Lecture 6
Physics 106
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• Angular Momentum
• Rolling

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Angular Momentum:

Definition:

\[ \mathbf{l} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) \]  

Angular Momentum for rotation

\[ l = I \cdot \omega \]

System of particles:

\[ \mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \ldots + \mathbf{L}_n = \sum_{i=1}^{n} \mathbf{L}_i \]

Torque:

\[ \tau = r \cdot \mathbf{F} \cdot \sin \phi \]

\[ \frac{d}{dt}(\mathbf{L}) = \mathbf{\tau} = I \mathbf{\alpha} \]

Conservation of Angular Momentum

Angular momentum of a solid body about a fixed axis

\[ \mathbf{L} = I \omega \]

Law of conservation of angular momentum

\[ \mathbf{L} = \text{const.} \Rightarrow \mathbf{L}_i = \mathbf{L}_f \]

(Valid from microscopic to macroscopic scales!)

If the net external torque \( \mathbf{L}_{\text{net}} \) acting on a system is zero, the angular momentum \( \mathbf{L} \) of the system remains constant, no matter what changes take place within the system

\[ \mathbf{F} = \frac{d}{dt}(\mathbf{p}) = m \mathbf{\alpha} \]

\[ \frac{d}{dt}(\mathbf{L}) = \mathbf{\tau} = I \mathbf{\alpha} \]

For rotating body:

\[ \mathbf{L} = I \omega \]

FOR ISOLATED SYSTEM: \( \mathbf{L} \) IS CONSERVED
Linear Momentum Conservation:

Both, elastic and Inelastic collisions

\[ \vec{p}_{i1} + \vec{p}_{i2} = \vec{p}_{f1} + \vec{p}_{f2} \]

\[ m_1v_{i1} + m_2v_{i2} = m_1v_{f1} + m_2v_{f2} \]

Angular Momentum Conservation:

"If the external torque is equal to zero, L is conserved"

1. Define a rotational axis and the origin
2. Calculate \( L \) before interaction or any changes in \( I \)
3. Compare with \( L \) after the interaction or any change in \( I \)

Example:

A horizontal disc of rotational inertia \( I = 1 \text{ kg.m}^2 \) and radius 100 cm is rotating about a vertical axis through its center with an angular speed of \( 1 \text{ rad/s} \). A wad of wet putty of mass 100 grams drops vertically onto the disc from above and sticks to the edge of the disk. What is the angular speed of the disk right after the putty sticks to it?

1. \( L_i = I \cdot \omega_i = 1 \text{ kg.m}^2 \cdot 1 \text{ rad/s} = 1 \text{ kg.m}^2/\text{s} \)
2. \( L_f = (I + mr^2) \cdot \omega_f = (1 \text{ kg.m}^2 + 0.1 \text{ kg.m}^2) \cdot 1 \text{ rad/s} \)
3. \( L_i = L_f \) (angular momentum conserv.)
4. \( \omega_f = \omega_i I / I = 1 \text{ rad/s} \cdot (1/1.1) = 0.91 \text{ rad/s} \)
Rolling
Smooth rolling motion
\[ v_{\text{com}} = \omega R \]

Rotation and Translation
Reference frame

Rolling of the train wheel
is it the same or slightly different?

Forces
A net force \( F_{\text{net}} \) acting on a rolling wheel speeds it up or slows it down and causes an acceleration.

There is a slipping tendency for the wheel, while the friction force prevents it.

Rolling Motion: without slipping
\[ S = R\theta \]
\[ v_c = \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega \]
\[ a_c = R\alpha \]

At any instant the wheel rotates about the point of contact.
Kinetic Energy

Stationary observer

Parallel–axis theorem

A rolling object has two types of kinetic energy: a rotational kinetic energy due to its rotation about its center of mass and a translational kinetic energy due to translation of its center of mass.

Sample Problem X12–1: A uniform solid cylindrical disk \((M = 1.4 \text{ kg}, r = 8.5 \text{ cm})\) roll smoothly across a horizontal table with a speed of 15 cm/s. What is its kinetic energy \(K\)?

What is more important:
Kinetic Energy Conservation or Angular Momentum Conservation?

Work of external and internal forces can change \(K\). \(K\) is a scalar variable, which has no direction

\[
\tau_{\text{int}}(\theta_f - \theta_i) = K_f - K_i = \text{Work}
\]

Only net external torque \(\tau_{\text{net}}\) can change the angular momentum. \(L\) is a vector, direction is important

\[
\frac{d}{dt}(\vec{L}) = \vec{\tau}
\]

Energy of Rolling

\[
K = \frac{1}{2} I_c \omega^2 + \frac{1}{2} M v_c^2 \quad v_c = R \omega
\]

\[
K = \frac{1}{2} I_c \left(\frac{v_c}{R}\right)^2 + \frac{1}{2} M v_c^2
\]

\[
K = \frac{1}{2} \left(\frac{I_c}{R^2} + M\right) v_c^2
\]

Forces

The acceleration tends to make the wheel slide.

A static frictional force \(\vec{f}\) acts on the wheel to oppose that tendency.
Torques on a Wheel

The Forces on a wheel

Gravity
Normal Force
Friction (so it won’t slide)

Center of Mass View

\[ \sum F_x = Mg \sin(\theta) - F_F = Ma_c \]
\[ \sum F_y = Mg \cos(\theta) - F_N = 0 \]
\[ \sum \tau = F_F R = I_C \alpha \]

Constraint
\[ a_c = \alpha R \]
Rolling without Slipping
\[ a_c = \frac{g \sin(\theta)}{1 + I_c/MR^2} \]

Free falling / sliding without friction:
\[ V_C = \sqrt{2gh} \]

Another View

The wheel rotates about the point of contact

No Torque - Normal Force
Friction

\[ \tau = MgR \sin(\theta) = I_p \alpha \]
\[ I_p = I_c + MR^2 \]
\[ MgR \sin(\theta) = (I_c + MR^2) \alpha \]

Same result
Don’t need x and y motion

Example 1

Kinetic Energy of Rolling

\[ K = \frac{1}{2} \left( \frac{I_c}{R^2} + M \right) v_c^2 \]

+ Energy conservation !!!

Kinetic Energy \( \leftrightarrow \) Potential Energy

\[ \Delta U + \Delta K = 0 \rightarrow U_{\text{initial}} = K_{\text{final}} \rightarrow MgH = \frac{1}{2} \left( \frac{I_c}{R^2} + M \right) v_{\text{com}}^2 \]

Disk:
\[ I_{\text{com}} = \frac{1}{2} MR^2 \]

Hoop:
\[ I_{\text{com}} = MR^2 \]

Sphere:
\[ I_{\text{com}} = \frac{2}{5} MR^2 \]

For disk:
\[ MgH = \frac{1}{2}(1/2M + M) v_{\text{com}}^2 \]
\[ v_{\text{com}} = \left( \frac{4}{3} gh \right)^{\frac{1}{2}} \]
Summary for rotational motion

$360 \text{ degrees} = 2\pi \text{ radians} = 1 \text{ revolution}$, $s = r\theta$, $v = r\omega$, $a = a_r$, $a_i = a = w^2 r = \omega^2 r$, $a_{br} = a_{br} = a_r^2 + a_i^2$

for rotation with constant angular acceleration:

$a = a_r + at$, $\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$, $\alpha = \frac{1}{2} \frac{d^2 \theta}{dt^2}$, $\omega = \omega_0 + \alpha t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$

$\bullet l = \Sigma m r^2$, $I_{cent} = M R^2$, $I_{knees} = 1/2 M R^2$, $I_{knees} = 1/2 M R^2$, $I_{cent} = 1/2 M L^2$

$\Sigma F = m a$, $\tau = \Sigma m r \times F$, $l = I_{cent} + M R^2$

$\tau = \text{force, moment arm} = F r \sin(\phi)$, $\tau = I \alpha, F \times r = \Sigma F = m a$, $\tau = r \times F$, $l = I_{cent} + M R^2$

$W_{ext} = \Delta K = K_f - K_i$, $W = \tau_{ext} \Delta \theta$, $K = K_f + K_i$, $E_{total} = K + U$, $P_{average} = \frac{\Delta W}{\Delta t}$

$P_{net} = \tau \omega$, $\tau_{ext} = 0 \text{ (isolated system)}$, $\tau_{net} = \omega r$ (rolling, no slipping)

$L = r \times p$, $p = mv$, $L = \Sigma L$, $\tau_{net} = dL/dt$, $L = I_{cent}$, $E_{knee} = m g \sin(\phi)$

For isolated systems: $\tau_{ext} = 0$, $L$ is constant, $\Delta L = 0$, $L_i = \Sigma \{l_0 \}$

$a \times b = -b \times a$, $a \times a = 0$, $|a \times b| = a.b \sin(\phi)$, $c = a \times b$ is perpendicular to plane of $a$ and $b$

$c_x = a_y b_z - a_z b_y$, $c_y = -a_z b_x + a_x b_z$, $c_z = a_x b_y - a_y b_x$

$i \times i = j \times j = k \times k = 0$, $i \times j = k$, $j \times k = i$, $k \times i = j$, etc.