) \]

subject to \( E[\Lambda(A)] \leq \Gamma \) and \( E[d(X, f(U, Y))] \leq D \)

- The two trees collapse as:

\[ I(X; A, U) = I(X; A) + I(X; U|A) \]

action (independent of \( Y_{i-1} \)) + estimate (dependent only on \( Y_i \)) as \( f(U, Y) \)

- (The two layers can be combined by setting \( A = g(U) \))
Controllable Causal Side Information
Permuter and Weissman ’11, Choudhuri and Mitra ’12

- For memoryless sources, rate-distortion-cost function

\[ R(D, \Gamma) = \min_{p(a,u|x), f(U,Y)} I(X; A, U) \]

subject to \( \mathbb{E}[\Lambda(A)] \leq \Gamma \) and \( \mathbb{E}[d(X, f(U, Y))] \leq D \)

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For memoryless sources, rate-distortion-cost function

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\]

\(\text{action (independent of } Y^{i-1})\) \hspace{1cm} \(\text{estimate (dependent only on } Y_i) \text{as } f(U, Y)\)

(The two layers can be combined by setting \(A = g(U)\))
Results summarized above based on the memorylessness assumption

\[
\left[ \prod_{i=1}^{n} P(x_i) \right] \cdot P(v^n, u^n|x^n) \cdot 1(a^n||v^n, 0y^{n-1})
\]

\[
\cdot 1(\hat{x}^n||u^n, y^n) \cdot \left[ \prod_{i=1}^{n} P(y_i|a_i, x_i) \right]
\]

Directed conditioning notation in [Kramer ’98]:

\[
P(w^n||v^n|t) = \prod_{i=1}^{n} P(w_i|v^i, t)
\]
Block Memory

- Block memory:

\[
\left[ \prod_{i=1}^{m} P(x_i^L) \right] P(v^n, u^n|x^n) 1(a^n\|v^n, 0y^{n-1}) \\
\cdot 1(\hat{x}^n\|u^n, y^n) \left[ \prod_{i=1}^{m} P(y_i^L\|a_i^L|x_i^L) \right]
\]

- Channel coding counterpart studied in [Kramer '12]
Block Memory

- Block memory $L = 2, n = 4$
Block Memory

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A triple \((R, D, \Gamma)\) is said to be achievable with distortion \(D\) and cost constraint \(\Gamma\), if, for all sufficiently large \(m\), there exist codetrees such that

\[
\frac{1}{mL} \sum_{i=1}^{m} E[d^L(X^L_i, \hat{X}^L_i)] \leq D + \epsilon
\]

and

\[
\frac{1}{mL} \sum_{i=1}^{m} E[\gamma^L(A^L_i, X^L_i)] \leq \Gamma + \epsilon
\]

for any \(\epsilon > 0\).
**Block Memory**

**Side Information Repeat Request**

- Upon the observation of $Y_i$, the decoder can decide whether to take a second measurement of the side information or not.
Block Memory

Block-feedforward Model

- Block-feedforward model: $Y_i = X_{i-1}$ for all $i$ not multiple of $L$ and $Y_i$ equal to a fixed symbol in $\mathcal{Y}$ otherwise

- Limited feedforward as compared to [Weissman and Merhav '03] [Venkataramanan and Pradhan '07] [Naiss and Permuter '12]
The rate-distortion-cost function can be expressed as the minimization

$$R(D, \Gamma) = \frac{1}{L} \min I(X^L; V^L, U^L)$$

where the joint distribution of the variables $X^L, Y^L, A^L, \hat{X}^L$ and of the codetrees $V^L$ and $U^L$ factorizes as

$$P(x^L)P(v^L, u^L|x^L)1(a^L||v^L, 0y^{L-1})$$
$$\cdot 1(\hat{x}^L||u^L, y^L)P(y^L||a^L|x^L),$$

and the minimization is performed over the conditional distribution $P(v^L, u^L|x^L)$ of the codetrees under the constraints

$$\frac{1}{L} E[d^L(X^L, \hat{X}^L)] \leq D \text{ and } \frac{1}{L} E[\gamma^L(A^L, X^L)] \leq \Gamma.$$
Main Results

- “Joint” codetree $j^{n+1}(w) = (j^1(w), j^2(w, y_1), \ldots, j^{n+1}(w, y^n))$

```
 i = 1  i = 2  i = 3  i = 4
```

```
\[ a_1 \]
```

```
\[ \hat{x}_1, a_2 \]
```

```
\[ y_1 = 1 \]
```

```
\[ \hat{x}_1, a_2 \]
```

```
\[ y_1 = 0 \]
```

```
\[ \hat{x}_2, a_3 \]
```

```
\[ y_2 = 0 \]
```

```
\[ \hat{x}_3 \]
```

```
\[ y_3 = 0^* \]
```

```
\[ \hat{x}_2, a_3 \]
```

```
\[ y_2 = 1 \]
```

```
\[ \hat{x}_3 \]
```

```
\[ y_3 = 1^* \]
```

```
\[ \hat{x}_3 \]
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```
\[ \hat{x}_3 \]
```

```
\[ y_3 = 1^* \]
```
The rate-distortion-cost function is given by

\[ R(D, \Gamma) = \frac{1}{L} \min I(X^L; J^{L+1}) \]

where the joint distribution of the variables \( X^L, Y^L, A^L, \hat{X}^L \) and of the codetree \( J^{L+1} \) factorizes as

\[
P(x^L)P(j^{L+1}|x^L)1(a^L||j^L, 0y^{L-1}) \\
\quad \cdot 1(\hat{x}^L||j_2^{L+1}, y^L)P(y^L||a^L|x^L),
\]

and the minimization is performed over the conditional distribution \( P(j^{L+1}|x^L) \) of the codetree under the constraints

\[
\frac{1}{L}E[d^L(X^L, \hat{X}^L)] \leq D \quad \text{and} \quad \frac{1}{L}E[\gamma^L(A^L, X^L)] \leq \Gamma
\]
Convex problem in the unknown $P(j^{L+1} | x^L)$

Extending [Dupuis et al ’04], possible to devise Blahut-Arimoto-type algorithm

Cardinality bounds in the paper

Actions and estimates can depend only on the side information samples within the same $L$-block
Convex problem in the unknown $P(j^{L+1}|x^L)$

Extending [Dupuis et al ‘04], possible to devise Blahut-Arimoto-type algorithm

Cardinality bounds in the paper

Actions and estimates can depend only on the side information samples within the same $L$-block
With action-independent side information ($P(y^L|a^L|x^L) = P(y^L|x^L)$):

$$R(D) = \frac{1}{L} \min I(X^L; U^L),$$

where the joint distribution of the variables $X^L, Y^L, \hat{X}^L$ and of the codetree $U^L$ factorizes as

$$P(x^L)P(u^L|x^L)1(\hat{x}^L|u^L, y^L)P(y^L|x^L),$$

such that $\frac{1}{L}E[d^L(X^L, \hat{X}^L)] \leq D$

- Extends [Weissman and El Gamal ’06] for $L = 1$
For the block-feedforward model, the rate-distortion function is given by

\[ R(D) = \frac{1}{L} \min I(\hat{X}^L \to X^L) \]

where the joint distribution of the variables \( X^L, Y^L \) and \( \hat{X}^L \) factorizes as

\[ P(x^L)P(\hat{x}^L|x^L)P(y^L|x^L), \]

subject to \( \frac{1}{L} E[d^L(X^L, \hat{X}^L)] \leq D \)

Consistent with [Venkataramanan and Pradhan '07]
Conclusions

- Source coding with in-block memory and causally controllable side information
- Characterization of rate-distortion-cost performance in terms of codetrees
- Special cases: side information request, block feedforward,...