Electromagnetism II

Instructor: Andrei Sirenko

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Spring 2013

Thursdays 1 pm – 4 pm

Spring 2013:

1. Book: D. J. Griffiths
Introduction to Electrodynamics
2. Syllabus: @ web.njit.edu/~sirenko
3. Course components:
5 HWs, 2 CQZs, Final Exam
20% 15%+15% 50% = 100%
50% D 60% C 65% C+ 70% B 75% B+ 80% A

Lecture 1

Prerequisites:
E&M I from Fall 2012
Chapters: 1, 2, 3, and 4

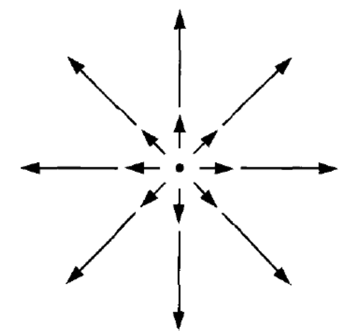
Vector algebra
Electrostatics
Laplace Eq. , Images, Expansions,…Diff. Eqs.
and Electric Fields in Matter

Ability to solve problems and use your knowledge

Vector Algebra

"del"

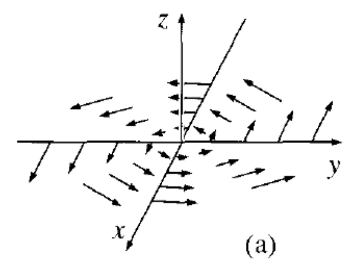
$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$



1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

"curl"

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$



(a)

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$.

(2) Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$.

The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

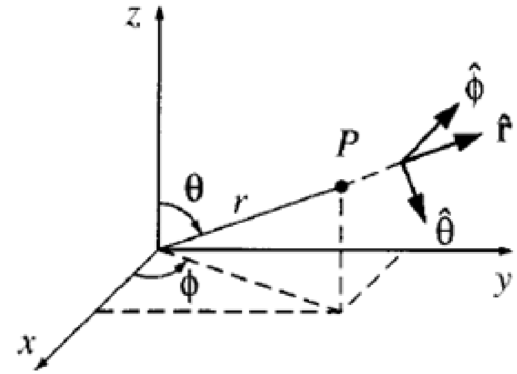
(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

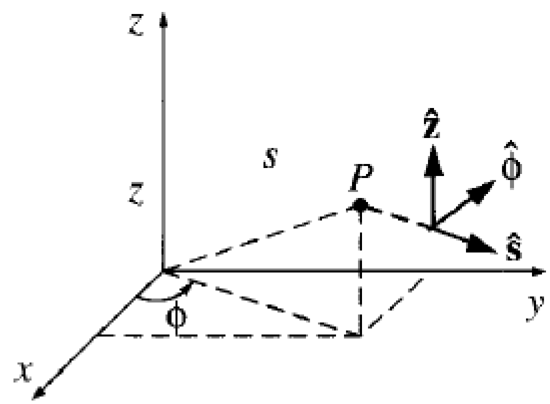
Also need
Boundary conditions!

Vector Algebra

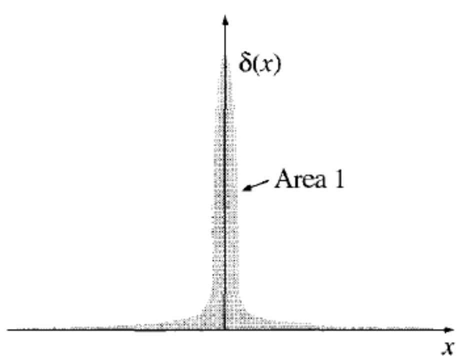
Spherical Polar Coordinates



Cylindrical Coordinates

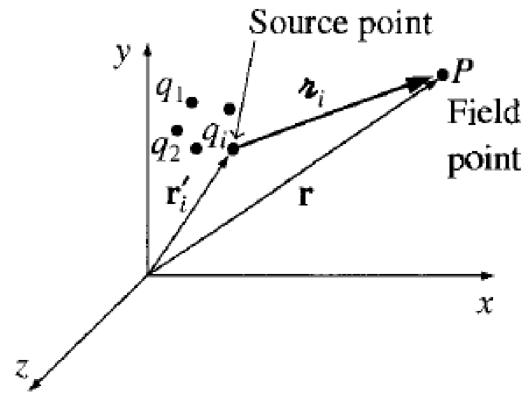


Dirac Delta Function



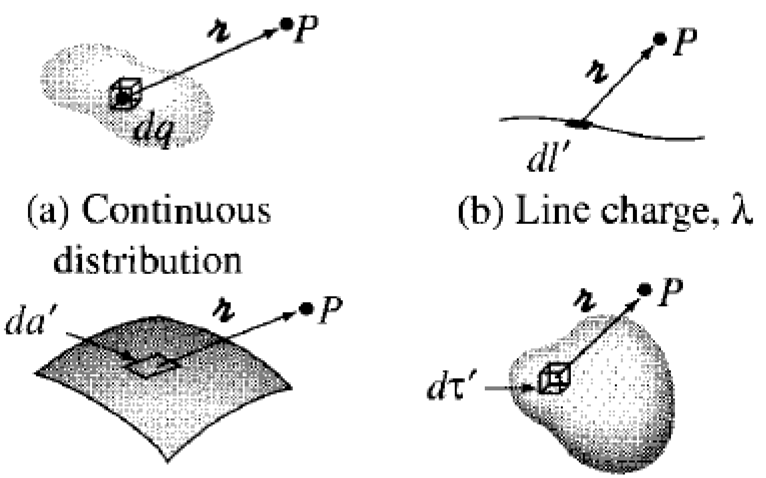
Electrostatics

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.}$$



$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i .$$

$$\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}.}$$

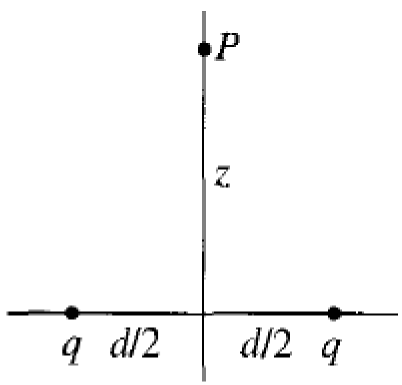


(a) Continuous distribution

(b) Line charge, λ

(c) Surface charge, σ

(d) Volume charge, ρ



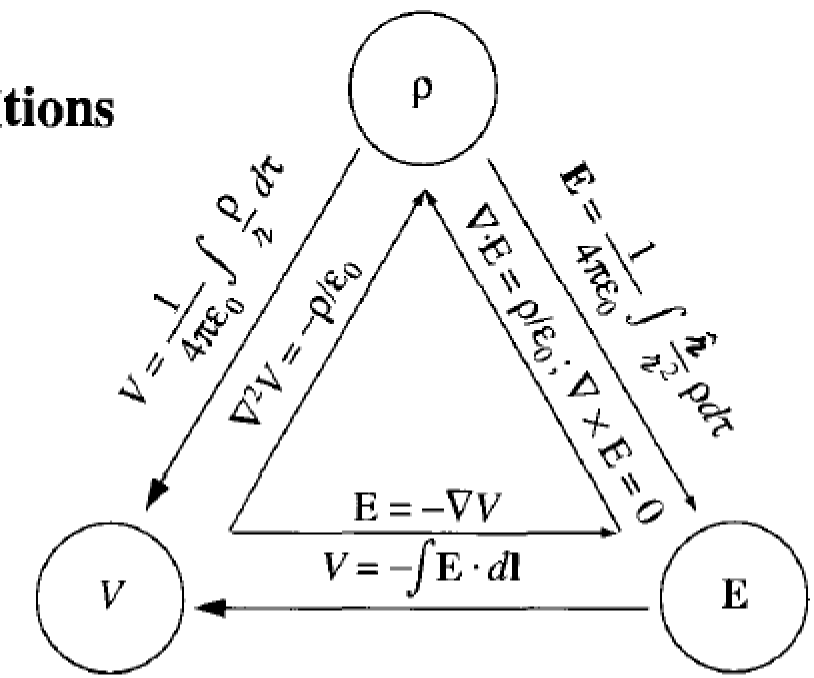
Electrostatics

Poisson's Equation and Laplace's Equation

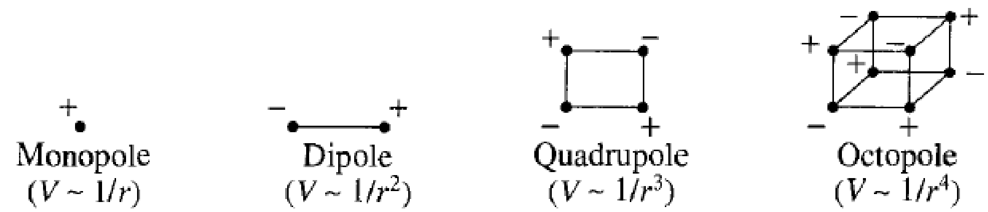
$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}.}$$

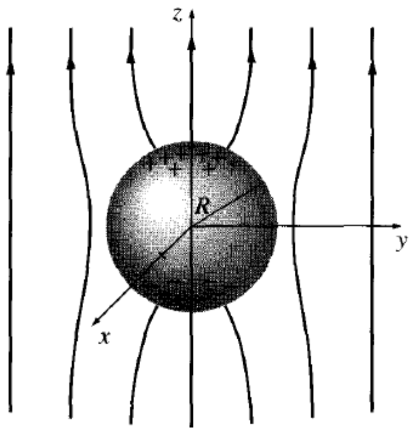
$$\nabla^2 V = 0.$$

Electrostatic Boundary Conditions



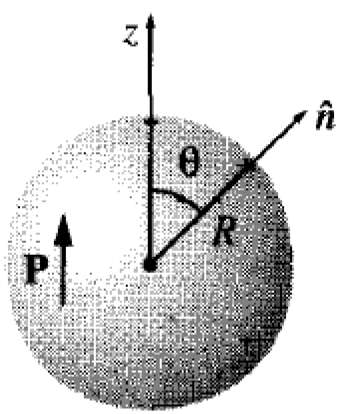
Electrostatics

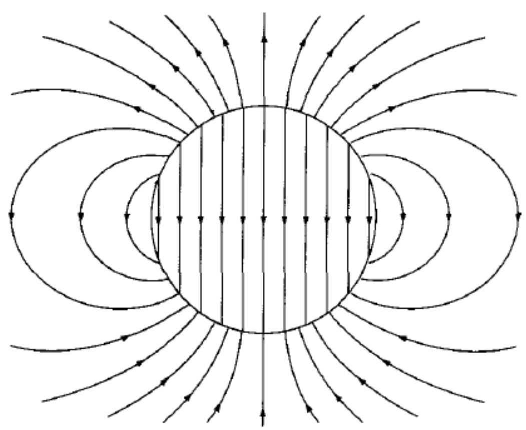


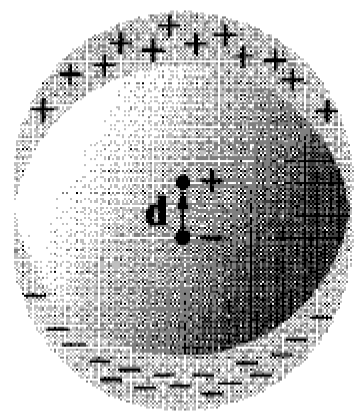


$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

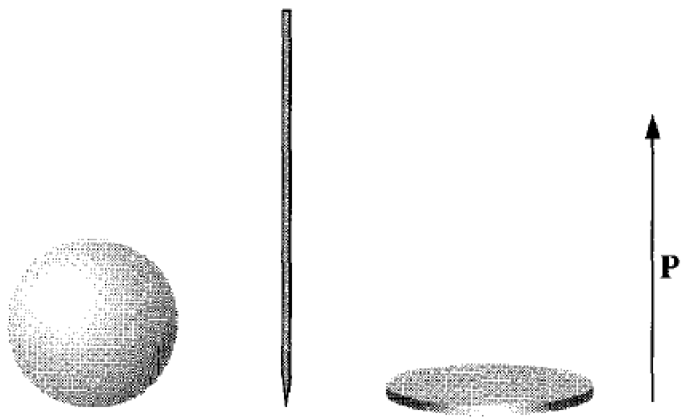
$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta.$$







Problems: Electrostatics



(a) Sphere (b) Needle (c) Wafer

$$P_x = \epsilon_0(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z)$$
$$P_y = \epsilon_0(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z)$$
$$P_z = \epsilon_0(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z)$$

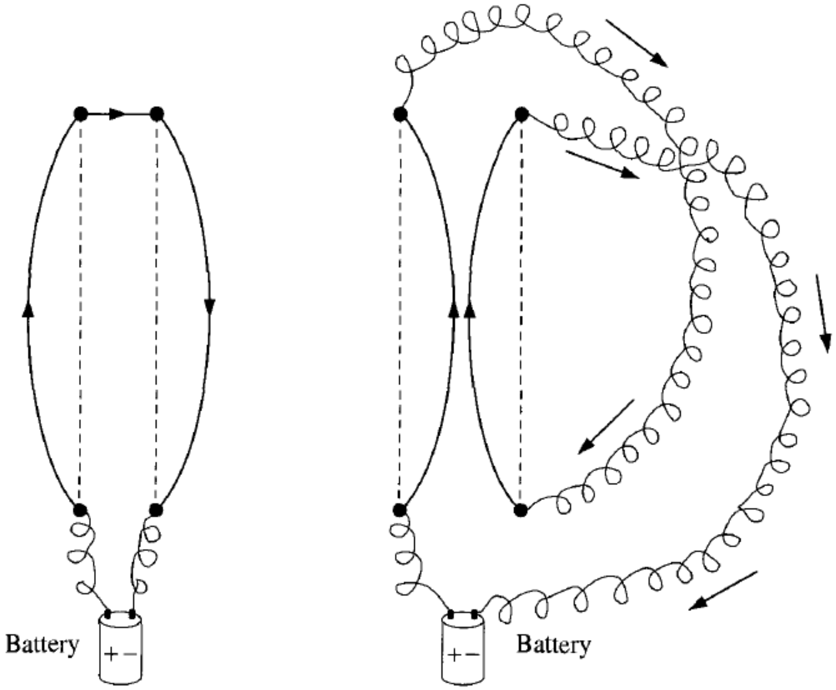
Syllabus for Spring 2013:

Chapters 5, 6, 7, 8, 9, 10, 11 +

Optics of Magnetoelectric materials

	week	Topic	Reading Materials	HW
1	Jan 24	Magnetostatics	Ch. 1 – 4 repeated Ch. 5	
2	Jan 31	Magnetostatics (cont.)	Ch. 5	HW1:
3	Feb 7	Magnetic Fields in Matter	Ch. 6	
4	Feb 14	Magnetic Fields in Matter	Ch. 6	HW2:
5	Feb 21	Electrodynamics	Ch. 7	
6	Feb 28	Electrodynamics / Maxwell Eqs.	Ch. 7	HW3:
7	March 7	Conservation Laws	Ch. 8	
8	March 14	Conservation Laws	Ch. 8	HW4:
9	March 28	Electromagnetic Waves	Ch. 9	
10	April 4	Electromagnetic Waves	Ch. 9	
11	April 11	EMW in Magnetoelectics		HW5:
12	April 18	Potentials and Fields	Ch. 10	
13	April 25	Radiation	Ch. 11	
14	May 2	REVIEW	Ch. 1 - 11	

Magnetostatics

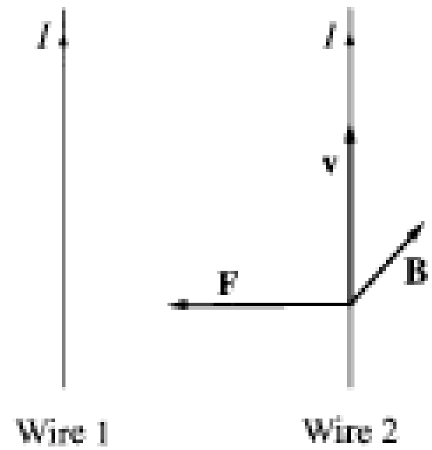


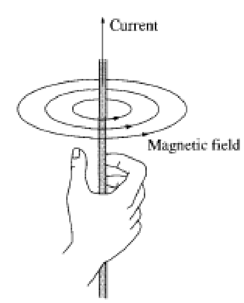
(a) Currents in opposite directions repel.

(b) Currents in same directions attract.

Two main questions:

- What is the m.f. produced by a current ?
- What is the force on a current due to the m.f. ?





Magnetic Forces

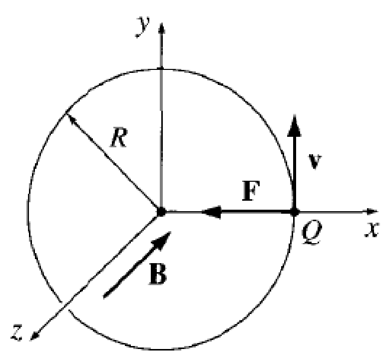
1. moving charge (e.g., electron)

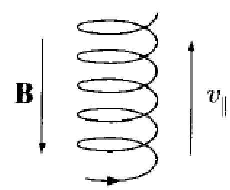
$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}).}$$

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$

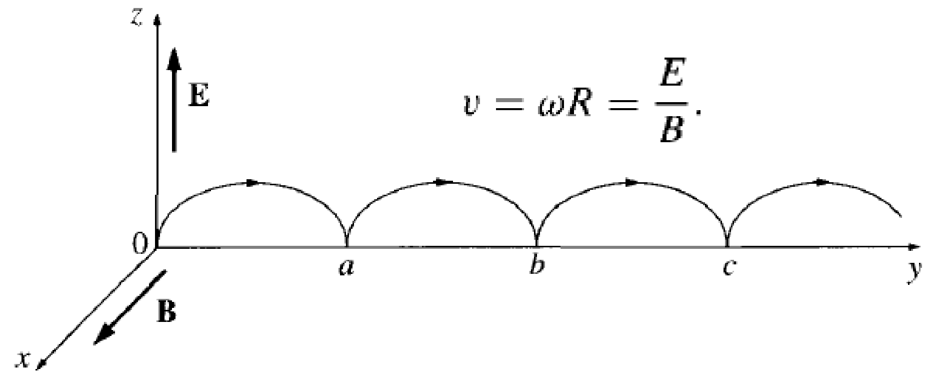
Cyclotron motion

$$QvB = m\frac{v^2}{R}, \quad \text{or} \quad p = QBR,$$





Cycloid Motion



$$v = \omega R = \frac{E}{B}.$$

$$\mathbf{v} = (0, \dot{y}, \dot{z}),$$

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}},$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}) = m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}. \qquad \omega \equiv \frac{QB}{m}.$$

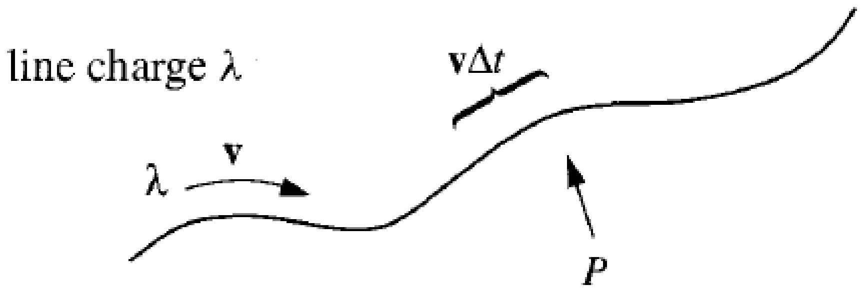
$$\ddot{y} = \omega\dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right).$$

$$\left.\begin{aligned} y(t) &= C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3, \\ z(t) &= C_2 \cos\omega t - C_1 \sin\omega t + C_4. \end{aligned}\right\}$$

$$\mathbf{v} = 0, \qquad y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t), \quad z(t) = \frac{E}{\omega B}(1 - \cos\omega t). \qquad R \equiv \frac{E}{\omega B},$$

$$(y - R\omega t)^2 + (z - R)^2 = R^2. \qquad v = \omega R = \frac{E}{B}.$$

Current



$$\mathbf{I} = \lambda \mathbf{v}$$

$$1\ \text{A} = 1\ \text{C/s}.$$

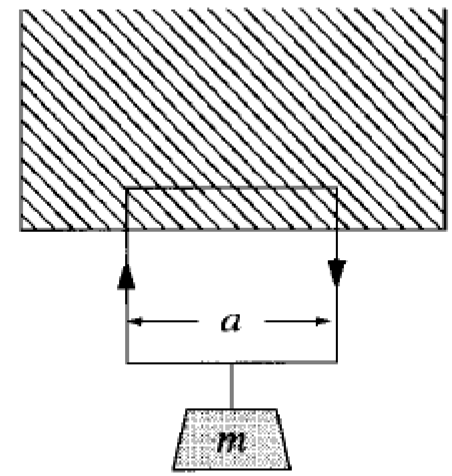
$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl.$$

$$\boxed{\mathbf{F}_{\text{mag}} = \int I\, (d\mathbf{l} \times \mathbf{B}).}$$

$$\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}).$$

Magnetic Forces do not work

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0.$$

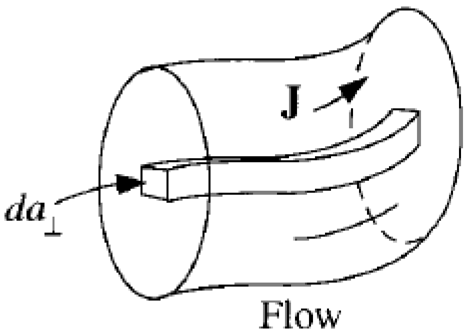


$$W_{\text{mag}} = F_{\text{mag}} h = IBah,$$

Current Density

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}.$$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau.$$



$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) \, d\tau.$$

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.}$$

i.e., Local charge conservation

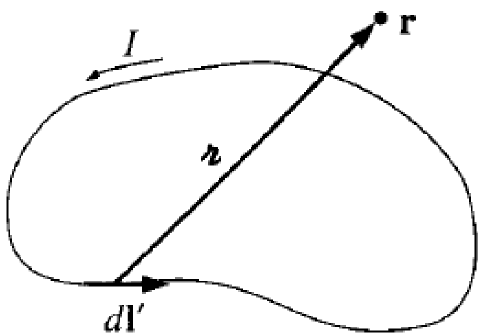
The Biot-Savart Law

Steady Currents

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

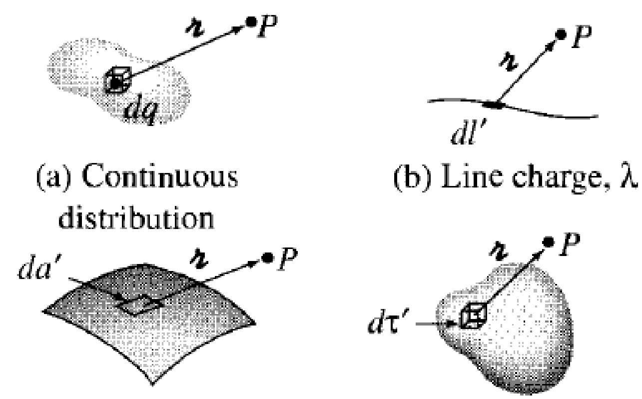
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\imath}}}{\imath^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}.$$

$1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}.$ $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}.$



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\imath^2}\hat{\boldsymbol{\imath}}\, d\tau'.$$

$$\epsilon_0 = 8.85 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N \cdot m^2}}.$$



(a) Continuous distribution (b) Line charge, λ (c) Surface charge, σ (d) Volume charge, ρ

The divergence and curl of the *magnetostatic* field are

$$
\begin{cases}
\nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\
\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).}
\end{cases}
$$

$$
\begin{cases}
\nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law)} \\
\nabla \times \mathbf{E} = 0, & \text{(no name).}
\end{cases}
$$

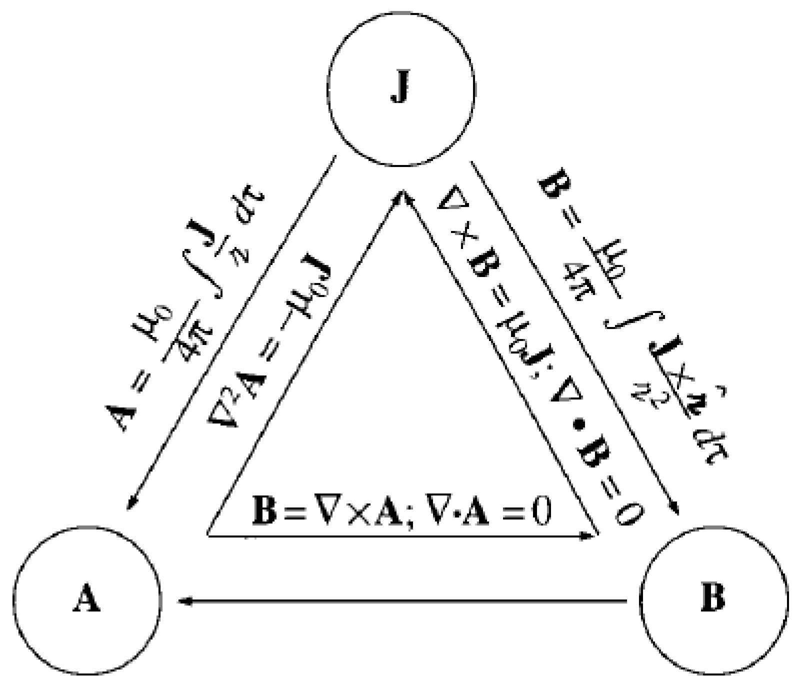
Stokes theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.}$$

Poisson's equation

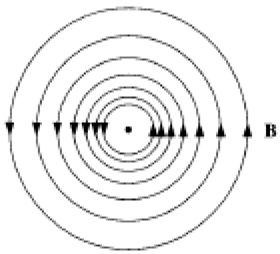
$$
\begin{cases}
\nabla^2 \vec{A} = -\mu_0 \vec{J} \\
\vec{B} = \nabla \times \vec{A} \\
\nabla \cdot \vec{A} = 0
\end{cases}
$$



$$
\begin{cases}
\text{Electrostatics :} & \text{Coulomb} & \rightarrow & \text{Gauss,} \\
\text{Magnetostatics :} & \text{Biot}-\text{Savart} & \rightarrow & \text{Ampère.}
\end{cases}
$$

The Divergence and Curl of B

Straight-Line Currents



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

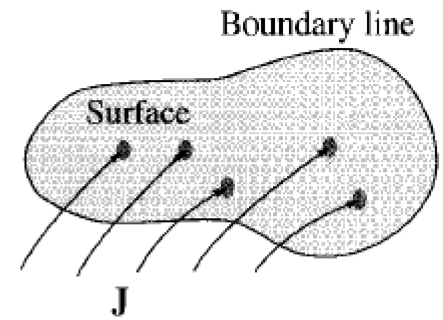
$$B = \frac{\mu_0 I}{2\pi s}. \qquad \oint \mathbf{B}\cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\,dl = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I.$$

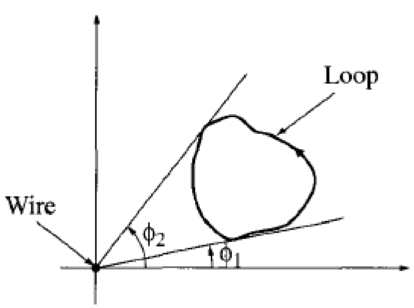
cylindrical coordinates (s, ϕ, z)

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}},$$

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

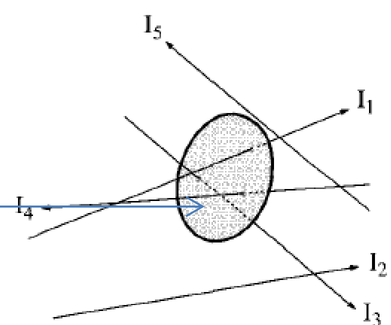
$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\,s\,d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I.$$





$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}},$$

$$I_{\text{enc}} = \int \mathbf{J}\cdot d\mathbf{a},$$



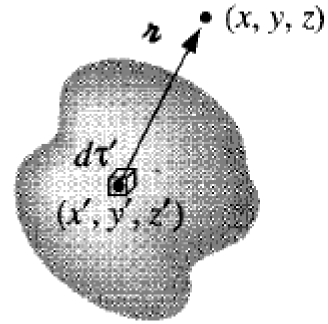
Stokes' theorem

$$\int (\nabla\times\mathbf{B})\cdot d\mathbf{a} = \mu_0 \int \mathbf{J}\cdot d\mathbf{a},$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}.$$

The Divergence and Curl of B

for the general case $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'.$



$\mathbf{B}$ is a function of (x, y, z),

$\mathbf{J}$ is a function of (x', y', z'),

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'.$$

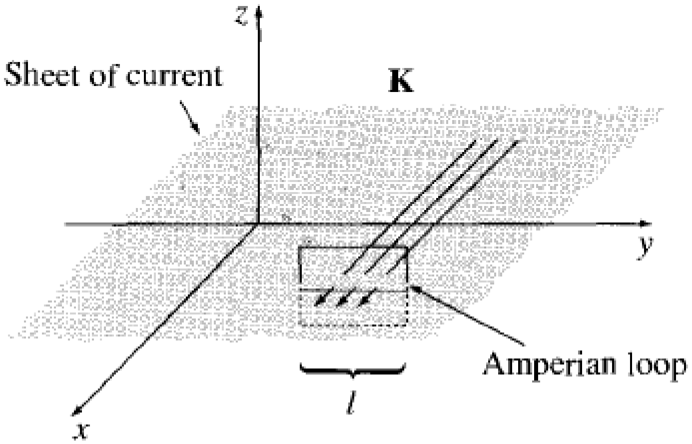
$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right).$$

$$\nabla \times \mathbf{J} = 0 \qquad \nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = 0$$

$$\nabla \times (r^n \hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^{n+2}) = \frac{1}{r^2}(n+2)r^{n+1} = \boxed{(n+2)r^{n-1}}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

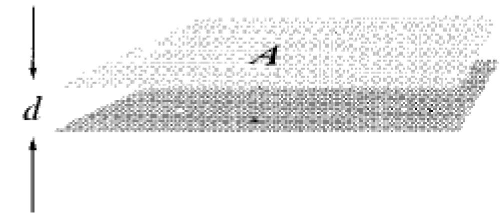
Find the magnetic field of an infinite uniform surface current $\mathbf{K} = K\,\hat{\mathbf{x}}$, flowing over the xy plane



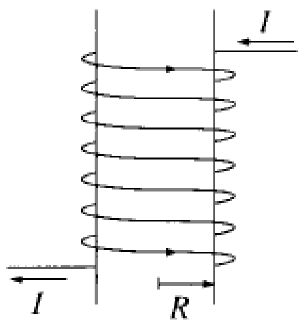
$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{enc} = \mu_0 K l,$$

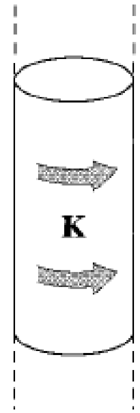
$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases}$$

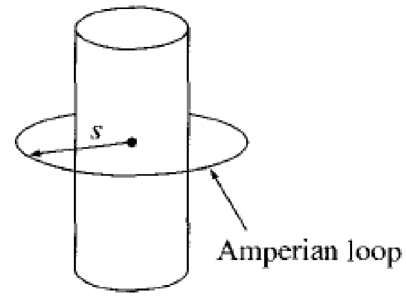
"parallel-plate capacitor"

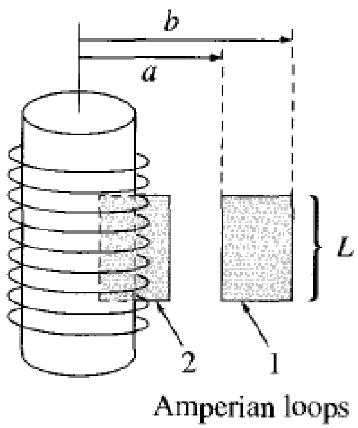


the magnetic field of a very long solenoid,







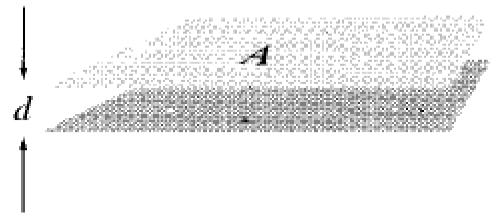


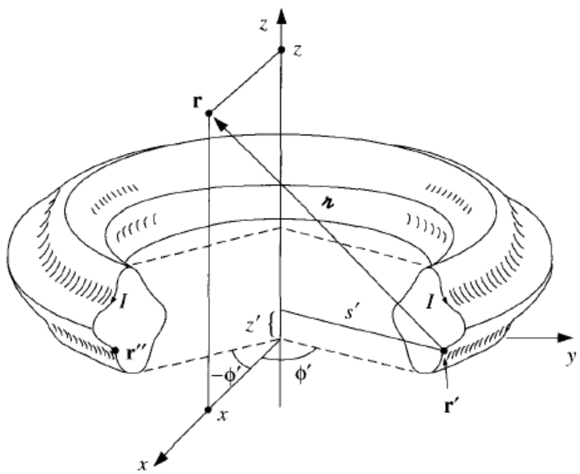
$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi (2\pi s) = \mu_0 I_{enc} = 0,$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{enc} = \mu_0 n I L,$$

$$\mathbf{B} = \begin{cases} \mu_0 n I \, \hat{\mathbf{z}}, & \text{inside the solenoid,} \\ 0, & \text{outside the solenoid.} \end{cases}$$

"parallel-plate capacitor"





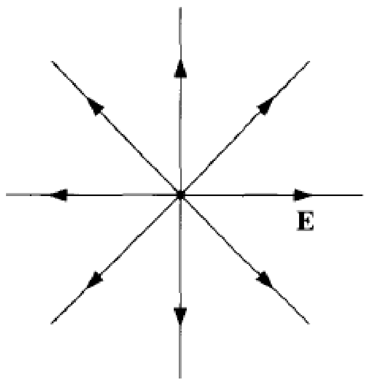
$$B 2\pi s = \mu_0 I_{\text{enc}},$$

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ 0, & \text{for points outside the coil,} \end{cases}$$

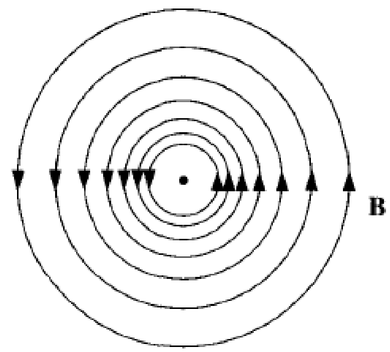
$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law)} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$

The divergence and curl of the *magnetostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$



(a) Electrostatic field of a point charge



(b) Magnetostatic field of a long wire

Stokes theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.}$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Magnetic Vector Potential

$$\mathbf{E} = -\nabla V,$$

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.}$$

$$\boxed{\nabla \cdot \mathbf{A} = 0.}$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Poisson's equation

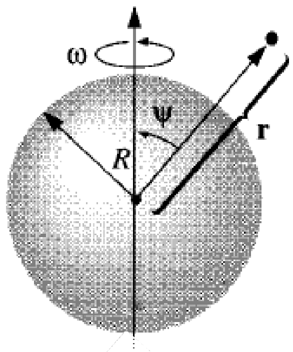
$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

3 of them for x,y,z

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'.}$$

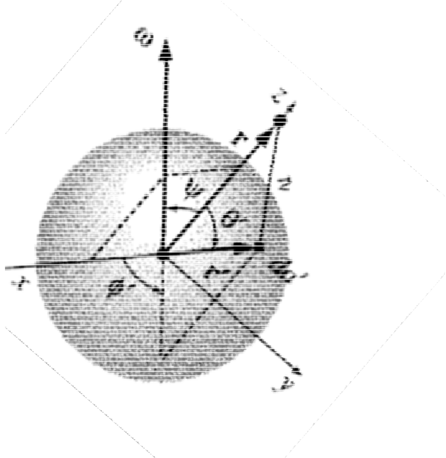
$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} \, d\mathbf{l}'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}} \, da'.$$

A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\, da',$$

where $\mathbf{K} = \sigma \mathbf{v}$, $\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2 \sin\theta'\, d\theta'\, d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega} \times \mathbf{r}'$; in this case,



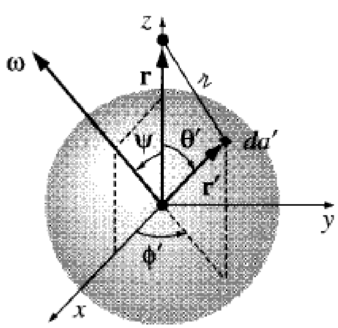
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi \sin\theta' \sin\phi')\,\hat{\mathbf{x}} + (\cos\psi \sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

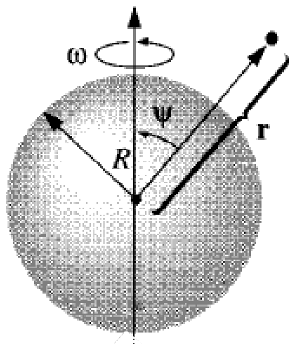
$$\int_0^{2\pi} \sin\phi'\, d\phi' = \int_0^{2\pi} \cos\phi'\, d\phi' = 0,$$

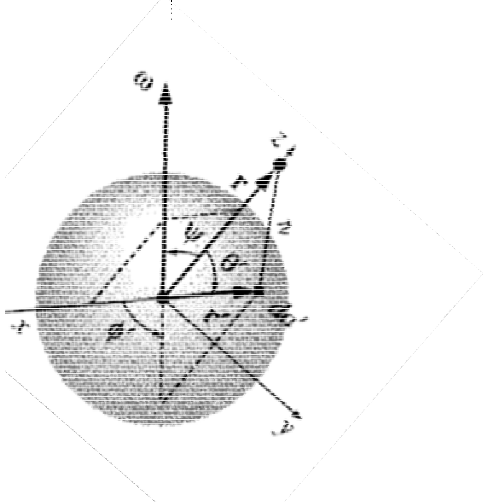
such terms contribute nothing. There remains

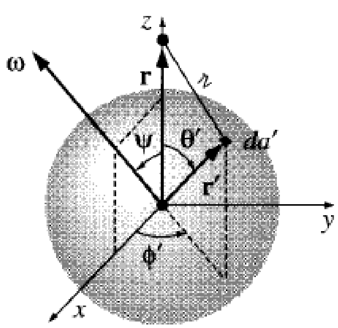


$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma\omega\sin\psi}{2}\left(\int_0^{\pi} \frac{\cos\theta'\sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\, d\theta'\right)\hat{\mathbf{y}}.$$

A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$







Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\Big].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega}\times\mathbf{r}) = -\omega r \sin\psi\,\hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } inside \text{ the sphere,} \\[2ex] \dfrac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } outside \text{ the sphere.} \end{cases} \tag{5.66}$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r \le R), \\[2ex] \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases} \tag{5.67}$$

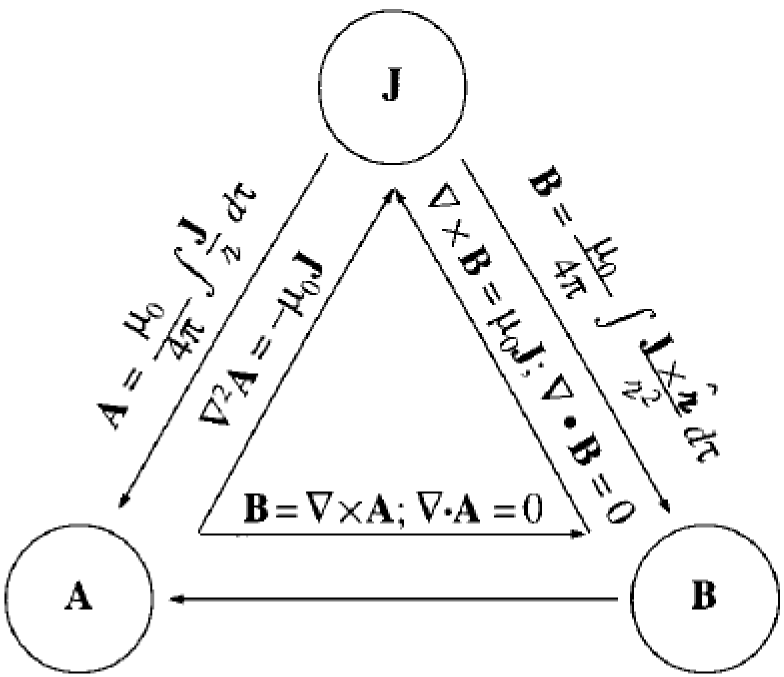
Curiously, the field inside this spherical shell is *uniform*:

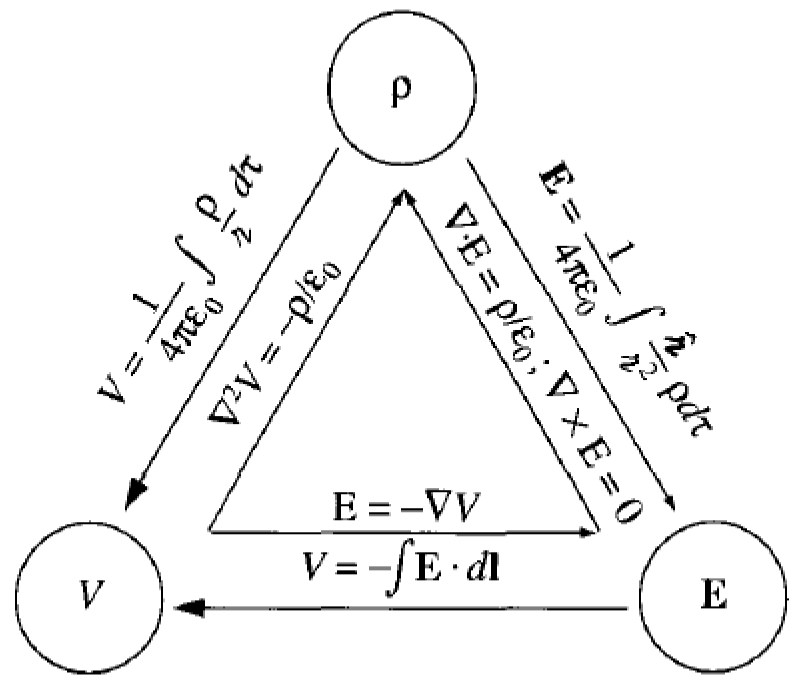
$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}. \tag{5.68}$$

The divergence and curl of the *magnetostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

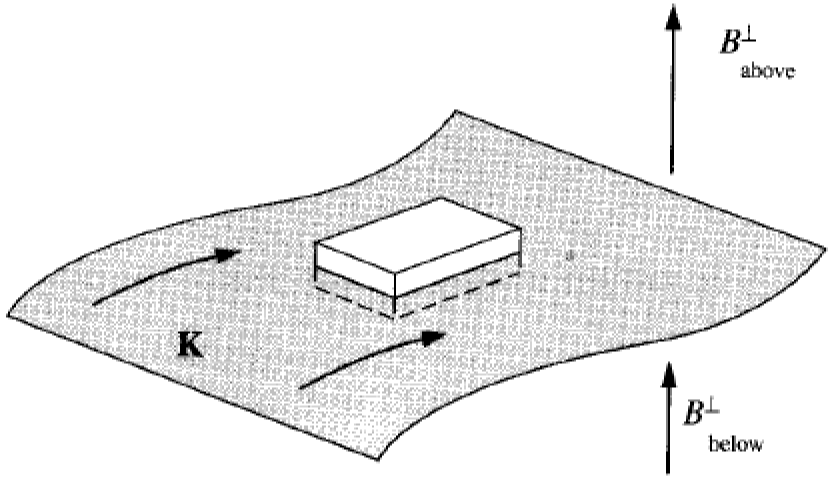
$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law)} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$





$$\begin{cases} \text{Electrostatics:} & \text{Coulomb} & \rightarrow & \text{Gauss,} \\ \text{Magnetostatics:} & \text{Biot–Savart} & \rightarrow & \text{Ampère.} \end{cases}$$

Boundary conditions

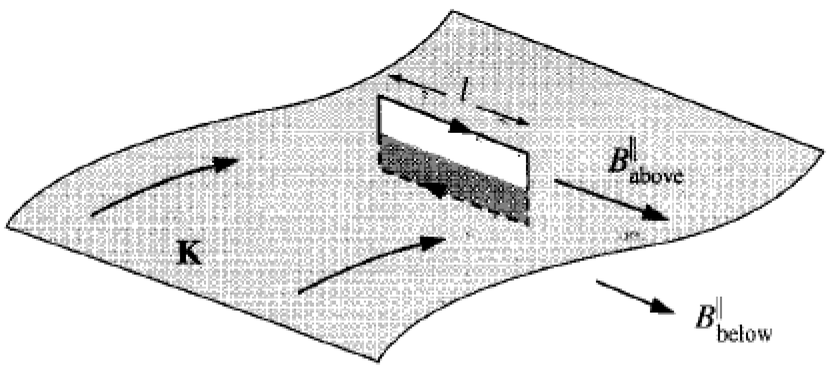


$$\oint \mathbf{B} \cdot d\mathbf{l} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})l = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K.$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}.$$

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}),$$



$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}},$$

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$