“Small-World” Effects of Shadowing in Pulse-Coupled Distributed Time Synchronization

Osvaldo Simeone, Member, IEEE, Umberto Spagnolini, Senior, IEEE, and Yeheskel Bar-Ness, Life Fellow, IEEE

Abstract—Performance of pulse-coupled distributed synchronization in wireless ad hoc networks is known to depend critically on the topology of the corresponding connectivity graph. This paper evaluates the impact of log-normal shadowing on the convergence of distributed discrete-time phase locked loops (PLLs). It is argued that the beneficial effect of shadowing on distributed synchronization can be seen as an instance of the "small-world" phenomenon. Evidence of this is further provided by evaluating average path length and clustering coefficient of the wireless network through simulations.

Index Terms—Distributed synchronization, small-world networks, shadowing.

I. INTRODUCTION

DISTRIBUTED synchronization based on pulse-coupled oscillators is currently being investigated for wireless networks as a promising alternative to the traditional packet-based approach [1] [2]. Its convergence properties have been shown to depend critically on the topology of the connectivity graph describing the underlying wireless network [3] [4], which in turns hinges on the considered propagation scenario.

Conventional models for ad hoc or sensor networks are based on random geometric graphs [5] (or equivalently on Boolean models [6]), wherein nodes are assumed to be randomly located in a given area (possibly infinite), and any two nodes are considered connected if their relative distance is smaller than a given transmission radius. Although useful abstractions for analysis, these models neglect the inherent randomness of wireless connections due to fading and shadowing. Recently, there have been a few attempts to consider more realistic scenarios where connection between any two randomly located nodes is not deterministically fixed by the network geometry, but occurs with a given probability which is a function of the relative distance. In particular, references [7] [8] [9] studied connectivity and coverage of the network in the presence of slow fading due to log-normal shadowing. Moreover, it was briefly stated in [7] [9] that shadowing effectively turns wireless networks into "small-world" networks [10] by breaking a few close connections and creating a small number of long-range links.

In this paper, we investigate the impact of log-normal shadowing on distributed synchronization by considering the system of distributed discrete-time Phase Locked Loops (PLLs) proposed in [4]. It is argued that the beneficial effects of shadowing on the performance of distributed synchronization can be interpreted as a manifestation of the "small-world" effect on distributed processing discussed in [11] [14]. This conclusion is corroborated by the evaluation of average path length and clustering coefficient [10] of the graph underlying the wireless network through simulation.

II. SHADOWING AND SMALL-WORLD AD HOC NETWORKS

In this section, we briefly discuss the model considered in [7] [8] [9] of wireless network with log-normal random connections (Sec. II-A). Then, we introduce the basic concepts and measures defining a small world network (Sec. II-B). Finally, Sec. II-C provides a numerical example to show the impact of shadowing on the topology of a wireless system and the relationship with small-world networks.

A. Modelling shadowing

Consider $K$ nodes randomly located in a given area. Focusing on any two nodes at a relative distance $d$, we define as $P(d)$ the power received over the distance $d$. The two nodes are considered connected if $P(d) > P_0$, where $P_0$ measures the sensitivity of the receiver. In random geometric graph models [5], that neglect channel randomness, $P(d)$ is deterministic. A typical choice is $P(d) \propto 1/d^\alpha$, where the path loss exponent is $\alpha = 2/4$. The previous condition then reads $d < r$, with $r$ usually referred to as the transmission radius. More realistic models account for the random nature of propagation, and model power $P(d)$ as a random variable.

In this case, the connectivity event $P(d) > P_0$ occurs with a given probability that depends on $d$ [6]. A typical and relevant case is that of log-normal shadowing, that models slow fading, with $P(d) = 10 \mu_i / d^\nu$, where random variable $\nu$ is Gaussian with zero average and variance $\sigma^2$ (see [7] [8] [9]). Notice that for $\sigma^2 \rightarrow \infty$, the model reduces to a random graph [12] while for $\sigma^2 \rightarrow 0$ it boils down to a random geometric graph [5].

B. Small-world networks

Small-world networks are characterized by the fact that, despite the large size, there exists a short chain of paths between any two nodes [10] [11]. A measure of the extent to which a network resembles a small world is the average path length, that is the average distance (in terms of hops) between any two nodes. Another important parameter is the clustering coefficient, that measures to what extent the network...
is localized in clusters. Consider any node \( i \), having a given number \( k_i \) of edges to \( k_i \) other nodes. If the all the \( k_i \) neighbors of node \( i \) were connected among themselves in a cluster, then there would be \( k_i(k_i - 1)/2 \) edges among them. The ratio between the actual number of edges \( E_i \) among the \( k_i \) neighbors of node \( i \) and \( k_i(k_i - 1)/2 \) is the so called clustering coefficient of node \( i \): 
\[
C_i = 2E_i/(k_i(k_i - 1)).
\]
The clustering coefficient of the network is the evaluated by averaging over all the nodes.

Small-world networks are usually characterized by both high clustering coefficient and small average path length. Such a scenario is made possible by the occurrence of a few long-range connections that link different clusters [10] [11]. As remarked in [7] [9] and further discussed below, log-normal shadowing makes this phenomenon possible. A related numerical investigation based on an artificial model where links are broken and rewired at random following [11] was reported in [13].

C. A numerical example

Due to the long tails of the log-normal distribution, while a few short-range connections are destroyed by shadowing, a small number of long-range connections are formed. Therefore, by increasing the standard deviation of shadowing \( \sigma \), it is expected that the network increasingly resembles a small world network. This is illustrated with the following numerical example, and through investigation of a distributed synchronization algorithm in the next section. Consider \( K = 100 \) nodes randomly located in a square with unit area and the connection model described in Sec. II-A with log-normal shadowing. The threshold is selected as \( P_0 = 4 \). Fig. 1 shows the average path length and clustering coefficient of the network, averaged over a large number of Monte Carlo iterations of random locations and shadowing, as a function of the shadowing standard deviation \( \sigma \). As expected, the average path length decreases with increasing \( \sigma \) and, moreover, it is enough to have a shadowing standard deviation of around \( \sigma = 3-4dB \) in order to get the most relevant average path length reduction. On the other hand, the average clustering does not vary as significantly as the average path length (similarly to [11] [13]) and slightly increases for increasing \( \sigma \) due to the formation of larger-scale clusters.

III. THE IMPACT OF SHADOWING ON DISTRIBUTED TIME SYNCHRONIZATION

As discussed in [11] [14] and references therein, small-world networks enhance the capacity of the participating nodes to perform distributed tasks, such as broadcasting a given information or attain consensus starting from a disorganized state. In this Section, we show that the same small-world effect is observed while studying distributed processing in wireless networks in the presence of log-normal shadowing. For this, we consider the system of distributed discrete-time PLLs for time synchronization studied in [4].

1) Distributed discrete-time PLLs: We consider a network of \( K \) clocks with different free oscillation frequencies \( 1/T_k \) \( k = 1 \) to \( K \), that communicate over the wireless network at hand. The clocks are defined by discrete-time functions \( t_k(n) \), that, in case of isolated nodes, evolve as 
\[
t_k(n) = nT_k + \theta_k(n),
\]
where index \( n = 1, 2, \ldots \) runs over the periods of the clock and \( 0 \leq \theta_k(n) < T_k \) is the instantaneous phase. Two synchronization conditions are of interest. We say the \( K \) clocks are frequency synchronized if \( t_k(n+1) - t_k(n) = T \) for each \( k \) and for sufficiently large \( n \), where \( 1/T \) is the common frequency. A more strict condition requires full frequency and phase synchronization, i.e., \( t_1(n) = \cdots = t_K(n) \) for \( n \) sufficiently large.

Towards the goal of achieving synchronization, clocks are coupled through the transmission by each node, say the \( k \)th, of a pulse at each time \( t_k(n) \). The topology of the network determines the power received by any \( k \)th node from the \( i \)th as \( P(d_{ki}) \), where \( d_{ki} = d_{ik} \) is the distance between the nodes. Only pulses received with sufficient power \( (P(d_{ki}) > P_0) \) are detected. Based on the detected pulses, any \( k \)th clock updates its instantaneous phase \( \theta_k(n) \). This operation is performed according to the discrete-time PLL shown in Fig. 2. A timing error detector estimates a convex weighted sum of the time differences between the local clock and the others, 
\[
\Delta t_k(n+1) = \sum_{i \in I_k} \alpha_{ki} \cdot (t_i(n) - t_k(n)),
\]
with \( \alpha_{ki} \geq 0 \) and \( \sum_{i \in I_k} \alpha_{ki} = 1 \). This measure is fed to a loop filter \( \mathcal{E}(z) \), whose output \( \Delta \theta_k(n+1) \) drives the local
\footnote{Propagation delays can be shown to contribute to frequency mismatch among nodes.}
Voltage Control Clock (VCC)

\[ t_k(n + 1) = t_k(n) + \Delta\theta_k(n + 1) + T_k. \quad (2) \]

The number of poles in the loop filter \( \varepsilon(z) \) determines the order of the loops as for conventional analog PLLs. Following the algorithm in [2], we select the weighting coefficients \( \alpha_{ki} \) as \( \alpha_{ki} = P(d_{ki})/\sum_{i \in T_k} P(d_{ki}) \).

2) Convergence: Reference [4] studies the convergence of the system of discrete-time PLLs (2) for second-order loops with filters \( \varepsilon(z) = \varepsilon_0(1 - \mu z^{-1}) \). Let us denote a possible common value for the frequency of all nodes as \( T \) (to be determined), i.e., \( t_k(n) - t_k(n - 1) = T \) for sufficiently large \( n \), so that the clock of the \( k \)th sensor can be written (for large \( n \)) as

\[ t_k(n) = nT + \tau_k(n), \quad (3) \]

where \( \tau_k(n) \) denotes the relative phase with respect to the common frequency. Moreover, let us define the vector \( \varepsilon = (\varepsilon_0, 0, \ldots, 0) \) and denote as \( L \) the Laplacian of the network (i.e., \( L_{ii} = 1 \) and \( L_{ij} = -\alpha_{ij} \) if \( i \neq j \)) and \( A = I - \varepsilon_0L \). It is shown in [4] that, if the gain \( \varepsilon_0 \) and the pole \( \mu \) are sufficiently small and the graph of the network is strongly connected\(^3\), then the system (2) synchronizes the clocks of the \( K \) nodes to the common period \( T = v^T T \), where \( T = [T_1, \ldots, T_K]^T \) and \( v \) is the normalised left eigenvector of matrix \( A \) corresponding to the eigenvalue \( \lambda(A) = 1 \) \( (A^T v = v \text{ with } 1^T v = 1) \). However, under the same assumptions, the timing phases \( \tau(n) \) remain generally mismatched and given for \( n \to \infty \) by

\[ \tau(n) \to \tau^* = 1 \cdot n + (1 - \mu) \frac{1}{\varepsilon} \Delta T, \quad (4) \]

with \( (\cdot)^\dagger \) denoting the pseudoinverse, \( \eta \) a given constant and the \( k \)th element of vector \( \Delta T \) being \( [\Delta T]^k = T_k - T \). Notice that [4] also shows that, in absence of frequency mismatch among the clocks \( \Delta T = 0 \), the network achieves full frequency and phase synchronization to the value \( \tau(n) \to \tau^* = 1 \cdot v^T \tau(0) \).

\(^3\)A graph is said to be strongly connected if there exists at least one path that links every pair of nodes.

A numerical example on distributed time synchronization

In this section, we evaluate the convergence of the synchronization algorithm discussed above on the random network with log-normal shadowing employed in Sec. II-C. We evaluate the standard deviation \( \xi(n) \) of the clocks, where \( \xi^2(n) = 1/K \cdot \sum_{k=1}^K (t_k(n) - 1/K \cdot \sum_{k=1}^K t_k(n))^2 \), versus time \( n \), averaged over random location of nodes and shadowing. The initial phases \( \tau(0) = t(0) \) are selected randomly in the set \( (0, 1) \) (and \( t(-1) = 0 \)), while the local free-oscillation frequencies are selected independently in the set \( 1 \pm 1\% \). The dashed lines in Fig. 3 correspond to the asymptotic result (4). It can be seen that increasing the amount of shadowing in the model (i.e., the variance \( \sigma^2 \)) improves both the convergence speed and the asymptotic phase error of the system of distributed PLLs. Moreover, as expected from the discussion in Sec. II-C, it is enough to have a standard deviation of around 3-4 dB to harness the most relevant advantages of shadowing.

IV. Concluding Remarks

This letter focused on the effect of log-normal shadowing on wireless ad hoc/ sensor networks in terms of "small-world" properties of the associated connectivity graphs. It has been shown, via numerical simulations, that a shadowing standard deviation of around 3-4 dB is enough to reduce significantly the average path length of the network. The beneficial impact of this phenomenon on a distributed synchronization algorithm has also been validated, thus confirming the notion that small-world networks enhance the capacity of participating nodes to perform distributed tasks.

REFERENCES