Please provide clear and complete answers.

1. (4 points) Consider the optimization problems P1, P2, P3 and P4 below, where $f_o(x)$ is the cost function and $f_1(x)$ defines the inequality constraint (i.e., the problem is "minimize $f_o(x)$ s.t. $f_1(x) \leq 0$"):

P1. $f_o(x) = x$, $f_1(x) = |x|$ with domain $D = \mathbb{R}$;

P2. $f_o(x) = x^3$, $f_1(x) = -x + 1$ with domain $D = \mathbb{R}$;

P3. $f_o(x) = x^3$, $f_1(x) = -x + 1$ with domain $D = \mathbb{R}^+$;

P4. $f_o(x) = x$, $f_1(x) = \begin{cases} 
-x - 2 & \text{for } x \leq -1 \\
-2 & \text{for } -1 \leq x \leq 1 \\
x & \text{for } x \geq 1
\end{cases}$ with domain $D = \mathbb{R}$.

a. For all the problems above, state whether the problem is convex and, if so, whether Slater’s condition holds.

b. For all the problems above, derive and plot the perturbation function $p(u)$, and identify $p^*$, $d^*$ along with $x^*$ and $\lambda^*$ if they exist (please provide all the necessary details).

Sol.:

P1. a. The problem is convex, and Slater’s condition does not hold.
b. The perturbation function is $p(u) = -u$ with $domp = \{u : u \geq 0\}$. We thus have $p^* = d^* = 0$ with $x^* = 0$ and $\lambda^* = 1$ (more precisely, any $\lambda^* \geq 1$ is dual optimal).

P2. a. The problem is not convex.
b. The perturbation function is $p(u) = (1 - u)^3$ with $domp = \mathbb{R}$. We thus have $p^* = 1$, while $d^* = -\infty$, with $x^* = 1$ (the value of $d^*$ can of course also double checked by solving the dual problem).

P3. a. The problem is convex and Slater’s condition is satisfied.
b. The perturbation function is $p(u) = (1 - u)^3$ for $u \leq 1$ and $p(u) = 0$ for $u > 1$. We thus have $p^* = 1$, while $d^* = 1$, with $x^* = 1$ and $\lambda^* = -dp(0)/du = 3$.

P4. a. The problem is not convex.
b. The perturbation function is $p(u) = -2 - u$ with $domp = \{u : u \geq -1\}$. We thus have $p^* = d^* = -2$, with $x^* = -2$ and $\lambda^* = 1$.

2. (2 points) Consider the following problem

$$\begin{align*}
\text{minimize } & \|Ax - b\|_2^2 \\
\text{s.t. } & Gx = h 
\end{align*}$$

with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n$ and $G \in \mathbb{R}^{p \times n}$ with $\text{rank}(G) = p$.

a. Calculate the dual function.
b. Write the KKT conditions. Do you expect to be able to find a solution?
c. Find the optimal multiplier vector and the optimal solution as a function of the optimal multiplier.
a. The Lagrangian function is
\[ \mathcal{L}(x, \nu) = \|Ax - b\|^2 + \nu^T(Gx - h) \]
\[ = xA^TAx + (G^T\nu - 2A^Tb)^T x - \nu^Th, \]
and the dual function is thus obtained by minimizing the above strictly convex function, obtaining
\[ g(\nu) = -\frac{1}{4}(G^T\nu - 2A^Tb)^T(A^TA)^{-1}(G^T\nu - 2A^Tb) - \nu^Th. \]
b. The KKT conditions are
\[ 2A^T(Ax^* - b) + G^T\nu^* = 0 \]
\[ Gx^* = h. \]
Since the problem is convex and satisfies Slater’s conditions (it is feasible), an optimal point exists if and only if the KKT have a solution. From Weierstrass theorem, an optimal solution exists, and therefore the KKT must have a solution.
c. From the first KKT condition, we calculate
\[ x^* = (A^TA)^{-1}(A^Tb - (1/2)G^T\nu^*), \]
which gives us from the second equation:
\[ \nu^* = -2(G(A^TA)^{-1}G^T)^{-1}(h - G(A^TA)^{-1}A^Tb). \]

3. (2 points) Consider a convex problem characterized by cost function \( f_o(x) \), inequality constraints \( f_i(x) \leq 0, \ i = 1, \ldots, m \) and equality constraints \( h_i(x) = 0, \ i = 1, \ldots, p \). Prove that the perturbation function \( p(u, v) = \inf f_o(x) \), where the infimum is taken under the constraints \( f_i(x) \leq u_i \) and \( h_i(x) = v_i \ (u = (u_1, \ldots, u_m), \ v = (v_1, \ldots, v_p)) \), is a convex function. Recall that you have to prove also that the domain is a convex set.

Sol: Consider the following function
\[ g(x, u, v) = \begin{cases} \tilde{f}_o(x) & \text{if } f_i(x) \leq u_i \\
\infty & \text{and } h_i(x) = v_i \\
\infty & \text{otherwise} \end{cases} \]
where \( \tilde{f}_o(x) \) is the extended value extension of \( f_o(x) \). The function \( g(x, u, v) \) is convex in \( x, u, v \), as it can be shown from the definition of convex function. To see this, define \( y = (x, u, v) \). Now consider any \( y_1, y_2 \) and a convex combination \( y = \theta y_1 + (1 - \theta) y_2 \ (0 \leq \theta \leq 1) \). We need to show that
\[ g(y) \leq \theta g(y_1) + (1 - \theta) g(y_2). \]
There are two cases. 1) If \( g(y) = \infty \), then necessarily we must have \( g(y_1) = \infty \) and/or \( g(y_2) = \infty \). In fact, if \( g(y_1) \) and \( g(y_2) \) were finite, then, by the convexity of \( \tilde{f}_o(x) \) and \( f_i(x) \) and the fact that \( h_i(x) \) is affine, \( g(y) \) would be finite too. 2) If \( g(y) \) is finite, then we have
two subcases. 2.a) \( g(y_1) = \infty \) and/or \( g(y_2) = \infty \): in this case, the inequality is apparent; 2.b) \( g(y_1) \) and \( g(y_2) \) are finite: in this case, we have

\[
g(y) = f_o(x) \leq \theta f_o(x_1) + (1 - \theta)f_o(x_2) = \theta g(y_1) + (1 - \theta)g(y_2),
\]

which concludes the proof.

Finally, by infimizing over \( x \) we thus obtain a convex function.

4. (2 points) Consider the multiobjective optimization problem (with respect to cone \( \mathbb{R}^2_+ \)) with objective function \( f_o(x) = [F_1(x) F_2(x)]^T \) where \( F_1(x) = x_1^2 + x_2^2 \) and \( F_2(x) = (2x_1 + 3)^2 \).

a. Evaluate all the Pareto optimal values and points via scalarization (give explicit expressions for both values and points). I

b. Solve the scalarization problems with either weight equal to zero. For both cases, are the solutions of the scalar problem also Pareto optimal?

Sol.: Since the problem is convex, we know that all Pareto optimal points can be obtained via scalarization with some weight vector \( \lambda \succeq 0 \). We now fix some \( \lambda \succ 0 \) and solve the problem

\[
\text{minimize } \lambda_1(x_1^2 + x_2^2) + \lambda_2(2x_1 + 3)^2,
\]

which is equivalent to

\[
\text{minimize } (\lambda_1 + 4\lambda_2)x_1^2 + \lambda_1x_2^2 + 12\lambda_2x_1 + 9\lambda_2.
\]

Any solution to this problem will give us a Pareto optimal point and value. Since the cost function is strictly convex, the corresponding Pareto optimal point is given as

\[
x^*(\lambda_1, \lambda_2) = \left[ \frac{-6\lambda_2}{\lambda_1 + 4\lambda_2} \right],
\]

which can be also restated in terms of \( \mu = \lambda_2/\lambda_1 \) as

\[
x^*(\mu) = \left[ \frac{-6\mu}{1 + 4\mu} \right],
\]

and the value is obtained by substituting the above into \( f_o(x) \), which yields

\[
F_1^*(\mu) = \left( \frac{-6\mu}{1 + 4\mu} \right)^2,
\]

\[
F_2^*(\mu) = \left( \frac{-12\mu}{1 + 4\mu} + 3 \right)^2.
\]

To calculate the remaining Pareto optimal points and values, we need to let \( \mu \to 0 \) and \( \mu \to \infty \). With \( \mu \to 0 \), we obtain \( x^* = 0 \) and \( f_o^* = (0, 9) \), which corresponds to minimization of the norm only. This point can also be obtained with \( \lambda_2 = 0 \) and \( \lambda_1 = 1 \). With \( \mu \to \infty \), we obtain \( x^* = (-3/2, 0)^T \) and \( f_o^* = (9/4, 0) \), which corresponds to minimizing the error with the solution with minimum norm. Note that if we solved the scalarization problem with \( \lambda_2 = 1 \) and \( \lambda_1 = 0 \), any point with \( x_1 = -3/2 \) is a solution, but is not necessarily a Pareto optimal point.
5. (2 points) Consider the game described by the payoff matrix below

\[
\begin{array}{cc}
1 & 2 \\
0 & 1 \\
2 & 1 \\
1 & 0 \\
\end{array}
\]

a. Identify the Pareto optimal points, and the Nash equilibria in pure strategies.

b. Calculate all Nash equilibria in mixed strategies.

**Sol.**

a. Pareto optimal values are (1,2) and (2,1), and the Nash equilibrium in pure strategy is (2,1).

b. Considering mixed strategies, the average utilities are

\[
\bar{U}_1(p_1, p_2) = p_1 p_2 + (1 - p_1)(2p_2 + (1 - p_2))
\]

\[
= p_1 p_2 + (1 - p_1)(p_2 + 1)
\]

\[
= -p_1 + (p_2 + 1)
\]

\[
\bar{U}_2(p_1, p_2) = p_2(2p_1 + (1 - p_1)) + (1 - p_2)p_1
\]

\[
= p_2(p_1 + 1) + (1 - p_2)p_1
\]

\[
= p_2 + p_1.
\]

This shows that player 1 always chooses \( p_1 = 0 \), irrespective of the action of player 2, and player 2 chooses always action \( p_2 = 1 \) irrespective of the action of player 1. This is also clear from the payoff matrix. Therefore, there are no mixed strategy equilibria, but only pure strategy equilibria.