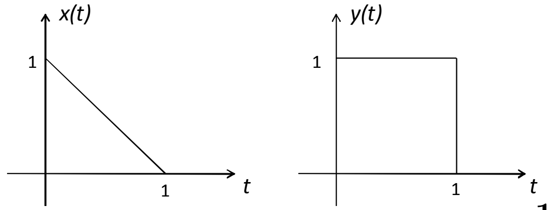
$\subset$ECE 232 - Circuits and Systems II
Final

Please provide clear and complete answers. Don't forget to specify the units of measure!

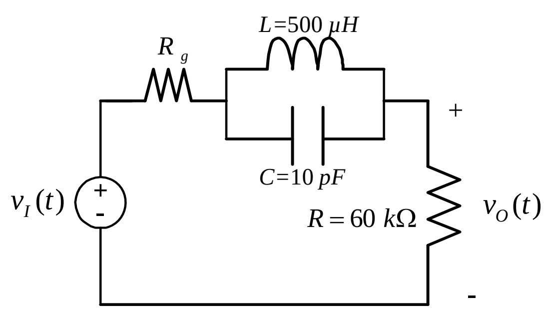


1. (2 points) Evaluate the convolution $z(t) = x(t) * y(t)$ between the two functions $x(t)$ and $y(t)$ plotted in the figure on the left (Hint: In a triangle with two 45° angles, the lengths of the two sides adjacent to the 90° angle are the same. Moreover, the area of one such triangle is simply the product of the lengths of such sides divided by two).

2. (2 points) Draw the amplitude Bode plot for the transfer function

$$H(s) = \frac{(s+0.2)}{(s+20)(s+2000)}.$$

Specify clearly the relevant values on the two axes.



3. (3 points) Consider the circuit in the figure on the left. Set $R_g = 0$ and answer the following.
a. Evaluate the transfer function between input $v_I(t)$ and $v_O(t)$. What type of filter is it?
b. Evaluate the central frequency, the maximum amplitude of the frequency response, the bandwidth and the quality factor.
c. Assuming that the input is $3u(t)$, evaluate the output in steady state.

4. (3 points) Now, assume that $R_g = 40k\Omega$ in the circuit studied at the previous point.
a. Evaluate the transfer function between input $v_I(t)$ and $v_O(t)$. Please express the transfer function by defining $K = \frac{R}{R+R_g}$ and $\alpha' = K\alpha$, where $\alpha = \frac{1}{2RC}$. What type of filter is it?
b. Evaluate the central frequency, the maximum amplitude of the frequency response, the bandwidth and the quality factor.
c. Assuming that the input is $3u(t)$, evaluate the output in steady state.

Sol.:

1. Using the hint and the usual geometrical considerations, we have:

$$
\begin{aligned}
& x(t) * y(t) \\
= & \begin{cases} 0 & \text{for } t \leq 0 \\ t(1-t) + \frac{t^2}{2} = t - \frac{t^2}{2} & \text{for } 0 \leq t \leq 1 \\ \frac{(2-t)^2}{2} & \text{for } 1 \leq t \leq 2 \\ 0 & \text{for } t \geq 3 \end{cases}
\end{aligned}
$$

2. We rewrite the transfer function as:

$$
\begin{aligned}
H(s) &= \frac{(s+0.2)}{(s+20)(s+2000)} \\
&= \frac{0.2\left(1+\frac{s}{0.2}\right)}{40000(1+\frac{s}{20})(1+\frac{s}{2000})} \\
&= \frac{1}{2}10^{-5}\frac{\left(1+\frac{s}{0.2}\right)}{(1+\frac{s}{20})(1+\frac{s}{2000})}.
\end{aligned}
$$

We thus have that the Bode plot for $\omega < 0.2$ rad/s has amplitude $20\log_{10}\left(\frac{1}{2}10^{-5}\right) = -6 - 100 = -106$ dB. The rest of the plot can be found as usual: 20 dB/decade slope between ω =0.2 and ω=20, zero slope between ω=20 and ω=2000 and -20dB/decade slope after ω=2000. We also have $|H(j0.2)| = -106dB$, $|H(j20)| = -66dB$, $|H(j2000)| = -66dB$.

3.
a. Using the voltage partition rule, we get:

$$
\begin{aligned}
H(s) &= \frac{R}{R+R_g+(sL||\frac{1}{sC})} \\
&= \frac{R}{R+R_g+(sL||\frac{1}{sC})} \\
&= \frac{R}{R+R_g+\frac{sL}{s^2LC+1}} \\
&= K\frac{s^2+\omega_0^2}{s^2+2\alpha' s+\omega_0^2},
\end{aligned}
$$

where we have defined K and α' as per question 4. From the transfer function, we recognize this to be a bandreject filter. Moreover,

$$
\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{500\cdot 10^{-6}\cdot 10\cdot 10^{-12}}} = \sqrt{2}\cdot 10^7 \text{ rad/s}.
$$

Setting $R_g = 0$, we further get

$$
\begin{aligned}
K &= 1 \\
\alpha' &= \alpha = \frac{1}{2RC} = \frac{1}{2\cdot 60\cdot 10^3\cdot 10\cdot 10^{-12}} = \frac{25}{3}\cdot 10^5,
\end{aligned}
$$

so that we obtain

$$H(s) = K\frac{s^2 + 2 \cdot 10^{14}}{s^2 + \frac{50}{3} \cdot 10^5 s + 2 \cdot 10^{14}}.$$

b. The central frequency is $\sqrt{2} \cdot 10^7 = 1.4 \cdot 10^7$ rad/s. The maximum amplitude of the frequency response is

$$|H(j0)| = |H(j\infty)| = K = 1.$$

The bandwidth is $\beta = 2\alpha' = 2\alpha = \frac{50}{3} \cdot 10^5 = 16.6 \cdot 10^5$ rad/s and the quality factor is $Q = \frac{\sqrt{2} \cdot 10^7}{\frac{50}{3} \cdot 10^5} = 6\sqrt{2} = 8.48$.
c. The output in steady state is

$$3|H(j0)|u(t) = 3u(t).$$

4. The solution follows the same calculations made above. In particular, we have the following.
a.

$$H(s) = K\frac{s^2 + \omega_0^2}{s^2 + 2\alpha' s + \omega_0^2}.$$

From the transfer function, we recognize this to be a bandreject filter. Moreover,

$$\omega_0 = \sqrt{2} \cdot 10^7 \text{ rad/s}.$$

Setting $R_g = 40k\Omega$, we further get

$$\begin{aligned} K &= \frac{R}{R + R_g} = \frac{60 \cdot 10^3}{40 \cdot 10^3 + 60 \cdot 10^3} = \frac{3}{5} \\ \alpha' &= K\alpha = 5 \cdot 10^5, \end{aligned}$$

so that we obtain

$$H(s) = \frac{3}{5}\frac{s^2 + 2 \cdot 10^{14}}{s^2 + 10 \cdot 10^5 s + 2 \cdot 10^{14}}.$$

b. The central frequency is $\sqrt{2} \cdot 10^7$ rad/s. The maximum amplitude of the frequency response is

$$|H(j0)| = |H(j\infty)| = K = \frac{3}{5} = 0.6.$$

The bandwidth is $\beta = 2\alpha' = 10 \cdot 10^5$ rad/s and the quality factor is $Q = \frac{\sqrt{2} \cdot 10^7}{10 \cdot 10^5} = 10\sqrt{2} = 14.14$. Adding the reistance R_g has decreased the quality factor of the filter.
c. The output in steady state is

$$3|H(j0)|u(t) = 3\frac{3}{5}u(t) = 1.8u(t).$$