Lecture 5

Physics 106

Spring 2006

Rotational Momentum
(Same as Angular Momentum)

http://web.njit.edu/~sirenko/

Angular Momentum

Linear motion $\vec{p} = m\vec{v}$ $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$

Using the correspondence with linear motion

Define "angular momentum"

$I \leftrightarrow m$
$\omega \leftrightarrow v$

$\vec{L} = I\vec{\omega}$ (must define around some origin)

$\vec{\tau}_{tot} = I\vec{\alpha} = \frac{d\vec{L}}{dt}$ If no torque, then L is a constant!

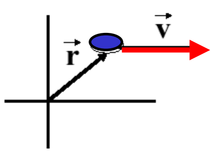
FOR ISOLATED SYSTEM: L IS CONSERVED

Angular Momentum of a particle

Can also define angular momentum for a particle with a linear velocity $\vec{v}$

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$

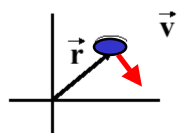
$\vec{r}$ is vector from origin to particle



$L = m \cdot r \cdot v \cdot \sin\phi$ or for a circular motion:
$L = m \cdot r^2 \cdot \omega \cdot \sin\phi$ ($\phi = \pi/2 = 90°$)
$L = I \cdot \omega$

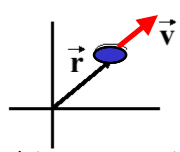
Examples:

$\vec{r} \perp \vec{v} \rightarrow$ L is big
L = mrv



Circular motion

$\vec{r} \parallel \vec{v} \rightarrow L = 0$



Linear motion

Angular Momentum

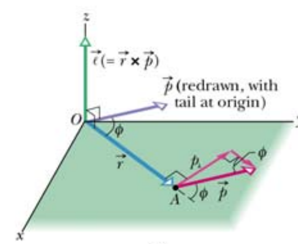
Definition

$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$ [kg m²/s]

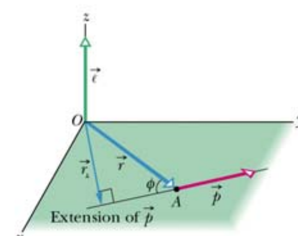
Angular counterpart of linear momentum!

System of particles

$\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$



(a)



(b)

Newton's 2nd Law

Angular Momentum of a particle:

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{L}}\right) = \frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{r}} \times \vec{\mathbf{p}}\right) = \vec{\mathbf{r}} \times \frac{\mathbf{d}\vec{\mathbf{p}}}{\mathbf{dt}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\boldsymbol{\tau}}$$

$$\boxed{\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{L}}\right) = \vec{\boldsymbol{\tau}}}$$

Newton's 2nd Law

Single particle
Linear form

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$

Single particle
Angular form

$$\vec{\tau}_{\text{net}} = \frac{d\vec{l}}{dt}$$

Proof

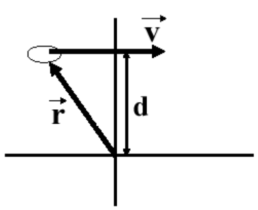
$$\begin{aligned} \vec{l} &= m(\vec{r} \times \vec{v}) \\ \frac{d\vec{l}}{dt} &= m\left(\vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v}\right) \\ &= m(\vec{r} \times \vec{a} + \vec{v} \times \vec{v}) \\ &= \vec{r} \times m\vec{a} = \vec{r} \times \vec{F}_{\text{net}} \\ &= \sum \vec{r} \times \vec{F} = \tau_{\text{net}} \end{aligned}$$

The (vector) sum of all torques acting on a particle is equal to the time rate of change of angular momentum of that particle!

$$\boxed{\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{L}}\right) = \vec{\boldsymbol{\tau}}}$$

EXAMPLE (Linear motion)

Constant velocity particle: Is L really constant?



$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \mathbf{m}\vec{\mathbf{r}} \times \vec{\mathbf{v}}$$

$L = m \cdot r \cdot v \cdot \sin\phi$ or

$L = m \cdot d \cdot v =$ *Const*

Conservation of Angular Momentum

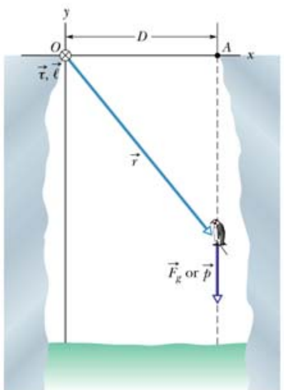
No torque: L is constant

L = Iω

if you change I, ω changes to keep L constant

This allows skaters and divers to spin really really fast (they studied their physics!)

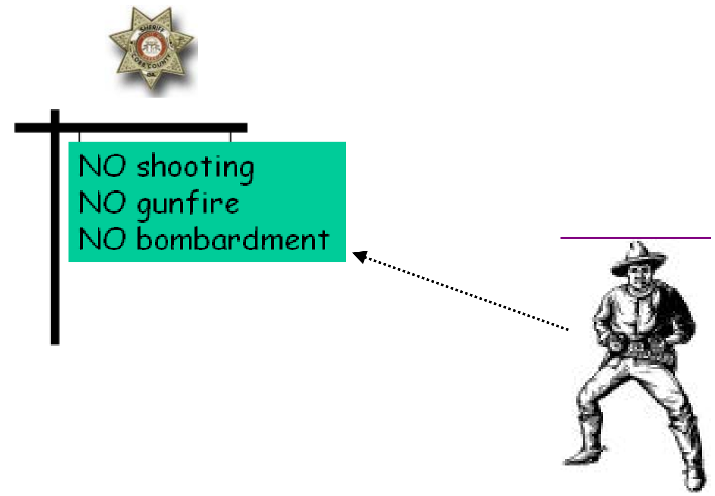
Sample Problem XII–5



A penguin of mass *m* falls from rest at point *A*, a horizontal distance *D* from the origin *O* of an *xyz* coordinate system.

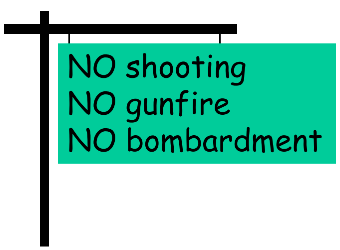
a) What is the angular momentum of $\underline{l}$ of the penguin about *O*?

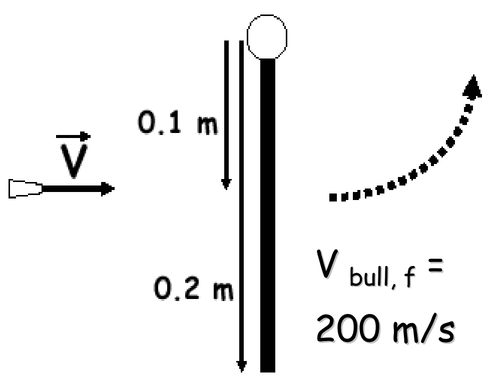
b) About the origin *O*, what is the torque $\underline{\tau}$ on the penguin due to the gravitational force $\underline{F}_g$?



Example

Bullet hits sign: how high does it go?





M_{bull} = 5 g
M_{sign} = 2.2 kg
v_{bull} = 300 m/s
I_{sign} = 0.03 kg m^2

Can it make a complete turn ?

Conservation of Angular Momentum

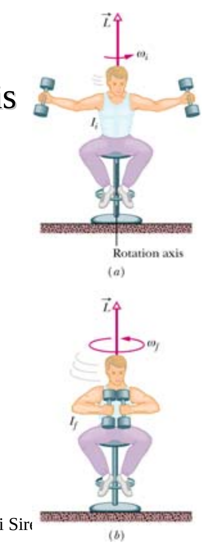
Angular momentum of a solid body about a fixed axis

$$L = I\omega$$

Law of conservation of angular momentum

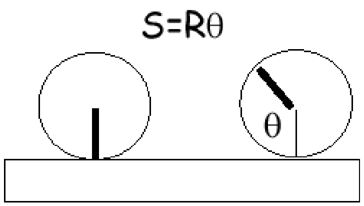
$$\vec{L} = \text{const.} \quad \Rightarrow \quad \vec{L}_i = \vec{L}_f$$

(Valid from microscopic to macroscopic scales!)



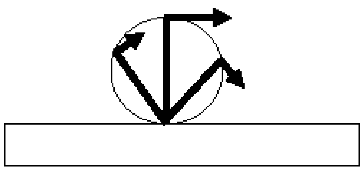
If the net external torque $\underline{\tau}_{net}$ acting on a system is zero, the angular momentum $\underline{L}$ of the system remains constant, no matter what changes take place within the system

Rolling Motion: without slipping



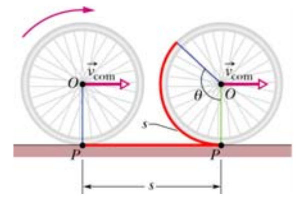
$$v_c = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_c = R\alpha$$



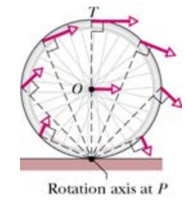
At any instant
the wheel rotates about
the point of contact

Rolling



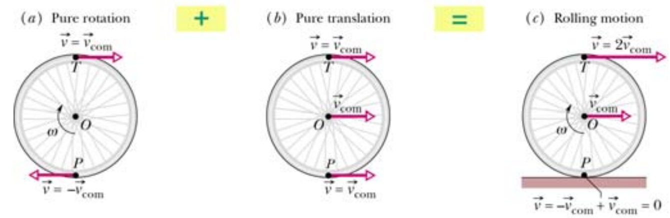
Smooth rolling motion

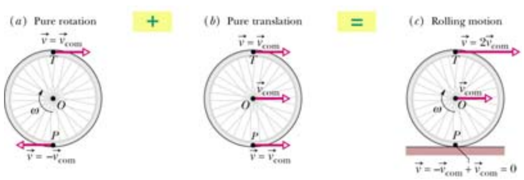
$$v_{com} = \omega R$$



Reference frame

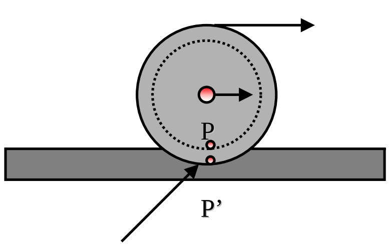
Rotation and Translation

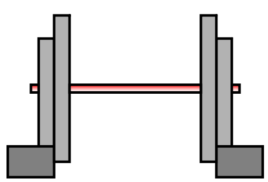




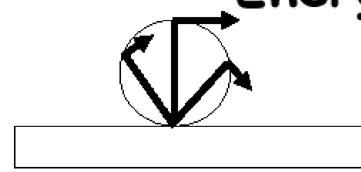
Rolling of the train wheel

is it the same or slightly different?





Energy of Rolling



$$K = \frac{1}{2} I_C\,\omega^2 + \frac{1}{2} M\,v_C^2 \qquad v_c = R\,\omega$$

$$K = \frac{1}{2} I_C \left(\frac{v_C}{R}\right)^2 + \frac{1}{2} M\,v_C^2$$

$$\mathbf{K} = \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\mathbf{I_C}}{\mathbf{R^2}} + \mathbf{M}\right)\mathbf{v_C^2}$$

Kinetic Energy

$$
\begin{aligned}
K &= \tfrac{1}{2} I_P \omega^2 \\
I_P &= I_{com} + MR^2 \\
K &= \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} MR^2\omega^2 \\
v_{com} &= \omega R \\
K &= \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} M v_{com}^2
\end{aligned}
$$

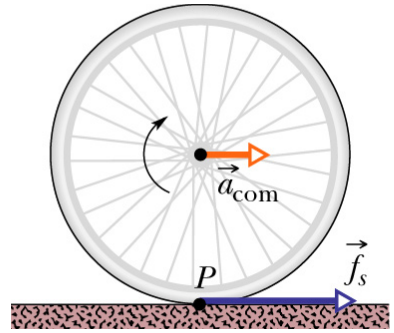
Stationary observer

Parallel–axis theorem

Sample Problem X12–1: A uniform solid cylindrical disk (M = 1.4 kg, r = 8.5 cm) roll smoothly across a horizontal table with a speed of 15 cm/s. What is its kinetic energy K?

A rolling object has two types of kinetic energy: a rotational kinetic energy due to its rotation about its center of mass and a translational kinetic energy due to translation of its center of mass.

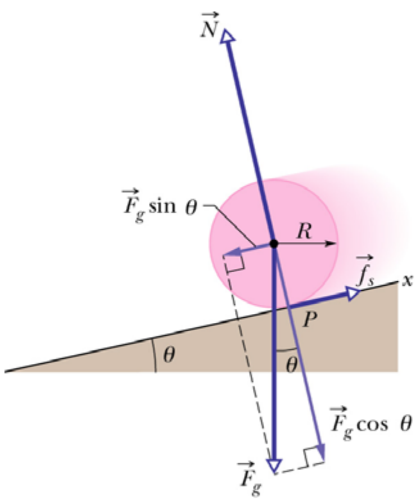
Forces



A net force $\underline{F}_{net}$ acting on a rolling wheel speeds it up or slows it down and causes an acceleration.

There is a slipping tendency for the wheel, while the friction force prevents it.

Forces



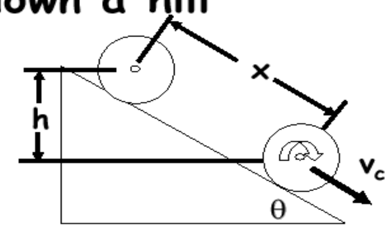
The acceleration tends to make the wheel slide.

A static frictional force f_s acts on the wheel to oppose that tendency.

Rolling down a hill

Conservation of Energy

$$\frac{1}{2}\left(\frac{I_C}{R^2} + M\right) v_C^2 = Mgh$$



$$v_C = \sqrt{\frac{2gh}{1 + I_C/MR^2}}$$

For a particle: $v_C = \sqrt{2gh}$

A Disc

$$I_C = \frac{1}{2} MR^2$$

A Ring

$$I_C = MR^2$$

$$v_C = \sqrt{\frac{2gh}{1 + \frac{1}{2}MR^2/MR^2}} = \sqrt{\frac{2}{3}2gh}$$

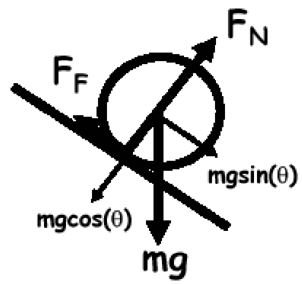
$$v_C = \sqrt{\frac{2gh}{1 + MR^2/MR^2}} = \sqrt{\frac{1}{2}2gh}$$

Free falling / sliding without friction: $V_C = \sqrt{2gh}$

Torques on a Wheel

The Forces on a wheel

- Gravity
- Normal Force
- Friction (so it won't slide)



Center of Mass View

$$\sum F_x = Mg\sin(\theta) - F_F = Ma_c$$

$$\sum F_y = Mg\cos(\theta) - F_N = 0$$

$$\sum \tau = \mathbf{F_F}\ \mathbf{R} = \mathbf{I_C}\ \alpha$$

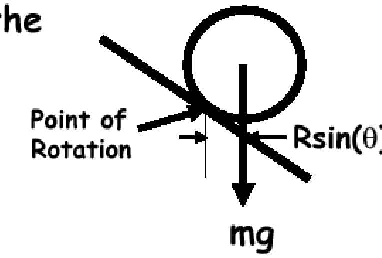
Constraint $a_c = \alpha R$

Rolling without Slipping

$$a_c = \frac{g\sin(\theta)}{1 + I_C/MR^2}$$

Another View

The wheel rotates about the point of contact



No Torque - Normal Force, Friction

$$\tau = \mathbf{MgRsin(\theta)} = \mathbf{I_P}\ \alpha$$

$$I_P = I_C + MR^2$$

$$\mathbf{MgRsin(\theta)} = \left(\mathbf{I_C} + \mathbf{MR^2}\right)\alpha$$

$$a_c = \frac{g\sin(\theta)}{1 + I_C/MR^2}$$

Same result

Don't need x and y motion

The Gyroscope

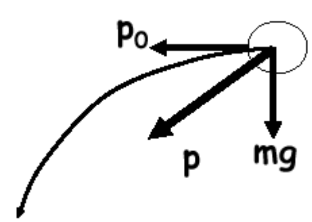
You are designing a cruise missile which makes lots of twists and turns and has no driver on board

How do you keep track of which way is up?

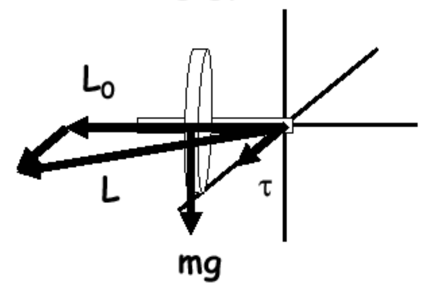
Start something spinning and protect it from any torque – $\vec{L}$ keeps pointing in same direction

Torque in three dimensions: the falling gyroscope

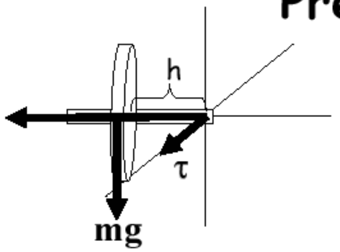
A falling rock



A falling gyro

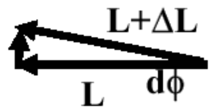


Precession



$$L = I\omega$$

$$\tau = mgh$$



$$\tau = \frac{dL}{dT}$$

$$\Delta L = \tau\ \Delta t$$

$$d\phi = \frac{dL}{L} = \frac{mgh\,dt}{L}$$

$$\boxed{\Delta L = mgh\ \Delta t}$$

$$\boxed{\omega_p = \frac{d\phi}{dt} = \frac{mgh}{I\omega}}$$

Torque

Revisit Chapter III!

Definition of Torque

$\vec{\tau} = \vec{r} \times \vec{F}$ Vector (cross) product

(Right–hand rule, order does matter!)