Lecture 6

Physics 106

Spring 2006

- **Angular Momentum**
- **Rolling**

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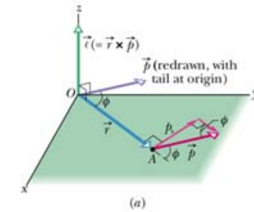
Angular Momentum:

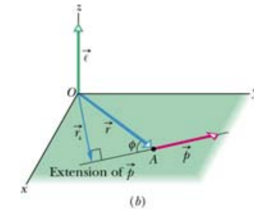
Definition: $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$ [kg m²/s]

$$l = r \cdot m \cdot v \cdot \sin\phi$$

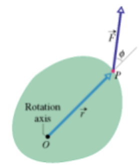


Angular Momentum for rotation $l = I \cdot \omega$

System of particles: $\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$



Torque:



$\vec{\tau} = [\vec{r} \times \vec{F}]$

$\tau = r \cdot F \cdot \sin\phi$

$$\frac{d}{dt}\left(\vec{L}\right) = \vec{\tau} = I\vec{\alpha}$$

Conservation of Angular Momentum

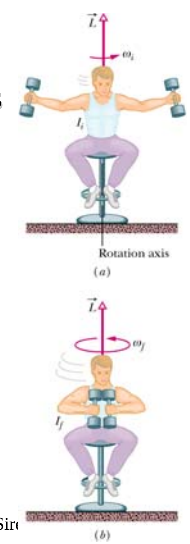
Angular momentum of a solid body about a fixed axis

$L = I\omega$

Law of conservation of angular momentum

$\vec{L} = \text{const.} \Rightarrow \vec{L}_i = \vec{L}_f$

(Valid from microscopic to macroscopic scales!)



If the net external torque τ_{net} acting on a system is zero, the angular momentum L of the system remains constant, no matter what changes take place within the system

Linear Momentum	**Angular Momentum**
$\vec{p} = m\vec{v}$	$\vec{L} = \vec{r} \times \vec{p}$ [kg m²/s]
[kg m/s]	$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$
	$L = m \cdot r \cdot v \cdot \sin\phi$

Both are vectors

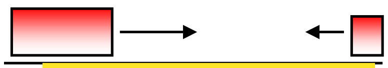
$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$	$\frac{d}{dt}\left(\vec{L}\right) = \vec{\tau} = I\vec{\alpha}$
	For rotating body: $L = I\omega$

m ↔ I

v ↔ ω

FOR ISOLATED SYSTEM: L IS CONSERVED

Linear Momentum Conservation:



$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

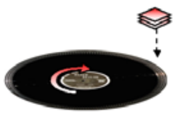
$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

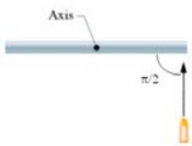
Both, elastic and Inelastic collisions

1. Define a reference frame
2. Calculate **P** before the collision
3. Compare with **P** after the collision

Angular Momentum Conservation:

"If the external torque is equal to zero, L is conserved"

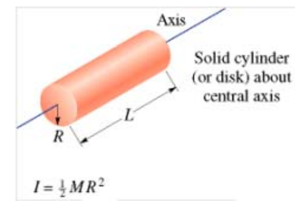




1. Define a rotational axis and the origin
2. Calculate **L** before interaction or any changes in I
3. Compare with **L** after the interaction or any change in I

Example:

1. Define a rotational axis and the origin
2. Calculate **L** before interaction or any change in I
3. Compare with **L** after the interaction or any change in I



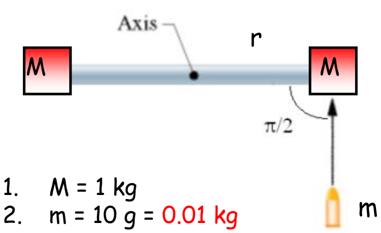
Example:

A horizontal disc of rotational inertia **I = 1 kg.m²** and radius **100 cm** is rotating about a vertical axis through its center with an angular speed of **1 rad/s**. A wad of wet putty of mass **100 grams** drops vertically onto the disc from above and sticks to the edge of the disk. What is the angular speed of the disk right after the putty sticks to it?

1. $L_i = I_i \cdot \omega_i = 1\ kg.m^2 \cdot 1\ rad/s = 1\ kg \cdot m^2/s$
2. $I_f = (I_i + mr^2) = (1\ kg.m^2 + 0.1\ kg.m^2)$
3. $L_i = L_f$ **(angular momentum conserv.)**
4. $\omega_f = \cdot\omega_i\ I_i /\ I_f = 1\ rad/s \cdot (1/1.1) =$ **0.91 rad/s**

Angular Momentum Conservation:

"If the external torque is equal to zero, L is conserved"

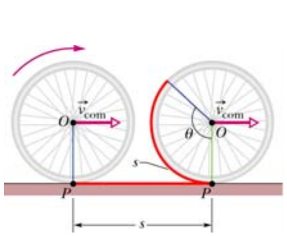


1. M = 1 kg
2. m = 10 g = 0.01 kg
3. r = 1 m
4. $\omega_i = 0$ $\omega_f = 1$ rad/s
5. $v_{bullet} = ?$ $K_f/K_i = ?$

1. Define a rotational axis and the origin
2. Calculate **L** before interaction or any changes in I
3. Compare with **L** after the interaction or any change in I

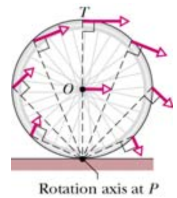
1. $L_i = L_{bullet} = m \cdot v \cdot r \cdot \sin(\pi/2) =$???
2. $L_f = I \cdot \omega = (Mr^2 + Mr^2 + mr^2)\ \omega_f = 2\ kg \cdot m^2/s$
3. $L_i = L_f$ (angular momentum conserv.)
4. $v_{bullet} = \omega_f \cdot (2Mr^2 + mr^2)/mr = 200$ m/s
5. $K_i = \frac{1}{2} m v^2_{bullet} = 200$ J
6. $K_f = \frac{1}{2} I \omega^2 = 1$ J
7. $K_f/K_i = 1/200$

Rolling



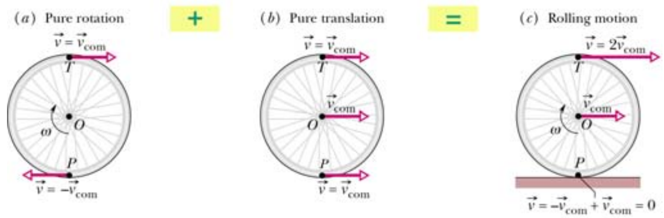
Smooth rolling motion

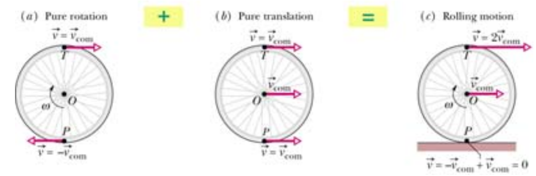
$v_{com} = \omega R$



Reference frame

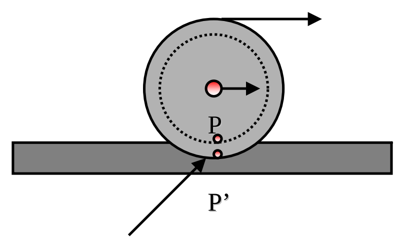
Rotation and Translation

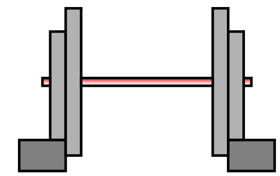




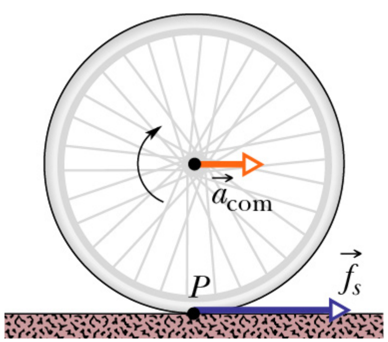
Rolling of the train wheel

is it the same or slightly different?





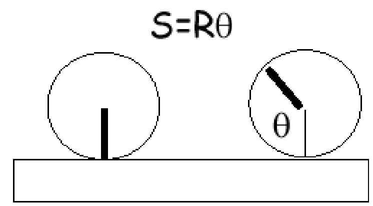
Forces



A net force $\underline{F}_{net}$ acting on a rolling wheel speeds it up or slows it down and causes an acceleration.

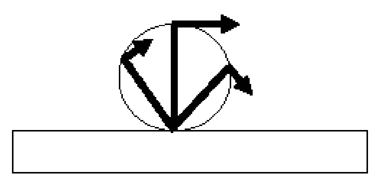
There is a slipping tendency for the wheel, while the friction force prevents it.

Rolling Motion: without slipping



$$v_c = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_c = R\alpha$$



At any instant
the wheel rotates about
the point of contact

Kinetic Energy

$$\begin{aligned} K &= \tfrac{1}{2} I_P \omega^2 \\ I_P &= I_{com} + MR^2 \\ K &= \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} MR^2\omega^2 \\ v_{com} &= \omega R \\ K &= \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} M v_{com}^2 \end{aligned}$$

Stationary observer

Parallel–axis theorem

A rolling object has two types of kinetic energy: a rotational kinetic energy due to its rotation about its center of mass and a translational kinetic energy due to translation of its center of mass.

Sample Problem X12–1: A uniform solid cylindrical disk (M = 1.4 kg, r = 8.5 cm) roll smoothly across a horizontal table with a speed of 15 cm/s. What is its kinetic energy K?

What is more important: Kinetic Energy Conservation or Angular Momentum Conservation ?

$$K = \tfrac{1}{2} I_{com}\omega^2$$

Work of external and internal forces can change **K**.

K is a scalar variable, which has no direction

$$\tau_{tot}(\theta_f - \theta_i) = K_f - K_i = \text{Work}$$

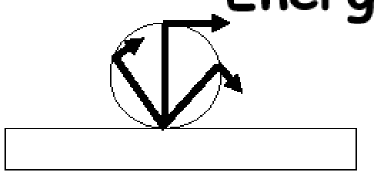
$$L = I\omega$$

Only net *external* torque τ_{net} can change the angular momentum.

L is a vector, direction is important

$$\frac{d}{dt}(\vec{L}) = \vec{\tau}$$

Energy of Rolling

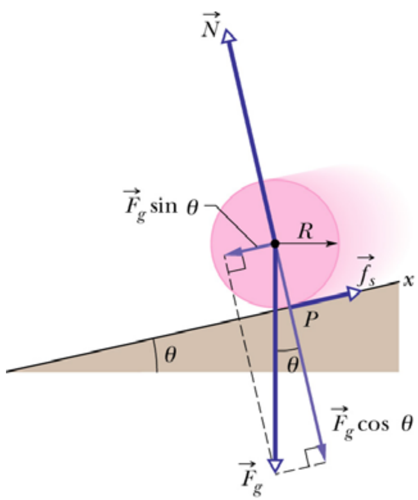


$$K = \frac{1}{2} I_C\, \omega^2 + \frac{1}{2} M v_C^2 \qquad v_c = R\,\omega$$

$$K = \frac{1}{2} I_C \left(\frac{v_C}{R}\right)^2 + \frac{1}{2} M v_C^2$$

$$\mathbf{K = \frac{1}{2}\left(\frac{I_C}{R^2} + M\right) v_C^2}$$

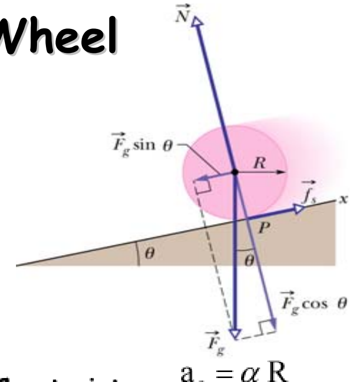
Forces



The acceleration tends to make the wheel slide.

A static frictional force f_s acts on the wheel to oppose that tendency.

Torques on a Wheel



The Forces on a wheel

- Gravity
- Normal Force
- Friction (so it won't slide)

Center of Mass View

$$\sum F_x = Mg\sin(\theta) - F_F = Ma_c$$

$$\sum F_y = Mg\cos(\theta) - F_N = 0$$

$$\sum \tau = F_F\ R = I_C\ \alpha$$

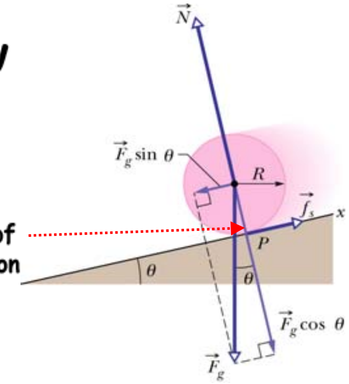
Constraint $\quad a_c = \alpha R$

Rolling without Slipping

$$a_c = \frac{g\sin(\theta)}{1 + I_C/MR^2}$$

Another View

The wheel rotates about the point of contact



Point of Rotation

No Torque - Normal Force, Friction

$$\tau = MgR\sin(\theta) = I_P\ \alpha$$

$$I_P = I_C + MR^2$$

$$MgR\sin(\theta) = \left(I_C + MR^2\right)\alpha$$

$$a_c = \frac{g\sin(\theta)}{1 + I_C/MR^2}$$

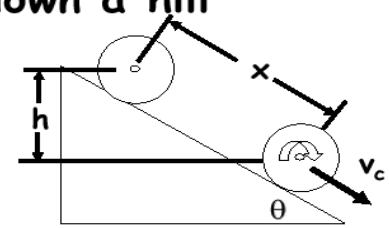
Same result

Don't need x and y motion

Rolling down a hill

Conservation of Energy

$$\frac{1}{2}\left(\frac{I_C}{R^2} + M\right)v_C^2 = Mgh$$



$$v_C = \sqrt{\frac{2gh}{1 + I_C/MR^2}}$$

For a particle: $v_C = \sqrt{2gh}$

A Disc

$$I_C = \frac{1}{2}MR^2$$

$$v_C = \sqrt{\frac{2gh}{1 + \frac{1}{2}MR^2/MR^2}} = \sqrt{\frac{2}{3}2gh}$$

A Ring

$$I_C = MR^2$$

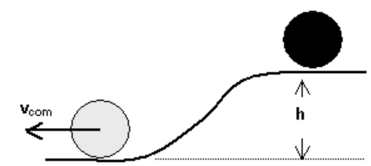
$$v_C = \sqrt{\frac{2gh}{1 + MR^2/MR^2}} = \sqrt{\frac{1}{2}2gh}$$

Free falling / sliding without friction: $V_C = \sqrt{2gh}$

Example 1

Kinetic Energy of Rolling

$$K = \frac{1}{2}\left(\frac{I_C}{R^2} + M\right)v_C^2$$



+ Energy conservation !!!

Kinetic Energy ←→ Potential Energy

$\Delta U + \Delta K = 0$ → $U_{initial} = K_{final}$ → $Mgh = \frac{1}{2}\left(\frac{I_C}{R^2} + M\right)v_{com}^2$

Disk:	Hoop:	Sphere:
$I_{com} = ½\ MR^2$	$I_{com} = MR^2$	$I_{com} = 2/5\ MR^2$

For disk: $Mgh = ½(1/2M + M)\ v^2_{com}$; $\quad v_{com} = (4/3\ gh)^{½}$

Summary for rotational motion

360 degrees = 2π radians = 1 revolution. $s = r\theta$ $v_t = r\omega$ $a_t = r\alpha$ $a_c = a_r = v_t^2/r = \omega^2 r$ $a_{tot}^2 = a_r^2 + a_t^2$

for rotation with constant angular acceleration:
$\omega = \omega_o + \alpha t$ $\theta - \theta_o = \omega_o t + \frac{1}{2}\alpha t^2$ $\omega^2 - \omega_o^2 = 2\alpha(\theta - \theta_o)$ $\theta - \theta_o = \frac{1}{2}(\omega + \omega_o)t$ $KE_{rot} = 1/2 I\omega^2$

$I = \Sigma m_i r_i^2$ $I_{point} = mr^2$ $I_{hoop} = MR^2$ $I_{disk} = 1/2\ MR^2$ $I_{sphere} = 2/5\ MR^2$ $I_{shell} = 2/3\ MR^2$ $I_{rod\ (center)} = 1/12\ ML^2$
$I_{rod\ (end)} = 1/3\ ML^2$

$\Sigma \mathbf{F} = m\mathbf{a}$ $\Sigma\tau = I\alpha$ $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ $I_p = I_{cm} + Mh^2$

τ = force$_x$moment arm = $F r \sin(\phi)$ $\boldsymbol{\tau}_{net} = \Sigma\tau = I\alpha$ $\mathbf{F}_{net} = \Sigma\mathbf{F} = m\mathbf{a}$ $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ $I_p = I_{cm} + Mh^2$

$W_{tot} = \Delta K = K_f - K_I$ $W = \tau_{net}\Delta\theta$ $K = K_{rot} + K_{cm}$ $E_{mech} = K + U$ $P_{average} = \Delta W/\Delta t$
$P_{instantaneous} = \tau.\omega$ (τ constant) $\Delta E_{mech} = 0$ (isolated system) $v_{cm} = \omega r$ (rolling, no slipping)

$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$ $\mathbf{p} = m\mathbf{v}$ $\mathbf{L} = \Sigma\, \boldsymbol{\ell}_i$ $\boldsymbol{\tau}_{net} = d\mathbf{L}/dt$ $L = I\omega$ $\boldsymbol{\ell}_{point\ mass} = mrv\sin(\phi)$
For isolated systems: $\tau_{net} = 0$ $\mathbf{L}$ is constant $\Delta\mathbf{L} = 0$ $L_0 = \Sigma\, I_0\omega_0 = L_f = \Sigma\, I_f\omega_f$

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ $\mathbf{a} \times \mathbf{a} = 0$ $|\mathbf{a} \times \mathbf{b}| = a.b.\sin(\phi)$ $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is perpendicular to plane of $\mathbf{a}$ and $\mathbf{b}$
$c_x = a_y \cdot b_z - a_z \cdot b_y$ $c_y = -a_x \cdot b_z + a_z \cdot b_x$ $c_z = a_x \cdot b_y - a_y \cdot b_x$
$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ etc.