(i) Consider the process \( X[n] = \log_2 U[n] \). Is this process IID (and therefore stationary)? Find the PMF of \( X[n] \) (\( p_X[x] = P[X[n] = x] \)), the average \( E[X[n]] \) and variance \( \text{var}(X[n]) \).

**Sol.** Yes, the process is IID since applying a transformation (here the logarithm) to each \( U[n] \) does not modify the statistical dependence among the variables of the process. Moreover, the range of \( X[n] \) is \( S_X = \{-1, 1\} \) and the PMF reads

\[
p_X[x] = \begin{cases} 
1/2 & x = -1 \\
1/2 & x = 1 
\end{cases}
\]

Average and variance are as follows:

\[
E[X[n]] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0
\]

\[
\text{var}(X[n]) = E[X[n]^2] - E[X[n]]^2 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1)^2 = 1.
\]

(ii) Consider now the process

\[ Y[n] = X[n] - X[n-2] \]

Is \( Y[n] \) WSS? In order to address this question, evaluate mean sequence \( \mu_Y[n] \) and correlation sequence \( c_Y[n, n+k] \). [notice that from the previous point \( E[X[n]] = 0 \) and \( \text{var}(X[n]) = 1 \)].

**Sol.:** Mean sequence:

\[
\mu_Y[n] = E[Y[n]] = E[X[n]] - E[X[n-2]] = 0.
\]

Covariance sequence:

\[
c_Y[n, n+k] = E[Y[n]Y[n+k]] - E[Y[n]]E[Y[n+k]] = \\
= E[Y[n]Y[n+k]] = E[(X[n] - X[n-2])(X[n+k] - X[n-2+k])] = \\
= E[X[n]X[n+k]] + E[X[n-2]X[n-2+k]] + \\
- E[X[n]X[n-2+k]] - E[X[n-2]X[n+k]]
\]

Evaluating the previous expression for different values of \( k \), we easily get

\[
c_Y(n, n+k) = 2\delta[k] - \delta[k-2] - \delta[k+2] = c_Y[k] = r_Y[k]. \tag{1}
\]
From the above calculations, it follows that $Y[n]$ is WSS.

(iii) Evaluate and plot the power spectral density of $Y[n]$, $P_Y(f)$.

**Sol.**: In order to calculate the power spectral density $P_Y(f)$, we need to evaluate the discrete Fourier transform of the correlation function $r_Y[k]$ as

$$P_Y(f) = \sum_k r_Y[k] \cdot e^{-j2\pi fk} = 2(1 - \cos(4\pi f)).$$

See figure for plot.

(iv) Evaluate the best linear predictor of $Y[n+1]$ given $Y[n]$ and the corresponding error.

**Sol.**: The correlation between $Y[n]$ and $Y[n+1]$ is zero from the answer to point (ii). Therefore, the best linear predictor of $Y[n+1]$ given $Y[n]$ is

$$\hat{Y}[n+1] = E[Y[n+1]] = 0,$$

and the corresponding mean square error is

$$mse = E[(\hat{Y}[n+1] - Y[n+1])^2] = var(Y[n+1]) = 2.$$

(v) Repeat the exercise above for the prediction of $Y[n+2]$ given $Y[n]$.

**Sol.**: The correlation between $Y[n]$ and $Y[n+2]$ is

$$cov(Y[n], X[Y+2]) = c_Y[2] = r_Y[2] = -1,$$

and the correlation coefficient reads

$$\rho[2] = \frac{cov(Y[n], Y[n+2])}{\sqrt{var(Y[n])}var(Y[n+2])} = \frac{r_Y[2]}{r_Y[0]} = \frac{-1}{2}.$$
Therefore, linear prediction is expected to be fairly effective. The predictor is

$$\hat{Y}[n + 2] = E[Y[n + 2]] + \frac{\text{cov}(Y[n], Y[n + 2])}{\text{var}(Y[n])}(Y[n] - E[Y[n]]) =$$

$$= \frac{r_Y[2]}{r_Y[0]} Y[n] = -\frac{1}{2} Y[n]$$

and the corresponding mean square error is

$$mse = E[(\hat{Y}[n + 2] - Y[n + 2])^2] = \text{var}(Y[n + 2])(1 - \rho[2]^2) =,$$

$$= 2 \cdot (1 - 1/4) = 1.5.$$ 

Linear prediction based on the knowledge of $Y[n]$ has reduced the mse from $\text{var}(Y[n+2]) = 2$ to 1.5.