

Random signal analysis I (ECE673)
Assignment 9

1) Consider the random process defined as

$$X[n] = 2U[n] - 4U[n - 1],$$

where $U[n]$ is a white noise with zero mean and variance $\sigma^2 = 1$.

(i) Is this process WSS? If so, evaluate, auto-correlation sequence (see previous assignment) and power spectral density.

(ii) Generate a realization of 1000 samples of $X[n]$ by using MATLAB. Based on this realization, estimate the power spectral density using the periodogram and plot the estimate. Compare the estimate with the true power spectral density.

(iii) Can you propose a method to improve the estimate at the previous point? Verify by using MATLAB that the proposed technique improves the performance by plotting the corresponding estimate.

Solution:

(i) In order to check if the process is WSS, we need to verify the following conditions on the mean and covariance sequences

$$\begin{aligned}\mu_X[n] &= \mu \\ c_X[n, n+k] &= c_X[k]\end{aligned}$$

or equivalently

$$\begin{aligned}\mu_X[n] &= \mu \\ E[X[n]X[n+k]] &= r_X[k].\end{aligned}$$

Let us calculate these moments for the random process at hand

$$\begin{aligned}\mu_X[n] &= E[2U[n] - 4U[n - 1]] = 0 \\ E[X[n]X[n+k]] &= E[(2U[n] - 4U[n - 1])(2U[n+k] - 4U[n+k - 1])] = \\ &= 4E[U[n]U[n+k]] + 16E[U[n-1]U[n+k-1]] - \\ &\quad - 8E[U[n]U[n+k-1]] - 8E[U[n-1]U[n+k]] \\ &= \begin{cases} 20 & k = 0 \\ -8 & k = \pm 1 \\ 0 & elsewhere \end{cases} = 20\delta[k] - 8\delta[k-1] - 8\delta[k+1].\end{aligned}$$

It can be concluded that the process is WSS (it is a specific kind of MA process) and we have mean sequence and correlation function as follows:

$$\begin{aligned}\mu_X[n] &= 0 \\ r_X[k] &= 20\delta[k] - 8\delta[k-1] - 8\delta[k+1].\end{aligned}$$

(ii) The true power spectral density is obtained as the Fourier transformation on the auto-correlation function $r_X[k]$

$$\begin{aligned} P_X(f) &= \sum_{k=-\infty}^{\infty} r_X[k] \exp(-j2\pi f k) = \\ &= 20 - 8(\exp(-j2\pi f) + \exp(j2\pi f)) = \\ &= 20 - 16 \cos(2\pi f). \end{aligned}$$

An estimate can be obtained by using the periodogram, i.e., the magnitude squared of the Fourier transform of a realization of the random process. A MATLAB code that performs this operation by employing the command FFT (Fast Fourier Transform) and compares the estimated to the true power spectral density is as follows:

```
N=1000;
u=randn(N,1);
for n=1:N
    if n==1 x(n)=2*u(n);
    else
x(n)=2*u(n)-4*u(n-1);
    end
end
f=[0:N-1]./N; %vector of frequencies at which the FFT command computes the Fourier Transform
P=1/N*abs(fft(x)).^2; %periodogram
plot(f-0.5,fftshift(P));
hold on;
plot((f-0.5),20-16*cos(2*pi*(f-0.5)),'-'); %true power spectral density
```

(iii) One way to improve the estimate is to partition the realization into subsequences, calculate the periodogram on each subsequence and finally average the periodograms obtained on all subsequences. The following code partitions the initial realization of $N = 1000$ samples into 10 subsequences of 100 samples each.

```
N=1000;
K=10; %number of subsequences
Ns=N/K; %length of each subsequence
P=0;
u=randn(N,1);
for n=1:N
    if n==1 x(n)=2*u(n);
    else
x(n)=2*u(n)-4*u(n-1);
    end
end
for k=1:K-1 %for each subsequence
xi=x(1+k*Ns:k*Ns+Ns);
```

```

Pi=1/Ns*abs(fft(xi)).^2;
P=P+Pi;
end
P=1/K*P;
f=[0:Ns-1]./Ns; %vector of frequencies at which the FFT command computes the Fourier
Transform (notice that the number of frequencies is the length of the subsequences)
plot(f-0.5,fftshift(P));
hold on;
plot((f-0.5),20-16*cos(2*pi*(f-0.5)),'-')

```

2) Let us consider the problem of prediction for the random process studied at the previous point. In particular, we would like to obtain the optimal linear estimate of $X[n+k]$ given the observation $X[n]$:

$$\hat{X}[n+k] = aX[n] + b.$$

- (i) Consider at first the prediction at one step, i.e., $k = 1$. Find the correlation coefficient between $X[n]$ and $X[n+1]$. Based on this calculation, do you expect linear prediction to be effective? Evaluate the optimal predictor $\hat{X}[n+1]$ and the corresponding mean square error.
- (ii) Let us now set $k = 2$. Find the correlation coefficient between $X[n]$ and $X[n+2]$. Based on this calculation, do you expect linear prediction to be effective? Evaluate the optimal predictor $\hat{X}[n+2]$ and the corresponding mean square error.
- (iii) How would you generalize the results at point (ii) for $k > 2$?

Solution:

(i) The correlation coefficient between $X[n]$ and $X[n+1]$ is

$$\rho_{X[n],X[n+1]} = \frac{\text{cov}(X[n], X[n+1])}{\sqrt{\text{var}(X[n])\text{var}(X[n+1])}} = \frac{r_X[1] - \mu^2}{r_X[0]} = \frac{r_X[1]}{r_X[0]} = \frac{-8}{20} = -\frac{2}{5},$$

therefore linear prediction is expected to be effective. The optimal predictor reads

$$\begin{aligned} \hat{X}[n+1] &= E[X[n+1]] + \frac{\text{cov}(X[n], X[n+1])}{\text{var}(X[n])}(X[n] - E[X[n]]) = \\ &= -\frac{2}{5}X[n], \end{aligned}$$

and the mean square error is

$$\begin{aligned} mse &= \text{var}[X[n+1]] - \frac{\text{cov}(X[n], X[n+1])^2}{\text{var}(X[n])} = r_X[0] - \frac{r_X[1]^2}{r_X[0]} = \\ &= 20 - \frac{64}{20} = \frac{84}{5} = 16.8 < \text{var}[X[n+1]] = 20. \end{aligned}$$

(ii) In this case, we have

$$\rho_{X[n],X[n+2]} = \frac{\text{cov}(X[n], X[n+2])}{\sqrt{\text{var}(X[n])\text{var}(X[n+1])}} = \frac{r_X[2] - \mu^2}{r_X[0]} = 0,$$

therefore linear prediction is not effective and the optimal predictor of $X[n+2]$ is simply

$$\hat{X}[n+2] = 0,$$

with mean square error

$$mse = var[X[n+2]] = 20.$$

(iii) The results obtained above for $k = 2$ generalize to any $K > 1$. In particular, we have

$$\begin{aligned}\hat{X}[n+k] &= 0 \\ mse &= var[X[n+k]] = 20\end{aligned}$$

for any $K > 1$.