1. Consider a deterministic broadcast channel $Y_1 = g_1(X)$ and $Y_2 = g_2(X)$ with $f_i(\cdot)$ given functions. The capacity region for this channel was found in class (see (7.20) in the booklet). Notice that capacity region needs to be evaluated by taking the union over all possible $P_X$. 

1.1. Find the distribution $P_{U_1,U_2}$ and function $X = f(U_1,U_2)$ that achieve such region for a given $P_X$ in the general Marton’s achievable region (see (7.18) in the booklet). In other words, find the assignment $P_{U_1,U_2}$ and $X = f(U_1,U_2)$ that reduces (7.18) to (7.20) (Hint: Fix a $P_X$...)

Let us see how we can achieve the corner rates $R_1 = H(Y_1)$ and $R_2 = H(Y_2|Y_1)$. For this, consider all the typical sequences $T^n_0(P_X)$ for some chosen $P_X$ and large $n$ (How many such sequences do we have?). Notice that a given $P_X$ induces a $P_{Y_1,Y_2}$.

1.2. How many distinct pairs of sequences $(Y_1^n,Y_2^n)$ corresponds to the input sequences in $T^n_0(P_X)$? How many distinct sequences $Y_1^n$? How many distinct sequences sequences $Y_2^n$? How many distinct sequences sequences $Y_2^n$ that appear with a given $Y_1^n = y_1^n$?

1.3. From the results at the point above, it follows that, to achieve $R_1 = H(Y_1)$, we need to assign a different index $w_1$ to all pairs with distinct $Y_1^n$. What is the maximum $R_2$ then and why? (Hint: It should follow immediately from the previous point).

2. Consider a Gaussian multiple access channel (MAC). Assume that the decoder is only interested in $W_1$ and not in $W_2$. We want to find a set of rates $(R_1,R_2)$ for which $W_1$ can be decoded correctly, while $W_2$ may or may not be decoded successfully (in other words, the probability of error is $P_e^n = \Pr[\hat{W}_1 \neq W_1]$). We expect this rate region to be larger than the MAC capacity region found in class (where both $W_1$ and $W_2$ needed to be decoded). To this end, consider a receiver that runs two decoders: 1) A joint typicality decoder for both $(W_1,W_2)$, as studied in class; 2) A decoder that treats $X_2^n(W_2)$ as noise and decodes only $W_1$.

2.1. What is the achievable rate region for such receiver (i.e., the rate region for which $W_1$ gets decoded with vanishingly small probability of error)?

2.2. For what rates $R_2$ should the receiver run decoder 1 or 2?

3. We have seen that the capacity of a broadcast channel depends only on the marginals $P_{Y_1|X}$ and $P_{Y_2|X}$. As an example, for the Gaussian broadcast channel

$$
Y_1 = X + Z_1 \\
Y_2 = X + Z_2,
$$

with $[Z_1,Z_2] \sim \mathcal{N}\left(0, \begin{bmatrix} N_1 & N_1 \rho \sqrt{N_1 N_2} / N_2 \\
\rho \sqrt{N_1 N_2} / N_2 & N_2 \end{bmatrix}\right)$ (and, say, $N_1 \leq N_2$), the capacity is the same irrespective of the correlation coefficient $\rho (-1 \leq \rho \leq 1)$. In this example we show that if $(Y_1,Y_2)$ are processed by a unique decoder to recover a common message $W_0$, then $\rho$ matters!

Consider then a point to point system where the receiver has available $(Y_1,Y_2)$.

3.1. Calculate the capacity of such system by maximizing $I(X;Y_1Y_2) = h(Y_1Y_2) - h(Y_1Y_2|X)$ (Hint: You need the maximum entropy theorem on p. 178. You can simplify the result by doing some algebraic manipulations, but this is not required here).
3.2. Plot (with MATLAB) the capacity above for $-1 \leq \rho \leq 1$. What happens at $\rho = -1$? What happens at $\rho = 1$?