Please write legibly and provide detailed answers.

We are interested in studying the connectivity of $M$ towns along a country road (see figure, where $M = 4$). Because of unreliable weather conditions, the road between one town and the following is accessible only a fraction of the time.

1. For a generic $M$, define a reasonable sample space to study the problem at hand. 
   **Sol.** Let us model each road as a different subexperiment and denote with "0" the probability of a road being closed and "1" open. The sample space then reads
   
   \[ S = \{(z_1, z_2, ..., z_{M-1}) : z_i \in \{0, 1\}\}. \]

2. Assume that the connectivity on a road does not depend on the availability of any other road. Moreover, consider that the probability of a road being open is 0.7 and there are $M = 4$ towns. What is the probability that the first two towns (1 and 2) are connected, and so are 3 and 4, but there is no way of going from the the first pair of towns to the last? 
   **Sol.** The requested probability reads
   
   \[ P[(1, 0, 1)] = P[1] \cdot (1 - P[1]) \cdot P[1] = 0.7 \cdot 0.3 \cdot 0.7 = 0.147. \]

3. Now assume that the quality of a road depends on the preceding (e.g., the connectivity between 2 and 3 depends on the connectivity between 1 and 2). In particular, consider that the probability of a road being available is 0.9 if the preceding is available, while is 0.2 if the preceding is not. Finally, assume that the first road (between 1 and 2) is open with probability 0.7. For $M = 4$, find the probability that the first two towns (1 and 2) are connected, and so are 3 and 4, but there is no way of going from the the first pair of towns to the last. Compare your result with question 2.
   **Sol.** Let us denote as $P[A_n|A_{n-1}]$ as the probability of some event $A_n$ relative to the $n$th link given event $A_{n-1}$ relative to the $n-1$ link. From the problem statement, we have
   
   \[ P[1|1] = 0.9 \]
   \[ P[1|0] = 0.2 \]

The probability at hand then reads

\[ P[(1, 0, 1)] = P[1] \cdot (1 - P[1|1]) \cdot P[1|0] = 0.7 \cdot 0.1 \cdot 0.2 = 0.014 \]
4. Consider a long road with many towns ($M = 10001$). Connection qualities are independent as in question 2. The weather in the region is so bad that the probability of a road being accessible is 0.001. What is the probability that five roads are open (you can use some approximation if you prefer)? What is the average number of accessible roads?

**Sol.** The real probability density function of $X = \sum_{k=1}^{M-1} z_k$ is $bin(M - 1, 0.001)$, i.e., $X \sim bin(10000, 0.001)$. Since the number of trials is large and the probability of "hit" is small, a good approximation is $X \sim Pois(10000 \cdot 0.001 = 10)$. The probability requested is

$$P[X = 5] = \exp(-10) \frac{10^5}{5!} = 0.038.$$ 

The average of $X$ is $E[X] = 10$.

5. Say that we have $M = 3$ towns with independent links, and that the probability of a road being accessible is 0.4. Write the equation and plot the probability mass function and cumulative distribution function of a random variable $Y$ that counts the number of available links.

6. Write a MATLAB code that generates the random variable $Y$ defined at the previous question.

**Sol.** We have $Y = \sum_{k=1}^{3} z_k \sim bin(2, 0.4)$ (see textbook and notes for plots and equations of PMF and CDF). A MATLAB code that generates this random variable, using the inverse CDF transform, is as follows:

```matlab
u = rand(1);
if (u <= 0.36)
    x = 0;
elseif (u > 0.36) && (u <= 0.84)
    x = 1;
elseif (u > 0.84)
    x = 2;
end
```

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