Optics in Stratified and Anisotropic Media: 4×4-Matrix Formulation

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A 4×4-matrix technique was recently introduced by Teitler and Henvis for finding propagation and reflection by stratified anisotropic media. It is more general than the 2×2-matrix technique developed by Jones and by Abelès and is applicable to problems involving media of low optical symmetry. A little later, we developed a 4×4 differential-matrix technique in order to solve the problem of reflection and transmission by cholesteric liquid crystals and other liquid crystals with continuously varying but planar ordering. Our technique is mathematically equivalent to that of Teitler and Henvis, but we used a somewhat different approach. We start with a 6×6-matrix representation of Maxwell's equations that can include Faraday rotation and optical activity. From this, we derive expressions for 16 differential-matrix elements so that a wide variety of specific problems can be attacked without repeating a large amount of tedious algebra. The 4×4-matrix technique is particularly well suited for solving complicated reflection and transmission problems on a computer. It also serves as an illuminating alternative way to rederive closed solutions to a number of less-complicated classical problems. Teitler and Henvis described a method of solving some of these problems, briefly in their paper. We give solutions to several such problems and add a solution to the Oseen–DeVries optical model of a cholesteric liquid crystal, to illustrate the power and simplicity of the 4×4-matrix technique.

INDEX HEADINGS: Reflection; Refraction; Crystals.

In many optical problems, there is a great advantage to reducing the number of electromagnetic-field variables to a minimum. The simplest case is that in which light of only one polarization need be considered at a time. It is necessary to carry two field variables, such as two orthogonal electric-field components (E_x and E_y) or the magnetic component of one and the electric component of the other, in order to describe changes of polarization. Matrix techniques for handling two variables have been used extensively since their introduction in optics by Jones[1] and Abelès.[2] In eliminating the other two electromagnetic-field components, Maxwell's first-order differential equations are combined to give second-order equations.

When the symmetry of a medium is low, the methods of Jones and of Abelès may become impractical. Teitler and Henvis first introduced the 4×4-matrix technique, retaining two electric and two magnetic field variables throughout the computation for solving such complicated problems.[3] A little later we developed an essentially equivalent technique to solve the problem of reflection and transmission of obliquely incident light by planar layers of cholesteric liquid crystals.[4–6] This is essentially the same problem, except that the strata become infinitesimal and vary continuously. This difference makes a differential formulation of the technique, rather than an integral formulation, useful. This paper is oriented more toward the differential formulation.

In spite of the size disadvantage of 4×4 matrices, we have found that the advantage of using first-order differential equations and of not having to resurrect the two suppressed field variables with auxiliary equations when matching fields at boundaries is often very helpful in computing optical properties of materials, even when low symmetry is not a problem. It seems to us that the 4×4-matrix technique often helps to keep the physical problem closer to the surface than is the case when some of the field variables are suppressed.

This paper is intended to be a general exposition of the method. A few examples of special simplicity are included to illustrate how the technique works. Specific solutions of the complicated problem of light falling obliquely on liquid-crystal films are published elsewhere.[4,7] We hope the examples will illustrate that the 4×4-matrix formulation can handle quite complicated problems with ease and clarity.

In order to facilitate the use of the technique, we have written out expressions for individual elements in certain matrices in full, even though their derivation is quite straightforward, because it is very easy to make errors in algebra in the process of deriving them. We refer in particular to Eqs. (21) and (24), (41), and (85)–(90).

DERIVATION OF MATRIX WAVE EQUATIONS

Maxwell's equations in gaussian units and rectangular coordinates may be written in the 6×6-matrix form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix} = -\frac{1}{c}\frac{\partial}{\partial t} \begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \end{pmatrix}, \quad (1)$$

where c is the velocity of light in vacuum and **E**, **H**, **D**, and **B** are the electromagnetic-field vectors. Equation (1) will be abbreviated as

$$\mathbf{R}\mathbf{G}=\frac{1}{c}\frac{\partial}{\partial t}\mathbf{C}. \tag{2}$$

Notice that **R** is a symmetrical matrix with nonzero elements only in the off-diagonal positions of the first (upper-right) and third (lower-left) quadrants. The first quadrant is the **curl** operator, operating on **H**, and the third is the negative **curl** operating on **E**.

If we ignore nonlinear optical effects, we may write a linear relation between **G** and **C** as

$$\mathbf{G}=\mathbf{M}\mathbf{C}. \tag{3}$$

The first and/or third quadrants of **M** are nonzero in media that are optically active. The second (upper-left) quadrant is the dielectric tensor:

$$\epsilon_{ij}=M_{ij},\quad i,\ j=1,\ 2,\ 3. \tag{4}$$

The fourth quadrant is the permeability tensor

$$\mu_{ij}=M_{i+3,j+3},\quad i,\ j=1,\ 2,\ 3. \tag{5}$$

To complete the breakdown, we shall define the first and third quadrants as the optical-rotation tensors

$$\rho_{ij}=M_{i,j+3}$$

and

$$\rho_{ij}'=M_{i+3,j},\quad i,\ j=1,\ 2,\ 3. \tag{6}$$

We shall define the time dependence of an optical disturbance by the exponential factor $\exp(-i\omega t)$. If the reader prefers the convention $\exp(+i\omega t)$, he should change the sign on every imaginary number in this paper. Combining Eqs. (2) and (3) we arrive at the spatial wave equation for frequency ω:

$$\mathbf{R}\boldsymbol{\Gamma}=\frac{-i\omega}{c}\mathbf{M}\boldsymbol{\Gamma}, \tag{7}$$

where $\boldsymbol{\Gamma}$ is the spatial part of **G**.

Drude[8] gave a classical derivation of equations for light propagation in certain optically active media. In the case of certain isotropic media, these equations are

$$\mathbf{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \tag{8}$$

and

$$-\mathbf{curl}\mathbf{H}=-\frac{\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{\gamma}{c}\frac{\partial}{\partial t}\mathbf{curl}\mathbf{E}, \tag{9}$$

where ϵ and γ may both depend on frequency ω. By inserting the time dependence, and eliminating **curlE** from the latter equation, we get

$$\mathbf{curl}\mathbf{E}=i\omega\mathbf{H}/c \tag{10}$$

and

$$-\mathbf{curl}\mathbf{H}=i\omega\epsilon\mathbf{E}/c-(i\omega\gamma/c)(i\omega\mathbf{H}/c). \tag{11}$$

Comparison of the last two equations with Eqs. (4)–(7) shows that, in our notation, the four quadrants of **M** for Drude's model are

$$\boldsymbol{\rho}=(-i\omega\gamma/c)\mathbf{I},\quad \boldsymbol{\varepsilon}=\epsilon\mathbf{I},\quad \boldsymbol{\rho}'=\mathbf{O},\quad\text{and}\quad \boldsymbol{\mu}=\mathbf{I}, \tag{12}$$

where **I** is the identity matrix and **O** is the null matrix.

Born[9] gave a simple expression to describe Faraday rotation of light when certain otherwise isotropic media are subjected to a magnetic field in the z direction. In our notation they reduce to

$$\boldsymbol{\rho}=\mathbf{O},\quad \boldsymbol{\varepsilon}=\begin{bmatrix}\epsilon & -i\gamma & 0\\ i\gamma & \epsilon & 0\\ 0 & 0 & \epsilon\end{bmatrix},\quad \boldsymbol{\rho}'=\mathbf{O},\quad\text{and}\quad \boldsymbol{\mu}=\mathbf{I}. \tag{13}$$

In this expression, γ is proportional to the applied magnetic field, and both ϵ and γ may be frequency dependent. (Born uses the symbol ϵ' for γ.)

In this paper, we shall pursue the solutions to Eq. (7) for the case of plane waves incident obliquely in the x,z plane on a region in which **M** is a function only of z. In that case, the component of the propagation vector in the x direction, ξ, is a constant and there is no y component. Solutions in this case have the common factor $\exp(i\xi x)$. The first quadrant of **R**, which is the **curl** operator, takes the special form

$$\mathbf{R}_1=\mathbf{curl}=\begin{bmatrix}0 & -\dfrac{\partial}{\partial z} & 0\\ \dfrac{\partial}{\partial z} & 0 & -i\xi\\ 0 & i\xi & 0\end{bmatrix}, \tag{14}$$

and the third quadrant $\mathbf{R}_3$ is the transpose of $\mathbf{R}_1$, which is $-$**curl**.

When Eq. (14) is substituted into the wave equation (7) the third and sixth rows are two linear algebraic equations in the six components of $\boldsymbol{\Gamma}$. They may be solved for two of the components in terms of the other four. We may thus eliminate these two variables and we are left with four first-order linear differential equations in the remaining four variables. We have chosen to eliminate $E_z=\Gamma_3$ and $H_z=\Gamma_6$. The four first-order linear differential equations from Eqs. (7) and (14) may be written as

$$-\frac{ic}{\omega}\frac{\partial}{\partial z}\Gamma_5=\sum_{j=1}^{6}M_{1j}\Gamma_j, \tag{15}$$

$$\frac{ic}{\omega}\frac{\partial}{\partial z}\Gamma_4=\sum_{j=1}^{6} M_{2j}\Gamma_j-\frac{c\xi}{\omega}\Gamma_6, \tag{16}$$

$$\frac{ic}{\omega}\frac{\partial}{\partial z}\Gamma_2=\sum_{j=1}^{6} M_{4j}\Gamma_j, \tag{17}$$

and

$$-\frac{ic}{\omega}\frac{\partial}{\partial z}\Gamma_1=\sum_{j=1}^{6} M_{5j}\Gamma_j+\frac{c\xi}{\omega}\Gamma_3. \tag{18}$$

The two linear algebraic equations are

$$-(c\xi/\omega)\Gamma_5=\sum_{j=1}^{6} M_{3j}\Gamma_j$$

and

$$(c\xi/\omega)\Gamma_2=\sum_{j=1}^{6} M_{6j}\Gamma_j. \tag{19}$$

These are to be solved for Γ_3 and Γ_6 in terms of the other variables. We get

$$\Gamma_3=a_{31}\Gamma_1+a_{32}\Gamma_2+a_{34}\Gamma_4+a_{35}\Gamma_5$$

and

$$\Gamma_6=a_{61}\Gamma_1+a_{62}\Gamma_2+a_{64}\Gamma_4+a_{65}\Gamma_5, \tag{20}$$

where

$$\begin{aligned}
a_{31}&=(M_{61}M_{36}-M_{31}M_{66})/d,\\
a_{32}&=[(M_{62}-c\xi/\omega)M_{36}-M_{32}M_{66}]/d,\\
a_{34}&=(M_{64}M_{36}-M_{34}M_{66})/d,\\
a_{35}&=[M_{65}M_{36}-(M_{35}+c\xi/\omega)M_{66}]/d,\\
a_{61}&=(M_{63}M_{31}-M_{33}M_{61})/d,\\
a_{62}&=[M_{63}M_{32}-M_{33}(M_{62}-c\xi/\omega)]/d,\\
a_{64}&=(M_{63}M_{34}-M_{33}M_{64})/d,\\
a_{65}&=[M_{63}(M_{35}+c\xi/\omega)-M_{33}M_{65}]/d,
\end{aligned} \tag{21}$$

in which

$$d=M_{33}M_{66}-M_{36}M_{63}.$$

By substituting Eqs. (20) and (21) into Eqs. (15)–(18) to eliminate Γ_3 and Γ_6, we get four first-order linear differential equations that may be written in 4×4-matrix form as

$$-\frac{ic}{\omega}\frac{\partial}{\partial z}\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}E_x\\E_y\\H_x\\H_y\end{bmatrix}=\mathbf{S}\begin{bmatrix}E_x\\E_y\\H_x\\H_y\end{bmatrix}. \tag{22}$$

The matrix on the left is its own inverse. Hence, multiplying both sides from the left by that matrix eliminates the matrix on the left. Then it is also convenient to change the order of the variables and the sign on H_x, so that we get an equation equivalent to Eq. (22), which is

$$\frac{\partial}{\partial z}\begin{bmatrix}E_x\\H_y\\E_y\\-H_x\end{bmatrix}=\frac{i\omega}{c}\begin{bmatrix}S_{41}&S_{44}&S_{42}&-S_{43}\\S_{11}&S_{14}&S_{12}&-S_{13}\\-S_{31}&-S_{34}&-S_{32}&S_{33}\\S_{21}&S_{24}&S_{22}&-S_{23}\end{bmatrix}\begin{bmatrix}E_x\\H_y\\E_y\\-H_x\end{bmatrix},$$

which will be abbreviated

$$\frac{\partial}{\partial z}\psi=\frac{i\omega}{c}\mathbf{\Delta}\psi. \tag{23}$$

[Some problems are more convenient to solve if the elements H_y and $-H_x$ in ψ are multiplied by i, and $\mathbf{\Delta}$ is changed accordingly.[4] This makes no significant difference in the theory but in some cases it results in real matrices where they would otherwise be complex. See Eq. (41).]

The elements of $\mathbf{S}$ and $\mathbf{\Delta}$ obtained from Eqs. (20), (21), and (15)–(18) are

$$\begin{aligned}
S_{11}&=\Delta_{21}=M_{11}+M_{13}a_{31}+M_{16}a_{61},\\
S_{12}&=\Delta_{23}=M_{12}+M_{13}a_{32}+M_{16}a_{62},\\
S_{13}&=-\Delta_{24}=M_{14}+M_{13}a_{34}+M_{16}a_{64},\\
S_{14}&=\Delta_{22}=M_{15}+M_{13}a_{35}+M_{16}a_{65},\\
S_{21}&=\Delta_{41}=M_{21}+M_{23}a_{31}+(M_{26}-c\xi/\omega)a_{61},\\
S_{22}&=\Delta_{43}=M_{22}+M_{23}a_{32}+(M_{26}-c\xi/\omega)a_{62},\\
S_{23}&=-\Delta_{44}=M_{24}+M_{23}a_{34}+(M_{26}-c\xi/\omega)a_{64},\\
S_{24}&=\Delta_{42}=M_{25}+M_{23}a_{35}+(M_{26}-c\xi/\omega)a_{65},\\
S_{31}&=-\Delta_{31}=M_{41}+M_{43}a_{31}+M_{46}a_{61},\\
S_{32}&=-\Delta_{33}=M_{42}+M_{43}a_{32}+M_{46}a_{62},\\
S_{33}&=\Delta_{34}=M_{44}+M_{43}a_{34}+M_{46}a_{64},\\
S_{34}&=-\Delta_{32}=M_{45}+M_{43}a_{35}+M_{46}a_{65},\\
S_{41}&=\Delta_{11}=M_{51}+(M_{53}+c\xi/\omega)a_{31}+M_{56}a_{61},\\
S_{42}&=\Delta_{13}=M_{52}+(M_{53}+c\xi/\omega)a_{32}+M_{56}a_{62},\\
S_{43}&=-\Delta_{14}=M_{54}+(M_{53}+c\xi/\omega)a_{34}+M_{56}a_{64},\\
S_{44}&=\Delta_{12}=M_{55}+(M_{53}+c\xi/\omega)a_{35}+M_{56}a_{65}.
\end{aligned} \tag{24}$$

It is not difficult to verify that if $\mathbf{M}$ is symmetric and if its first and third quadrants are zero, then $\mathbf{S}$ is also symmetric. Hence $\mathbf{S}$ is symmetric in ordinary media that are free of optical activity and Faraday rotation.

If the matrix $\mathbf{\Delta}$ is approximately independent of z over some short interval δz, then there will be four periodic solutions of Eq. (23), which are of the form

$$\psi_j(\delta z)=e^{iq_j\delta z}\psi_j(0). \tag{25}$$

The four eigenvalues of q_j may be obtained analytically by solving the quartic polynomial equation in q resulting from expanding the determinant in the secular equation, which is derived by substituting Eq. (25) into Eq. (23),

$$\mathrm{Det}\left(\frac{\omega}{c}\boldsymbol{\Delta}-q\mathbf{I}\right)=0. \tag{26}$$

In this secular equation, $\mathbf{I}$ is the 4×4 identity matrix. Eigenvectors $\boldsymbol{\psi}_j(0)$ may then be found for each eigenvalue by solving three of the four simultaneous equations represented by the matrix equation

$$\frac{\omega}{c}\boldsymbol{\Delta}\,\boldsymbol{\psi}_j(0)=q_j\boldsymbol{\psi}_j(0). \tag{27}$$

If the matrix $\boldsymbol{\Delta}$ is independent of z over some finite distance h in the direction of the z axis, then Eq. (23) may be integrated to give

$$\boldsymbol{\psi}(z+h)=\mathbf{P}(h)\boldsymbol{\psi}(z). \tag{28}$$

A closed expression for $\mathbf{P}(h)$ can always be found. First, solve Eq. (26) for the four eigenvalues of q. Then insert each q_j into Eq. (27) to find an eigenvector $\boldsymbol{\psi}_j$. Make a 4×4 matrix $\boldsymbol{\Psi}$ with elements ψ_{ij} out of the four eigenvectors so obtained. Then notice that Eq. (28) can be applied to $\boldsymbol{\Psi}$ to give

$$\boldsymbol{\Psi}\mathbf{K}(h)=\mathbf{P}(h)\boldsymbol{\Psi}, \tag{29}$$

where $\mathbf{K}(h)$ is a diagonal matrix with elements

$$K_{jj}=\exp(iq_jh). \tag{30}$$

Equation (24) may be solved for $\mathbf{P}(h)$ to give

$$\mathbf{P}(h)=\boldsymbol{\Psi}\mathbf{K}(h)\boldsymbol{\Psi}^{-1}. \tag{31}$$

An alternative method for finding $\mathbf{P}(h)$ is simply to integrate Eq. (23). The solution is

$$\begin{aligned}\boldsymbol{\psi}(z+h)&=\mathbf{P}(h)\,\boldsymbol{\psi}(z)=\exp(i\omega h\boldsymbol{\Delta}/c)\,\boldsymbol{\psi}(z)\\&=\left[\mathbf{I}+i\frac{\omega h}{c}\boldsymbol{\Delta}-\left(\frac{\omega h}{c}\right)^2\boldsymbol{\Delta}:\boldsymbol{\Delta}/2!-\cdots\right]\boldsymbol{\psi}(z).\end{aligned} \tag{32}$$

In some simple cases, a general expression for $(\boldsymbol{\Delta})^n$ can be written down, in which case it may be possible to recognize a closed exponential expression for each of the 16 terms in $\mathbf{P}(h)$. However, this series solution is useful mainly for numerical problems in which $\omega h/c$ is small enough that the first few terms of the series give sufficient accuracy.

Certain useful symmetry properties of $\mathbf{P}(h)$ are evident from Eq. (32). In particular,

$$\mathbf{P}(mh)=[\mathbf{P}(h)]^m, \tag{33}$$

where m is any positive or negative integer or zero. A secular equation equivalent to Eq. (26) is

$$\mathrm{Det}[\mathbf{P}(h)-e^{iq_jh}\mathbf{I}]=0. \tag{34}$$

This equation is not often useful in itself, because eigenvalues may be obtained directly from Eq. (26), but we put it here for comparison with later results for media with variable $\mathbf{M}$.

EXAMPLES

An example of a problem that can be solved exactly for $\mathbf{P}(h)$ using Eq. (32) is the case of light propagated in a slab of orthorhombic crystal with its principal axes parallel to the x, y, and z coordinate axes defined just before Eq. (14). For such a crystal, $\mathbf{M}$ is a constant, diagonal matrix. Equations (20) and (21) give only two nonzero coefficients

$$a_{35}=-(c\xi/\omega)/M_{33}$$

and

$$a_{62}=(c\xi/\omega)/M_{66}. \tag{35}$$

Then Eqs. (24) give only four nonzero elements

$$\begin{aligned}&\Delta_{21}=S_{11}=M_{11}=\epsilon_{11},\\&\Delta_{43}=S_{22}=M_{22}-(c\xi/\omega)^2/M_{66}=\epsilon_{22}-(c\xi/\omega)^2/\mu_{33},\\&\Delta_{34}=S_{33}=M_{44}=\mu_{11},\\&\Delta_{12}=S_{44}=M_{55}-(c\xi/\omega)^2/M_{33}=\mu_{22}-(c\xi/\omega)^2/\epsilon_{33}.\end{aligned} \tag{36}$$

For brevity, we may write the $\boldsymbol{\Delta}$ matrix from Eq. (36) as

$$\boldsymbol{\Delta}=\begin{bmatrix}0 & a^2 & 0 & 0\\ b^2 & 0 & 0 & 0\\ 0 & 0 & 0 & u^2\\ 0 & 0 & v^2 & 0\end{bmatrix}. \tag{37}$$

Hence

$$\boldsymbol{\Delta}^2=\begin{bmatrix}a^2b^2 & 0 & 0 & 0\\ 0 & a^2b^2 & 0 & 0\\ 0 & 0 & u^2v^2 & 0\\ 0 & 0 & 0 & u^2v^2\end{bmatrix}. \tag{38}$$

It is easy to write a general expression for $\boldsymbol{\Delta}^n$ in this case. For example,

$$(\Delta^n)_{11}=\begin{cases}0 & \text{for } n \text{ odd}\\ (ab)^n & \text{for } n \text{ even}\end{cases} \tag{39}$$

and

$$(\Delta^n)_{12}=\begin{cases}(ab)^n a/b & \text{for } n \text{ odd}\\ 0 & \text{for } n \text{ even.}\end{cases} \tag{40}$$

These and similar relations together with Eq. (32) show that

$$\mathbf{P}(h)=\begin{bmatrix}\cos x_1 & (ia/b)\sin x_1 & 0 & 0\\ (ib/a)\sin x_1 & \cos x_1 & 0 & 0\\ 0 & 0 & \cos x_2 & (iu/v)\sin x_2\\ 0 & 0 & (iv/u)\sin x_2 & \cos x_2\end{bmatrix}, \tag{41}$$

where

$$x_1 = ab\omega h/c, \qquad x_2 = uv\omega h/c.$$

[$\mathbf{P}(h)$ would have been real if ψ had been defined as suggested below Eq. (23), and if a, b, u, and v were real.]

The eigenvalues of the secular equation (26) for the case where $\mathbf{\Delta}$ is given by Eq. (41) are

$$q_j = \pm ab\omega/c \quad \text{and} \quad \pm uv\omega/c. \tag{42}$$

The eigenvectors can be found by substituting these eigenvalues into Eq. (27). The eigenvectors corresponding to the first pair of roots are the π-polarized plane waves

$$\psi_\pi = E_x e^{i(ab\omega/c)z}\begin{bmatrix}1\\ b/a\\ 0\\ 0\end{bmatrix}$$

and (43)

$$E_x e^{-i(ab\omega/c)z}\begin{bmatrix}1\\ -b/a\\ 0\\ 0\end{bmatrix}.$$

The other two are the σ-polarized plane waves

$$\psi_\sigma = E_y e^{i(uv\omega/c)z}\begin{bmatrix}0\\ 0\\ 1\\ v/u\end{bmatrix}$$

and (44)

$$E_y e^{-i(uv\omega/c)z}\begin{bmatrix}0\\ 0\\ 1\\ -v/u\end{bmatrix}.$$

It may be noted that the same eigenvalues satisfy Eq. (34) and the same eigenvectors satisfy Eq. (32).

Now consider Drude's model of an optically active isotropic substance, for which Eqs. (12) describe the $\mathbf{M}$ matrix. For brevity, we shall define

$$X \equiv c\xi/\omega \tag{45}$$

and

$$g \equiv \omega\gamma/c. \tag{46}$$

Then Eqs. (20), (21), and (24) yield

$$\mathbf{\Delta} = \begin{bmatrix}0 & 1-X^2/\epsilon & igX^2/\epsilon & 0\\ \epsilon & 0 & 0 & ig\\ 0 & 0 & 0 & 1\\ 0 & -ig & \epsilon - X^2 & 0\end{bmatrix}. \tag{47}$$

The power-series method of finding $\mathbf{P}(h)$ is impractical in this case, because the general expression for $\mathbf{\Delta}^n$ is very complicated. The eigenvalues of Eq. (26) are easy to find, however. The secular equation may be written

$$(cq/\omega)^4 - 2a_2(cq/\omega)^2 + a_4 = 0, \tag{48}$$

where

$$a_2 = \epsilon - X^2 + g^2/2 \tag{49}$$

and

$$a_4 = (\epsilon - X^2)^2 - g^2X^2/\epsilon. \tag{50}$$

The roots are

$$q_j = \pm\frac{\omega}{c}[a_2 \pm (a_2^2 - a_4)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{51}$$

The eigenvector problem is particularly simple when X is zero, that is, when propagation is in the z direction. Then the eigenvalues of q are

$$q_{1,2,3,4} = \pm\omega r_1/c, \quad \pm\omega r_2/c, \tag{52}$$

where

$$r_1 = [\epsilon + (g/2)]^{\frac{1}{2}} - g/2 \tag{53}$$

and

$$r_2 = [\epsilon + (g/2)]^{\frac{1}{2}} + g/2. \tag{54}$$

The eigenvectors are the four columns of the matrix

$$\mathbf{\Psi} = \begin{bmatrix}1 & 1 & 1 & 1\\ r_1 & -r_1 & r_2 & -r_2\\ i & -i & i & -i\\ ir_1 & ir_1 & ir_2 & ir_2\end{bmatrix}. \tag{55}$$

The first-column eigenvector in Eq. (55) represents left-circularly polarized light moving in the positive z direction and the second, left-circularly polarized light moving at the same speed in the negative z direction. The third represents right-circularly polarized light moving in the positive z direction with a different speed, and the fourth is right-circularly polarized light moving at that speed in the negative z direction. By right-circularly polarized, we mean light whose electric-field vector forms a right-handed helix at any instant in time.

Note that the diagonal elements of $\mathbf{K}$ in Eq. (30) can be obtained from Eq. (52). By inverting $\mathbf{\Psi}$ from Eq. (55) and then doing the matrix multiplication indicated in Eq. (30) we may obtain an exact expression for $\mathbf{P}(h)$, relating any electromagnetic-field vector ψ at one position in such a medium with that at a distance h along the z axis from that position. Although the eigenvalues are easy to compute from Eq. (50), the eigenvectors are much more complicated for light moving oblique to the z axis because right- and left-circularly polarized components are not propagated in the same direction when they have the same x component of propagation, ξ.

As another example, consider Born's model of Faraday rotation, for which Eqs. (13) define the $\mathbf{M}$ matrix. Equations (20), (21), and (24) yield

$$\mathbf{\Delta} = \begin{bmatrix}0 & 1-X^2/\epsilon & 0 & 0\\ \epsilon & 0 & -i\gamma & 0\\ 0 & 0 & 0 & 1\\ i\gamma & 0 & \epsilon - X^2 & 0\end{bmatrix}, \tag{56}$$

where X is defined by Eq. (45). The secular equation is again of the form shown in Eq. (48) with roots shown by Eq. (51), but in this case

$$a_2 = \epsilon - X^2 \tag{57}$$

and

$$a_4 = (\epsilon - X^2)(\epsilon - X^2 - \gamma^2/\epsilon). \tag{58}$$

Here again, the solution is quite complicated except when X is zero, in which case the four roots are

$$q_{1,2,3,4}=\pm(\epsilon+\gamma)^{\frac{1}{2}}\omega/c, \quad \pm(\epsilon-\gamma)^{\frac{1}{2}}\omega/c, \tag{59}$$

and the corresponding eigenvectors are the columns of the matrix

$$\mathbf{\Psi}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ (\epsilon+\gamma)^{\frac{1}{2}} & -(\epsilon+\gamma)^{\frac{1}{2}} & (\epsilon-\gamma)^{\frac{1}{2}} & -(\epsilon-\gamma)^{\frac{1}{2}} \\ i & i & -i & -i \\ i(\epsilon+\gamma)^{\frac{1}{2}} & -i(\epsilon+\gamma)^{\frac{1}{2}} & -i(\epsilon-\gamma)^{\frac{1}{2}} & i(\epsilon-\gamma)^{\frac{1}{2}} \end{bmatrix}. \tag{60}$$

Notice that for Faraday rotation, the first-column eigenvector represents left-circularly polarized light traveling in the positive z direction, as in the matrix for optically active material, but the second represents right-circularly polarized light moving in the negative z direction with the same speed, unlike the case for optically active media, for which the same speed was always associated with the same sign of helicity.

There are a few cases in which exact analytic solutions can be found for problems in which $\mathbf{M}$ is periodic. One such example is that of a single domain of cholesteric liquid illuminated through a flat surface normal to its axis of helicity.[10–12] Although there must be a small amount of ordinary optical activity in such media, the effect of that activity is generally negligible compared to other optical effects. A simple optical model of such a liquid crystal, in which ordinary optical activity is neglected, was first proposed by Oseen.[10] The model can be described by unit permeability μ, zero optical activity ρ and ρ', and the dielectric tensor

$$\boldsymbol{\varepsilon}=\begin{bmatrix} \epsilon+\delta\cos 2\beta z & \delta\sin 2\beta z & 0 \\ \delta\sin 2\beta z & \epsilon-\delta\cos 2\beta z & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}. \tag{61}$$

The dielectric ellipsoid representing this tensor has two principal axes normal to the z axis. The axis of length $(\epsilon+\delta)$ is in the x direction at $z=0$, and it spirals with a pitch

$$\lambda=2\pi/\beta$$

as the value of z is increased.

The differential propagation matrix computed from Eqs. (20), (24), and (61) is

$$\mathbf{\Delta}(z)=\begin{bmatrix} 0 & 1-X^2/\epsilon_3 & 0 & 0 \\ \epsilon+\delta\cos 2\beta z & 0 & \delta\sin 2\beta z & 0 \\ 0 & 0 & 0 & 1 \\ \delta\sin 2\beta z & 0 & \epsilon-X^2-\delta\cos 2\beta z & 0 \end{bmatrix}, \tag{62}$$

where X is defined in Eq. (45).

Numerical solutions have been obtained when X is not zero.[4,5,7] More recently, analytical solutions by another method have also been reported.[6] The results are very complicated. Much earlier, Oseen[10] and DeVries[11] obtained simple analytic solutions for the case of normal incidence, when X is zero. More recently Marathay[12] used the 2×2-matrix method to obtain an analytic solution when X is zero. We believe it is instructive to obtain the simple analytic solutions with $X=0$ using our 4×4-matrix notation. When X is zero, it is reasonably evident from symmetry consideration that eigenvectors $\boldsymbol{\psi}(z)$ should have a form such that E_x and H_y are the same at $z=0$ as are E_y and $-H_x$ at $z=\lambda/4$, except for a phase factor. This spiral symmetry is characterized by eigenvectors

$$\boldsymbol{\psi}=e^{iqz}\begin{bmatrix} Ae^{i\beta z}+Be^{-i\beta z} \\ A'e^{i\beta z}+B'e^{-i\beta z} \\ Ae^{i\beta(z-\pi/2\beta)}+Be^{-i\beta(z-\pi/2\beta)} \\ A'e^{i\beta(z-\pi/2\beta)}+B'e^{-i\beta(z-\pi/2\beta)} \end{bmatrix} = e^{i(q+\beta)z}\begin{bmatrix} A \\ A' \\ -iA \\ -iA' \end{bmatrix} + e^{i(q-\beta)z}\begin{bmatrix} B \\ B' \\ iB \\ iB' \end{bmatrix}, \tag{63}$$

in which the A's and B's and eigenvalues of the propagation phase factor q are to be determined. By matrix multiplication when X is zero, we get

$$\mathbf{\Delta}\boldsymbol{\psi}=e^{i(q+\beta)z}\begin{bmatrix} A' \\ \epsilon A+\delta B \\ -iA' \\ -i(\epsilon A+\delta B) \end{bmatrix} + e^{i(q-\beta)z}\begin{bmatrix} B' \\ \epsilon B+\delta A \\ iB' \\ i(\epsilon B+\delta A) \end{bmatrix}. \tag{64}$$

By differentiation, we get

$$-i\frac{c}{\omega}\frac{\partial\boldsymbol{\psi}}{\partial z}=e^{i(q+\beta)z}\begin{bmatrix} A \\ A' \\ -iA \\ -iA' \end{bmatrix}\frac{c}{\omega}(q+\beta) + e^{i(q-\beta)z}\begin{bmatrix} B \\ B' \\ iB \\ iB' \end{bmatrix}\frac{c}{\omega}(q-\beta). \tag{65}$$

Equation (23) shows that the left-hand sides of the two preceding equations are equal.

Equating coefficients of like terms, we get the following four equations (actually the equations are repeated twice because, on the basis of symmetry, we put some information in the form of the assumed eigenvector that could have been obtained by using a less specific form):

$$A'=(q+\beta)Ac/\omega, \tag{66}$$

$$B'=(q-\beta)Bc/\omega, \tag{67}$$

$$\epsilon A+\delta B=(q+\beta)A'c/\omega=[(q+\beta)c/\omega]^2A, \tag{68}$$

$$\epsilon B+\delta A=(q-\beta)B'c/\omega=[(q-\beta)c/\omega]^2B. \tag{69}$$

The last two equations have nontrivial solutions only when q has one of the four eigenvalues

$$q=\pm\{\beta^2\pm 2\beta(\omega/c)[\epsilon+(\delta^2/\epsilon)(\omega/c)^2/4\beta^2]^{\frac{1}{2}} +(\omega/c)^2\epsilon\}^{\frac{1}{2}}. \tag{70}$$

If an arbitrary value is assigned to either A or B, the remaining one may be evaluated with Eq. (68) or (69). Then A' and B' can be evaluated with Eqs. (66) and (67).

For small values of ω, the eigenvectors corresponding to the eigenvalues of q obtained when X is zero represent approximately circularly polarized light and the material acts in some ways like an optically active substance. However, when

$$\omega \approx c\beta/\sqrt{\epsilon}, \tag{71}$$

that is, when the wavelength in the medium is near half the pitch length, the eigenvectors are no longer approximately circularly polarized. Moreover, one pair of the eigenvalues becomes imaginary over a band near that frequency. Hence two of the four eigenvectors are attenuated modes. If ϵ is real, which is practically true for many cholesteric liquid crystals, the part of the light that excites an attenuated mode is totally reflected if the sample is thick and perfectly ordered. The light in the other mode is transmitted through the crystal. This band of reflection is similar to the Bragg-reflection band for x rays in crystals, but it occurs at optical frequencies and causes a brightly colored appearance in many cholesteric liquid crystals. Although there is only a single reflection band for light at normal incidence, the band breaks up for obliquely incident light. Other weak bands near higher integral multiples of the frequency of the fundamental band also appear when X is nonzero.[4–7]

NUMERICAL SOLUTIONS WHEN M DEPENDS ON z

We originally developed the technique presented here for the purpose of solving problems in which $\mathbf{M}$ is a periodic function of z, rather than a constant. We were particularly interested in the optical properties of single-domain cholesteric liquid-crystal films.[4,7] The technique can be applied to any medium in which $\mathbf{M}$ depends on z, however.

Equations (1)–(7) and (14)–(24) are valid whether or not $\mathbf{M}$ depends on z. Equations (2)–(34) also hold in the limit of small values of h, so long as $\mathbf{M}$ is a continuous function of z.

When $\mathbf{M}$ is variable, the matrix $\mathbf{P}$, which relates ψ at two values of z separated by h, will be written as $\mathbf{P}(z,h)$, where z is the point at which $\mathbf{\Delta}$ is to be evaluated and h is a distance sufficiently small that $\mathbf{\Delta}(z+h)$ differs from $\mathbf{\Delta}(z)$ by a negligible amount. Because Eq. (28) applies over each such interval, we may use the equations repeatedly to obtain the relation between vectors ψ separated by a finite distance s:

$$\begin{aligned}\psi(z+s) &\equiv \mathbf{F}(z,s)\psi(z)\\ &\approx \mathbf{P}(z+s-h_m, h_m)\cdots\mathbf{P}(z+h_1+h_2, h_3)\\ &\qquad\times\mathbf{P}(z+h_1, h_2)\mathbf{P}(z,h_1)\psi(z),\end{aligned} \tag{72}$$

where s has been subdivided into intervals $h_1, h_2, \ldots, h_m$.

In the case where $\mathbf{\Delta}$ is a continuous function of z, Eq. (72) gives better and better approximations to $\mathbf{F}$ as the intervals h_j become shorter. Since $\mathbf{P}$ is much more difficult to determine than $\mathbf{\Delta}$ in the general case, one obvious method of finding $\mathbf{F}$ is to approximate each value of $\mathbf{P}$ by using only the first few terms of the power series in Eq. (32), which converges very rapidly if h is small. It is not hard to show that it is necessary and sufficient to use the second-order term to get a converging result from Eq. (72). In the general case, if only the first order in $\mathbf{\Delta}$ is used, a finite cumulative error occurs in Eq. (72) even with infinitesimal intervals h_j.

By setting all values of h alike and using Eq. (33) with $m=-1$, we may change Eq. (72) to

$$\begin{aligned}\mathbf{F}(z,s) \approx \mathbf{P}(z+s-h, h)\mathbf{P}^{-1}(z+s-h, -h)\cdots\\ \mathbf{P}(z+h, h)\mathbf{P}^{-1}(z+h, -h).\end{aligned} \tag{73}$$

This alternating-product form is convergent even if only the first-order approximation to $\mathbf{P}$ is used. The exact inverse of the approximation to $\mathbf{P}$, and not just a first-order approximation to the inverse, must be used in Eq. (73).

In the Oseen model of a cholesteric liquid crystal without optical absorption, the inverse of the first-order approximation to $\mathbf{P}$ has the same elements as $\mathbf{P}$ itself, but they are switched around in a particular pattern. In cases as simple as that, Eq. (73) provides an efficient and accurate way to evaluate $\mathbf{F}$ if values of h are kept quite small. In more-complicated cases, it is more efficient to use Eq. (72) with somewhat larger values of h, and to evaluate enough terms in the series in Eq. (32) that residual terms are negligible. The accuracy of computations with Eqs. (32) and (72) is increased somewhat if the matrix $\mathbf{M}$ is evaluated at the center of each interval h, rather than at one end.

Another useful approach is to write $\mathbf{P}$ as the sum of two terms, one of which, $\mathbf{P}_0$, describes the effect of a constant average value of $\mathbf{M}$ exactly through the whole path, and hence is always the same for a series of equal intervals, and the other of which corrects $\mathbf{P}$ for the difference between the local value of $\mathbf{M}$ and its average value. In our study of cholesteric liquid crystals[4,7] we found this method to be satisfactory even when the correction term was only a first-order approximation and when Eq. (72) was used. However, we ultimately used this approximation together with Eq. (73) in order to obtain accurate results with fewer steps. The exact expression we used for $\mathbf{P}_0$ in that work is given by Eq. (41). Equation (41) is valid if the effect of optical activity is negligible, which we found it to be.

If $\mathbf{M}$ is an arbitrary function of z, either continuous or discontinuous, Eq. (72) or (73) gives the basic technique for writing the relation between the electromagnetic-field variables at the two ends of an interval.

If $\mathbf{M}$ is a periodic function of z, as it is in a cholesteric liquid crystal, many interference filters and certain biological materials, Eq. (72) or (73) need be used over only one period or cycle. If we define the product of matrices in Eq. (72) over one cycle of length λ as $\mathbf{F}_1$, and if there are m cycles, then

$$\psi(z+m\lambda)=\mathbf{F}_1{}^m\psi(z)=\mathbf{F}(z,m\lambda)\psi(z). \tag{74}$$

In such a periodic medium, there are four normal modes of propagation, as was the case in a medium with constant $\mathbf{M}$. These modes may be defined by the eigenvalue equation

$$\mathbf{F}_1\psi_j(z)=e^{iQ_j\lambda}\psi_j(z)=\psi_j(z+\lambda). \tag{75}$$

The eigenvalues of Q_j may be determined by solving the secular equation

$$\mathrm{Det}(\mathbf{F}_1-e^{iQ_j\lambda}\mathbf{I})=0. \tag{76}$$

Equations (75) and (76) are equivalent to Eqs. (32) and (34) in media in which $\mathbf{M}$ is invariant, so that the periodic length λ is an arbitrary number. In a given periodic medium, the eigenvalues Q_j are unique. However, the eigenvectors and the matrix $\mathbf{F}_1$ depend on the starting point z at which the series in Eq. (72) is initiated. If a slab of such a periodic medium did not have a whole number of cycles, the residual part of a cycle would have to be included as a separate factor in $\mathbf{F}(z,s)$.

REFLECTANCE AND TRANSMITTANCE IN FINITE SLABS BETWEEN ISOTROPIC MEDIA

The 4×4-matrix method for finding reflectance and transmittance of a finite slab is analogous to the method introduced by Jones[1] for slabs with normally incident light using 2×2 matrices, but boundary conditions are defined in terms of electric and magnetic field components and involve no derivatives.

Suppose we have a slab or layer of material of thickness s. Let z be zero at the first surface and s at the second. Then, in general, the two electric and two magnetic-field components at and parallel to the first surface, defining $\psi(0)$, are related to those at the second by the matrix equation

$$\psi(s)=\mathbf{F}(0,s)\psi(0), \tag{72'}$$

where $\mathbf{F}$ is defined by Eq. (72) in general, or by Eq. (73) for periodic media with an integral number of cycles, or it equals $\mathbf{P}(s)$ for a medium in which $\mathbf{M}$ is invariant.

An equivalent alternative formulation is

$$\psi(0)=\mathbf{F}^{-1}(0,s)\psi(s)=\mathbf{F}(s,-s)\psi(s). \tag{77}$$

[Compare Eq. (33) when m is a negative integer.]

The field vector at the first surface is made up of two parts, the incident- and the reflected-wave contributions,

$$\psi(0)=\psi_i+\psi_r. \tag{78}$$

The field at the second surface matches only a single transmitted wave field

$$\psi(s)=\psi_t. \tag{79}$$

Suppose that the incident beam is a plane wave of frequency ω and its direction of propagation in the first medium is at an angle θ_1 from the normal to the surface of the slab. Let the first medium be nonabsorbing and let it have refractive index n_1. Only the two electric-field components are needed to complete the definition of the incident wave, since the magnetic-field components can be defined easily in terms of the electric components in isotropic media. We shall write

$$\psi_i=\begin{bmatrix} E_x \\ r_xE_x \\ E_y \\ r_yE_y \end{bmatrix}. \tag{80}$$

It is easy to show that

$$r_x=n_1/\cos\theta_1 \quad \text{and} \quad r_y=n_1\cos\theta_1. \tag{81}$$

We shall write the reflected and transmitted waves as

$$\psi_r=\begin{bmatrix} R_x \\ -r_xR_x \\ R_y \\ -r_yR_y \end{bmatrix},\quad \psi_t=\begin{bmatrix} T_x \\ r_x'T_x \\ T_y \\ r_y'T_y \end{bmatrix}, \tag{82}$$

where r_x' and r_y' are defined by equations similar to Eq. (81) with the refractive index n_2 of the second medium in place of n_1. From Snell's law, the angle θ_2 replacing θ_1 is

$$\theta_2=\sin^{-1}[(n_1/n_2)\sin\theta_1]. \tag{83}$$

The values of X and ξ are also defined in terms of the incident-wave frequency and direction by

$$\xi=n_1\sin\theta_1=X\omega/c. \tag{84}$$

Now let F_{ij} represent the elements of the matrix $\mathbf{F}(s,-s)$ in Eq. (77). Then we may expand that matrix equation into four linear equations:

$$\begin{aligned}
E_x+R_x&=(F_{11}+F_{12}r_x')T_x+(F_{13}+F_{14}r_y')T_y \\
&\equiv 2(g_{11}T_x+g_{12}T_y),\\
r_x(E_x-R_x)&=(F_{21}+F_{22}r_x')T_x+(F_{23}+F_{24}r_y')T_y \\
&\equiv 2(g_{21}T_x+g_{22}T_y)r_x,\\
E_y+R_y&=(F_{31}+F_{32}r_x')T_x+(F_{33}+F_{34}r_y')T_y \\
&\equiv 2(g_{31}T_x+g_{32}T_y),\\
r_y(E_y-R_y)&=(F_{41}+F_{42}r_x')T_x+(F_{43}+F_{44}r_y')T_y \\
&\equiv 2(g_{41}T_x+g_{42}T_y)r_y,
\end{aligned} \tag{85}$$

where the elements g_{ij} are implicitly defined. If we define

$$D=(g_{11}+g_{21})(g_{32}+g_{42})-(g_{21}+g_{22})(g_{31}+g_{41}), \tag{86}$$

then the solutions to these equations are

$$T_x = [(g_{32}+g_{42})E_x - (g_{12}+g_{22})E_y]/D, \quad (87)$$

$$T_y = [-(g_{31}+g_{41})E_x + (g_{11}+g_{21})E_y]/D, \quad (88)$$

$$R_x = (g_{11}-g_{21})T_x + (g_{12}-g_{22})T_y, \quad (89)$$

and

$$R_y = (g_{31}-g_{41})T_x + (g_{32}-g_{42})T_y. \quad (90)$$

When the thickness s of the slab is reduced to zero, so that **F** becomes the identity matrix, Eqs. (87)–(90) reduce to Fresnel's transmission- and reflection-amplitude equations for the interface between media of refractive index n_1 and n_2.[13]

REFLECTANCE AND TRANSMITTANCE BY A SEMI-INFINITE MEDIUM IN AN ISOTROPIC MEDIUM

There is no general solution of the problem of reflectance or transmittance by a semi-infinite medium in which **M** depends in an arbitrary way on z, because every variation of **M** contributes to the reflectance. However, if the dependence of **M** on z is periodic or if **M** does not depend on z, then solutions are not hard to obtain. We may treat the case where **M** is constant as a special case of the more-general problem in which **M** is periodic.

If **M** is periodic with period λ, eigenvalues and eigenvectors of the propagating electromagnetic fields are determined by Eqs. (74) and (75). We shall set $z=0$ at the interface between the isotropic and the periodic medium and define $\mathbf{F}_1$ as the matrix relating the field at a distance of one period into the medium, $\psi(\lambda)$, to that at the surface, $\psi(0)$. From the four eigenvalues of Q_j given by the secular Eq. (75) we select those two that have positive imaginary parts if they are complex, or positive real parts if they are real, so that they correspond to modes that originate at $z=0$. We find the two eigenvectors ψ_1 and ψ_2 corresponding to the two selected eigenvalues Q_1 and Q_2 by solving the simultaneous equations from Eq. (74).

Equation (78) holds adjacent to the interface in the isotropic medium, just as it did in the problem of the finite slab. The electromagnetic boundary conditions require that

$$\psi_i + \psi_r = A_1\psi_1 + A_2\psi_2, \quad (91)$$

where A_1 and A_2 are complex constants that define the amplitudes of the two transmitted modes. Again we have four equations and four unknowns, R_x, R_y, A_1, and A_2, which may be written

$$\begin{aligned} E_x + R_x &= A_1\psi_{11} + A_2\psi_{12} = 2(g_{11}A_1 + g_{12}A_2), \\ r_x(E_x - R_x) &= A_1\psi_{21} + A_2\psi_{22} = 2(g_{21}A_1 + g_{22}A_2)r_x, \\ E_y + R_y &= A_1\psi_{31} + A_2\psi_{32} = 2(g_{31}A_1 + g_{32}A_2), \\ r_y(E_y - R_y) &= A_1\psi_{41} + A_2\psi_{42} = 2(g_{41}A_1 + g_{42}A_2)r_y, \end{aligned} \quad (92)$$

where the variables g_{ij} are implicitly defined. Equations (92) are identical in form to Eqs. (85) but A_1 replaces T_x and A_2 replaces T_y. With these two replacements, Eqs. (86)–(90) give the solutions for the reflected- and transmitted-wave amplitudes.

The problem for media in which **M** is independent of z can be solved in exactly the same way except that the choice of λ is arbitrary. The problem may be simplified by allowing λ to be infinitesimal and using Eqs. (25) and (26) rather than Eqs. (84) and (75) or their equivalents for isotropic media, Eqs. (32) and (34). The simplification is that solutions can be found by evaluating $\mathbf{\Delta}$ without evaluating its exponential $\mathbf{P}(\lambda)$, which is a difficult problem in many cases. [See Eqs. (31) and (32).]

We wish to reiterate that Eq. (73) is valid only when m is an integer in a periodic medium. If the electromagnetic-field vector components at some intermediate location $(m\lambda+h)$ are needed, they can be determined by finding ψ at $z=m\lambda$ and then multiplying that electromagnetic-field vector by $\mathbf{F}(m\lambda,h)$, which would be evaluated using Eq. (72).

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