Scaling Law of the Sum-Rate for Multi-Antenna Broadcast Channels with Deterministic or Selective Binary Feedback

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Abstract—The sum-capacity of the multi-antenna Gaussian broadcast channel is known to be achieved by Dirty Paper Coding techniques, that require full channel state information at the base station. It has been recently shown that a sum-rate having the same scaling law of the sum-capacity with respect to the number of users \( n \) for a fixed signal to noise ratio (i.e., \( M \log \log n \) where \( M \) is the number of transmitting antennas) can be achieved by using reduced feedback (or equivalently reduced channel state information at the transmitter). In particular, it has been proved that \( n \) real and \( n \) integer numbers are enough to guarantee the optimal scaling law.

In this paper, the optimal scaling law of the sum-rate is shown to be achievable with an even smaller amount of feedback and, more precisely, with

1) \( n \log_2(M + 1) \) bits, if a deterministic feedback scheme is employed;

2) an average number of feedback bits that scales as \( M \log_2 M \log n \) with the number of users \( n \), if a selective (random) feedback scheme is employed.

I. INTRODUCTION

In the process of defining the evolution of current cellular communication standards and setting the ground for future wireless systems, researchers have recently focused on advanced techniques for improving the performance of broadcast (downlink) channels [1]. As witnessed by already finalized standards for high data rate downlink [2], the crucial factor that allows performance enhancement is the availability of channel state information (CSI) at the base station (BS). In fact, based on CSI, scheduling and coding can be designed at the BS so as to take advantage of the instantaneous fading states of different users, an effect usually referred to as multiuser diversity [3] [4]. Clearly, the increased downlink capacity is obtained at the expense of a dedicated uplink channel for CSI feedback by the mobile terminals to the BS.

From an information theoretic standpoint, the capacity region of a general Gaussian multi-antenna broadcast channel has been recently derived in [5], following earlier references [6] [7], where the sum-capacity was computed. These results show that the capacity achieving transmission scheme for the channel at hand is the Dirty Paper Coding (DPC) technique proposed in [8]. In [9], a simpler strategy based on zero-forcing beamforming is proved to be optimal in terms of sum-capacity in the limit of a large number of users. However, both DPC and zero-forcing beamforming require full CSI at the transmitter, that in turn requires a large rate feedback channel. For this reason, recent analyses have focused on the impact of reduced feedback, or equivalently of reduced CSI at the transmitter, on the performance of multi-antenna broadcast systems.

In [10], a scheme based on the opportunistic beamforming principle is proposed that requires the feedback of only \( n \) real and \( n \) integer numbers, where \( n \) is the number of users. Each user measures the signal to noise plus interference ratios (SINRs) on different randomly selected beams and feeds back the index of the best beam and the corresponding SINR. The authors show that the sum-rate \( R \) of this scheme presents the same scaling law with respect to the number of users \( n \) (for fixed SNR) of optimal DPC, namely \( R \) scales as \( M \log \log n \). This result proves that multiuser diversity gains can be obtained even with reduced feedback. Notice that in this paper we focus for simplicity on a multi-antenna broadcast channel where each user is equipped with one antenna. Attempts to further reduce the amount of feedback have followed two main directions: (i) quantization: the \( M \) SINRs per users are quantized to a fixed number of bits [11]. For instance, a very coarse quantization with only one bit was investigated in [12]–[14]; (ii) selective feedback: only the users with a sufficiently large SINR transmit a feedback signal [15]. Notice that in this second case, the amount of feedback is random, depending on the number of users that for a given fading realization satisfy a threshold condition. However, it is still not clear whether the optimal scaling law for the sum-rate can be obtained by using a feedback of less than \( n \) real and \( n \) integer numbers.

A. Main results

In this paper, we show that the optimal scaling law of the sum-rate for multi-antenna Gaussian broadcast channel can be achieved with a smaller amount of feedback than already reported. In particular, we show that: 1) the optimal scaling law \( M \log \log n \) can be attained by using a deterministic feedback of \( n \log_2(M + 1) \) bits. This result is proved by evaluating the scaling law of the sum-rate achievable by the following
transmission scheme. The BS randomly selects $M$ beams, while each user feeds back $\log_2(M + 1)$ bits in order to identify the beam with the best SINR that crosses a pre-selected threshold (one bit quantization)\(^1\). The BS then selects randomly which user to serve on each beam among the ones that have signaled the corresponding SINR to be above the threshold. Notice that this scheme differs from the one proposed in [13], in that therein only one beam out of $M$ is employed for transmission; 2) if the number of feedback bits is allowed to be a random variable\(^2\), the optimal scaling law $M \log \log n$ can be guaranteed by using an average number of feedback bits that scales as $M \log_2 M \log n$ with the number of users $n$. In this case, similarly to the selective feedback principle [15], the transmission scheme that is proved to attain the stated result works as described above with the only modification that no signal is fed back by a user if there is no beam that crosses the threshold. The rest of the paper is organized as follows. A random multi-beam transmission scheme with limited (deterministic or selective) feedback is presented in Sec. II. This scheme is proved in Sec. III to be able to achieve the optimal scaling law with respect to the number of users (see Theorem 1). The amount of feedback bits required on average by the selective version of the transmission scheme is evaluated in Sec. IV along with its scaling law (see Theorem 2).

II. MULTI-BEAM TRANSMISSION WITH LIMITED FEEDBACK

This section describes the transmission scheme employed to prove the achievability of the optimal scaling law of the sum-rate with limited feedback. We consider a downlink in which the BS uses a set of $M$ random orthonormal beams $\mathcal{U} = \{u_1, \ldots, u_M\}$, generated from an isotropic distribution [16], and constructs the transmitted signal $x$ as

$$ x = \sum_{m=1}^{M} u_m s_m, \quad (1) $$

where we assume that $s_m$’s are letters from a capacity-achieving Gaussian codebook. The signal received by the $i$th user is given by

$$ y_i = \sqrt{\rho} h_i^T x + n_i, \quad (2) $$

where the transmitted vector $x \in \mathbb{C}^{M \times 1}$ has a power constraint $\mathbb{E}[\|x\|^2] = M$. The AWGN $n_i$ has unit variance $CN(0, 1)$. Hence, the average SNR at the receiver is $M \rho$. The channel $h_i$ is a $M \times 1$ dimensional vector of i.i.d. zero mean circularly symmetric complex Gaussian random variables with unit variance $CN(0, 1)$, independent among different users. It is assumed that the channel is perfectly known at the receiver and that communication spans a large number of channel coherence periods (ergodic model). Let us assume that the $r$th beam is intended for user $i$. The received signal in (2) may now be restated as

$$ y_i = \sqrt{\rho} h_i^T u_r s_r + \sqrt{\rho} \sum_{m \neq r} h_i^T u_m s_m + n_i, \quad (3) $$

and the corresponding SINR for the $i$th user is given by

$$ S_{i,r} = \frac{|h_i^T u_r|^2}{\sum_{m \neq r} |h_i^T u_m|^2 + \rho}. \quad (4) $$

Prior to transmission, every user, say the $i$th, calculates the SINR for all beams $(S_{i,m} \ m = 1, \ldots, M)$ and identifies the beam $r$ that has maximum SINR,

$$ r = \arg\max_m S_{i,m}. \quad (5) $$

Then, user $i$ compares the corresponding $S_{i,r}$ to a certain threshold $\alpha$, which is a network parameter. If $S_{i,r} > \alpha$, the user feeds back the beam index $r$ to the transmitter, signaling that it is suited for transmission along beam $u_r$. Otherwise, if $S_{i,r} < \alpha$, the message is $r = 0$, for the case of deterministic feedback, or no feedback at all when employing selective feedback. Once the feedback is received, the BS schedules on each beam a user randomly selected among all users who signaled that beam to be above the threshold. Notice that there is a small probability that a certain beam is not requested by any user, i.e., that no user measures a strong enough SINR on the beam. In this event, we assume that the BS communicates through the unrequested beam to a user picked randomly from the entire set of users. This assumption ensures that equation (4) holds at all times, which is mathematically convenient for our analysis.

The number of possible messages for deterministic feedback is $M+1$, which requires $\log_2(M + 1)$ bits of feedback per user or $n \log_2(M + 1)$ bits per cell. For selective feedback, the average feedback per cell is $b_n = \bar{n} \log_2(M)$, where $\bar{n}$ is the average number of users sending feedback bits in any given coherence period. The average number of bits $b_n$ will be evaluated in Sec. IV, along with its scaling law with respect to $n$. In the following section we evaluate the scaling law of the sum-rate of multi-beam transmission with either deterministic or selective feedback.

III. SCALING LAW OF THE SUM-RATE OF MULTI-BEAM TRANSMISSION

The SINR at user $i$ for beam $m$ $(S_{i,m})$ is identically distributed\(^3\) for all $i$ and $m$, due to the symmetry of the setup, and is characterized by the cumulative distribution function [10]

$$ F_s(x) = 1 - \frac{e^{-x/\rho}}{(1 + x)^{M-1}}. \quad (6) $$

Lemma 1: For $\alpha > 1$, the sum-rate of the multi-beam transmission scheme described in Sec. II is lower bounded by

$$ R \geq M \left(1 - (F_s(\alpha))^{\bar{n}}\right) \mathbb{E}[\log(1 + S) \mid S > \alpha]. \quad (7) $$

\(^1\)The “+1” in $\log_2(M + 1)$ account for the possibility that no beam crosses the threshold.

\(^2\)by, e.g., using a contention channel for the uplink feedback.

\(^3\)but not independent among beams of the same user.
The proof of Lemma 1 is presented in the Appendix. For the case of $M = 1$, (7) reduces to the expression obtained in [14].

**Theorem 1:** Let $M$ and $\rho$ be fixed. The scaling law of sum-rate achievable by the multi-beam scheme in Sec. II satisfies

$$\lim_{n \to \infty} \frac{R}{M \log \log n} = 1,$$

which concludes the proof.

**Corollary 1:** Let $g_n = b_n/n$ be the average feedback per user. A direct consequence of Theorem 2 is that

$$\lim_{n \to \infty} g_n = 0.$$

Therefore, the average number of feedback bits per user approaches zero as $n$ grows to infinity. This result can be intuitively explained by noticing that in order to fully capitalize on multiuser diversity, it is enough that only the users with the best SINR feed back information to the BS. Therefore, the fraction of users using the feedback channel vanishes for an asymptotically large number of users.

**V. CONCLUSION**

In this paper, it has been shown that the optimal scaling law of the sum-rate for a multi-antenna broadcast channel with respect to the number of users can be attained with a smaller amount of feedback (from the terminals to the base station) than already reported. In particular, it is proved that either a deterministic feedback of $n \log_2 (M + 1)$ bits per cell or an average number of feedback bits that scales as $M \log_2 M \log n$ as a function of $n$ is enough to achieve the optimal scaling law for fixed SNR.

**VI. APPENDIX: PROOF OF LEMMA 1**

The sum-rate of the multi-beam scheme is lower bounded by

$$R \geq \sum_{m=1}^{M} P_m \mathbb{E} \left[ \log (1 + S_m) | S_m > \alpha \right],$$

where $P_m$ is the probability of at least one user signaling beam $m$ to be above the threshold. Since the probability is equal for all beams, we henceforth drop its dependence on $m$. Expression (7) is only a lower bound on the actual achievable rate since it neglects the time instants where user $i$ is scheduled on a beam that no user has requested (recall the description of the scheduling process in Sec. II). Notice that the probability that no user measures a strong enough SINR on a given beam (for an appropriately selected threshold $\alpha$) is expected to be small for large number of users $n$. The SINR is identically distributed for all beams, thus the sum-rate in (18) becomes

$$R \geq P \mathbb{E} \left[ \log (1 + S) | S > \alpha \right].$$

According to [10], the SINRs measured by each user can be larger than one on at most one beam. Therefore, for $\alpha > 1$, $P$ equals the probability of at least one user having $S_{i,r} > \alpha$:

$$P = 1 - (F_s(\alpha))^n = 1 - \left( 1 - \frac{e^{-\alpha/\rho}}{(1 + \alpha)^{M-1}} \right)^n.$$

Substituting equation (20) in equation (19) we conclude the proof. The fact that this expression is only valid for $\alpha > 1$ is not an issue, since in all the proofs included in this paper the threshold $\alpha$ is selected to be strictly greater than one.

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REFERENCES


