Lecture 5  
Physics 106  
Fall 2006

Rotational Momentum  
(Same as Angular Momentum)

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Angular Momentum

Linear motion \[ \vec{p} = m \vec{v} \quad \vec{F} = \frac{d\vec{p}}{dt} = ma \]

Using the correspondence with linear motion

Define “angular momentum” \[ I \leftrightarrow m \quad \omega \leftrightarrow v \]

\[ \vec{L} = I \vec{\omega} \] (must define around some origin)

If no torque, then \( L \) is a constant!

FOR ISOLATED SYSTEM: \( L \) IS CONSERVED

Angular Momentum of a particle

Can also define angular momentum for a particle with a linear velocity \( \vec{v} \)

\[ \vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} \]

\( \vec{r} \) is vector from origin to particle

\[ L = m \cdot r \cdot v \cdot \sin \phi \quad \text{or for a circular motion:} \]

\[ L = m \cdot r^2 \cdot \omega \cdot \sin \phi \quad (\phi = \pi/2 = 90^\circ) \]

Linear motion

Circular motion

Examples:

\[ \vec{r} \perp \vec{v} \rightarrow L \text{ is big} \]

\[ L = mrv \]

\[ \vec{r} \parallel \vec{v} \rightarrow L = 0 \]

Angular Momentum

Definition

\[ \vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \quad [\text{kg m}^2/\text{s}] \]

Angular counterpart of linear momentum!

System of particles

\[ \vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i \]

Excursion of \( \vec{p} \)
Newton’s 2nd Law

Angular Momentum of a particle:

\[
\frac{d}{dt}(\vec{L}) = \frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}
\]

\[
\frac{d}{dt}(\vec{L}) = \vec{\tau}
\]

EXAMPLE (Linear motion)

Constant velocity particle: Is L really constant?

\[
\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}
\]

\[
L = m \cdot r \cdot v \cdot \sin \phi \quad \text{or} \quad L = m \cdot d \cdot v = \text{Const}
\]

Conservation of Angular Momentum

No torque: L is constant

\[
L = I\omega
\]

if you change I, \( \omega \) changes to keep L constant

This allows skaters and divers to spin really fast (they studied their physics!)
Sample Problem XII–5

A penguin of mass $m$ falls from rest at point $A$, a horizontal distance $D$ from the origin $O$ of an $xyz$ coordinate system.

a) What is the angular momentum of $I$ of the penguin about $O$?

b) About the origin $O$, what is the torque $\tau$ on the penguin due to the gravitational force $F_g$?

Example

Bullet hits sign: how high does it go?

NO shooting NO gunfire NO bombardment

$M_{\text{bull}} = 5 \, \text{g}$
$M_{\text{sign}} = 2.2 \, \text{kg}$
$v_{\text{bull}} = 300 \, \text{m/s}$
$I_{\text{sign}} = 0.03 \, \text{kg m}^2$

$V = 0.1 \, \text{m}$
$V = 0.2 \, \text{m}$
$v = 200 \, \text{m/s}$

Can it make a complete turn?

Conservation of Angular Momentum

Angular momentum of a solid body about a fixed axis

$L = I \omega$

Law of conservation of angular momentum

$\vec{L} = \text{const.} \Rightarrow \vec{L}_i = \vec{L}_f$

(Valid from microscopic to macroscopic scales!)

If the net external torque $\tau_{net}$ acting on a system is zero, the angular momentum $L$ of the system remains constant, no matter what changes take place within the system.
Rolling Motion: without slipping

\[ S = R \theta \]

\[ v_c = \frac{ds}{dt} = R \frac{d\theta}{dt} = R \omega \]

\[ a_c = R \alpha \]

At any instant, the wheel rotates about the point of contact

Rolling

Smooth rolling motion

\[ v_{com} = \omega R \]

Rotation and Translation

Reference frame

Rolling of the train wheel is it the same or slightly different?

Energy of Rolling

\[ K = \frac{1}{2} I_c \omega^2 + \frac{1}{2} M v_c^2 \quad v_c = R \omega \]

\[ K = \frac{1}{2} I_c \left( \frac{v_c}{R} \right)^2 + \frac{1}{2} M v_c^2 \]

\[ K = \frac{1}{2} \left( \frac{I_c}{R^2} + M \right) v_c^2 \]
Kinetic Energy

\[ K = \frac{1}{2} I \omega^2 \]
\[ I = I_{com} + MR^2 \]
\[ K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} MR^2 \omega^2 \]
\[ v_{com} = \omega R \]
\[ K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} Mv_{com}^2 \]

Stationary observer

Parallel–axis theorem

A rolling object has two types of kinetic energy: a rotational kinetic energy due to its rotation about its center of mass and a translational kinetic energy due to translation of its center of mass.

Sample Problem X12–1: A uniform solid cylindrical disk \((M = 1.4 \text{ kg}, r = 8.5 \text{ cm})\) roll smoothly across a horizontal table with a speed of 15 cm/s. What is its kinetic energy \(K\)?

Forces

A net force \(F_{net}\) acting on a rolling wheel speeds it up or slows it down and causes an acceleration.

There is a slipping tendency for the wheel, while the friction force prevents it.

The acceleration tends to make the wheel slide.

A static frictional force \(f_s\) acts on the wheel to oppose that tendency.

Conservation of Energy

\[ \frac{1}{2} \left( \frac{I_c}{R^2} + M \right) v_c^2 = Mgh \]

For a particle:

\[ v_c = \frac{\sqrt{2gh}}{\sqrt{1 + \frac{1}{2} \frac{I_c}{MR^2}}} \]

A Disc

\[ I_c = \frac{1}{2} MR^2 \]

A Ring

\[ I_c = MR^2 \]

Free falling / sliding without friction:

\[ v_c = \sqrt{\frac{2gh}{3}} \]
Torques on a Wheel

The Forces on a wheel

Gravity
Normal Force
Friction (so it won’t slide)

Center of Mass View

\[ \sum F_x = Mg \sin(\theta) - F_F = Ma_c \]

Constraint
Rolling without Slipping

\[ a_c = \alpha R \]

\[ \sum F_y = Mg \cos(\theta) - F_N = 0 \]

\[ \sum \tau = F_F R = I_c \alpha \]

Another View

The wheel rotates about the point of contact

No Torque - Normal Force Friction

Point of Rotation

\[ \tau = MgR\sin(\theta) = I_P \alpha \]

\[ I_P = I_C + MR^2 \]

\[ MgR\sin(\theta) = (I_C + MR^2) \alpha \]

Same result

Don’t need x and y motion

The Gyroscope

You are designing a cruise missile which makes lots of twists and turns and has no driver on board

How do you keep track of which way is up?

Start something spinning and protect it from any torque - L keeps pointing in same direction

Torque in three dimensions: the falling gyroscope

A falling rock

\[ p_0 \]

\[ p \]

\[ mg \]

A falling gyro

\[ L_0 \]

\[ L \]

\[ \tau \]

\[ mg \]
**Definition of Torque**

Vector (cross) product

(Right-hand rule, order does matter!)

**Rotational Inertia**

<table>
<thead>
<tr>
<th>Axis</th>
<th>Thin spherical shell about any diameter</th>
<th>Thin rod about axis through center perpendicular to length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hoop about central axis</td>
<td>$I = MR^2$</td>
<td>$I = \frac{1}{2} M I$</td>
</tr>
<tr>
<td>Annular cylinder (or ring) about central axis</td>
<td>$I = \frac{1}{2} M (R_2^2 - R_1^2)$</td>
<td>$I = \frac{1}{2} M R^2$</td>
</tr>
<tr>
<td>Solid cylinder (or disk) about central axis</td>
<td>$I = \frac{1}{2} M R^2$</td>
<td>$I = \frac{1}{2} M R^2 + \frac{1}{2} M L^2$</td>
</tr>
<tr>
<td>Solid sphere about any diameter</td>
<td>$I = \frac{2}{5} M R^2$</td>
<td>$I = \frac{1}{2} M R^2$</td>
</tr>
<tr>
<td>Hoop about any diameter</td>
<td>$I = \frac{1}{2} M R^2$</td>
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