Q1 (1 point). Consider a Markov chain $X_1 - X_2 - X_3$ with $|X_1| = 3$, $|X_2| = 2$, $|X_3| = 4$ prove that $I(X_1; X_3) \leq 1$.

_Sol._ We have

$$I(X_1; X_3) \leq I(X_1; X_2) \leq H(X_2) \leq \log_2 |X_2| = 1,$$

where the first inequality follows from the data processing inequality.

Q2 (1 point). Consider an i.i.d. process $X^n$ with pdf $f(x) = c \cdot 2^{-x^4}$, where $c$ is an appropriate constant. Prove that

$$\Pr \left[ \frac{1}{n} \sum_{i=1}^{n} x_i^4 \leq h(X) + \log_2 c + \epsilon \right] \rightarrow 1 \text{ as } n \rightarrow \infty$$

for any $\epsilon > 0$ (Hint: What is the set of typical sequences?)

_Sol._ The set of typical sequences is

$$A^{(n)}_{\epsilon} = \left\{ x^n \in \mathbb{R}^n : h(X) - \epsilon \leq -\frac{1}{n} \log_2 f(x^n) \leq h(X) + \epsilon \right\}$$

but $f(x^n) = c^n \prod_{i=1}^{n} 2^{-x_i^4}$, so $-\frac{1}{n} \log_2 f(x^n) = -\log_2 c + \frac{1}{n} \sum_{i=1}^{n} x_i^4$, so that we can write

$$A^{(n)}_{\epsilon} = \left\{ x^n \in \mathbb{R}^n : \log_2 c + h(X) - \epsilon \leq \frac{1}{n} \sum_{i=1}^{n} x_i^4 \leq \log_2 c + h(X) + \epsilon \right\},$$

which answers the question.

Q3 (1 point). Consider an additive white Gaussian noise channel $Y = X + Z$ with $Z \sim \mathcal{N}(0, N)$ subject only to an output power constraint of $P$ (no constraint is imposed on the input!). Evaluate the corresponding capacity

$$C = \max_{f(x): E[Y^2] \leq P} I(X; Y)$$

_Sol._ We have

$$I(X; Y) = h(Y) - h(Y|X) = h(Y) - \frac{1}{2} \log_2 (2\pi e N)$$

$$\leq \frac{1}{2} \log_2 (2\pi e P) - \frac{1}{2} \log_2 (2\pi e N)$$

$$= \frac{1}{2} \log_2 \left( \frac{P}{N} \right).$$
where the inequality follows from the fact that the entropy $h(Y)$ is maximized by a Gaussian distribution under power constraint $E[Y^2] \leq P$.

**Q4** (1 point). Consider the i.i.d. Gaussian vector process $[X_1 \ X_2]$ with zero-mean, independent entries and powers $E[X_1^2] = 1$ and $E[X_2^2] = 3$. Focusing on the MSE distortion, find the rate-distortion function using reverse waterfilling.

**Sol.** Using reverse waterfilling, we find that

- If $D \geq 4$, $R(D) = 0$;
- If $2 \leq D \leq 4$, $R(D) = \frac{1}{2} \log_2 \frac{3}{D-1}$;
- If $D \leq 2$, $R(D) = \frac{1}{2} \log_2 \frac{1}{D/3} + 1/2 \log_2 \frac{3}{D/3}$.

**P1** (2 points) Consider the channel

$$p(y|x) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

where rows correspond to different $x \in \{0,1,2\}$ and columns to different $y \in \{0,1,2\}$. The number of channel uses is $n$.

a. Argue that symbol $x = 2$ should not be used.

b. Using only the input set $\{0,1\}$ find the optimal input distribution and the capacity.

c. Assume that we want to operate the channel with a (suboptimal) code (of length $n$) that is randomly generated with $p_X(0) = 3/4$ and $p_X(1) = 1/4$. What is the rate achievable with this code?

d. To achieve the rate $R$ obtained at the previous point, consider the following transmission scheme. An i.i.d. binary code $B^{2n}$ of length $2n$ and rate $2R$ is constructed i.i.d. with $p_B(0) = 1/2$ and $p_B(1) = 1/2$. Every two bits of $B^{2n}$ are mapped into one input bit of the transmitted sequence $X^n$. Find one such mapping so that $X^n$ has the desired distribution ($p_X(0) = 3/4$ and $p_X(1) = 1/4$).

**Sol.**

a. Transmitting $x = 2$ may result in any output with the same probability, so that the decoder has no means to find out whether $x = 2$ was transmitted.

b. The channel is (weakly) symmetric. In fact, it is an erasure channel with erasure probability $1/3$. Therefore, the optimal input distribution is $p_X(0) = 1/2$. The capacity is

$$C = 1 - 1/3 = 2/3.$$ 

c. The rate achievable by this code is

$$R = I(X;Y) = H(Y) - H(1/3) = H(1/2, 1/3, 1/6) - 0.92 = 1.46 - 0.92 = 0.54,$$
where we have calculated $p_Y(0) = 3/4 \cdot 2/3 = 1/2$, $p_Y(1) = 1/3 \cdot 3/4 + 1/3 \cdot 1/4 = 1/3$, $p_Y(2) = 1/6$.

d. A mapping that satisfies the requirement is

\[
\begin{align*}
00 &\rightarrow 0 \\
01 &\rightarrow 0 \\
10 &\rightarrow 0 \\
11 &\rightarrow 1.
\end{align*}
\]

**P2** (2 points) Consider a horse race with four horses with winning probabilities $p = \{1/8, 1/8, 1/4, 1/2\}$. The odds fair with respect to the uniform distribution $r = \{1/4, 1/4, 1/4, 1/4\}$. You start with $1$.

a. Consider the betting strategy $b = \{1/4, 1/4, 1/4, 1/4\}$. What is the (long-term) doubling rate of this strategy? Approximately, how much money you would have after 100 races?

b. What is the optimal betting strategy $b_1, b_2, b_3, b_4$ that maximizes the (long-term) doubling rate? Find the corresponding doubling rate. Approximately, how much money you would have after 100 races?

**Sol.** a. The doubling rate is

\[
W = D(p||r) - D(p||b)
= \sum_k p_k \log_2 \left( \frac{b_k}{r_k} \right) = 0,
\]

therefore the amount of money after 100 races is about

\[
S_n \simeq 1 \cdot 2^{100W} = 1.
\]

b. The optimal betting strategy is $b = p$. This leads to the doubling rate

\[
W^* = D(p||r)
= 1/4
\]

so that

\[
S_n \simeq 1 \cdot 2^{\frac{100}{4}}
= 33554432.
\]

**P3** (2 points) Consider the additive Gaussian noise channel

\[
Y_i = X_i + Z_i,
\]

where we have power constraint on $X^n$ equal to $P = 3$. The noise $Z^n$ has independent Gaussian samples but with possibly non-zero and time-varying mean.

a. Assume that $Z_i \sim \mathcal{N} \left( \frac{1}{2}, 1 \right)$ and find the capacity.

b. Assume that $Z_i \sim \mathcal{N} \left( Q_i, 1 \right)$ with $Q_i$ being a random variable with $Q_i \sim \mathcal{N} \left( 0, 2 \right)$. Find the capacity.
c. For the scenario at point a, consider the transmission of an i.i.d. Gaussian process $V^n$ with $V_i \sim \mathcal{N}(0, 1)$. Can we construct a source-channel coding scheme that enables transmission at MSE equal to 0.75?

d. Repeat the point above with MSE equal to 0.2.

**Sol:**

a. By evaluating the mutual information, one obtains $C = \frac{1}{2} \log_2(1 + 3) = 1$.

b. We can write

$$Z_i = Q_i + U_i,$$

where $U_i \sim \mathcal{N}(0, 1)$ and independent of $Q_i$. Therefore, $Z_i \sim \mathcal{N}(0, 3)$ and $C = \frac{1}{2} \log_2(1 + 3/3) = 1/2$.

c. We know that this is possible if and only if

$$R(D) = \frac{1}{2} \log_2 \frac{1}{D} \leq C = 1.$$

Therefore, $D = 0.75$ is feasible since $R(0.75) = 0.2$.

d. This is instead not possible since $R(0.2) = 1.16$. 