Chapter 16 Waves I

In this chapter we will start the discussion on wave phenomena. We will study the following topics:

Types of waves
Amplitude, phase, frequency, period, propagation speed of a wave
Mechanical waves propagating along a stretched string
Wave equation
Principle of superposition of waves
Wave interference
Standing waves, resonance

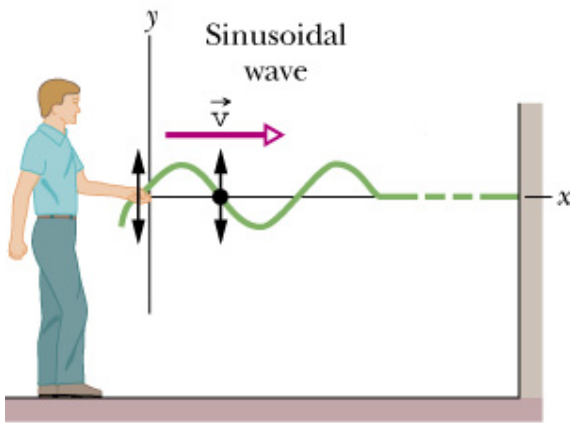
A **wave** is defined as a disturbance that is self-sustained and propagates in space with a constant speed

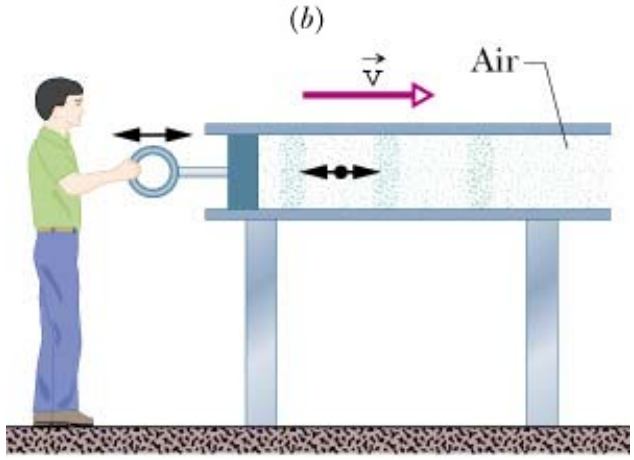
Waves can be classified in the following three categories:

1. Mechanical waves. These involve motions that are governed by Newton's laws and can exist only within a material medium such as air, water, rock, etc. Common examples are: sound waves, seismic waves, etc.

2. Electromagnetic waves. These waves involve propagating disturbances in the electric and magnetic field governed by Maxwell's equations. They do not require a material medium in which to propagate but they travel through vacuum. Common examples are: radio waves of all types, visible, infra-red, and ultra-violet light, x-rays, gamma rays. All electromagnetic waves propagate in vacuum with the same speed $c = 300{,}000$ km/s

3. Matter waves. All microscopic particles such as electrons, protons, neutrons, atoms etc have a wave associated with them governed by Schroedinger's equation.





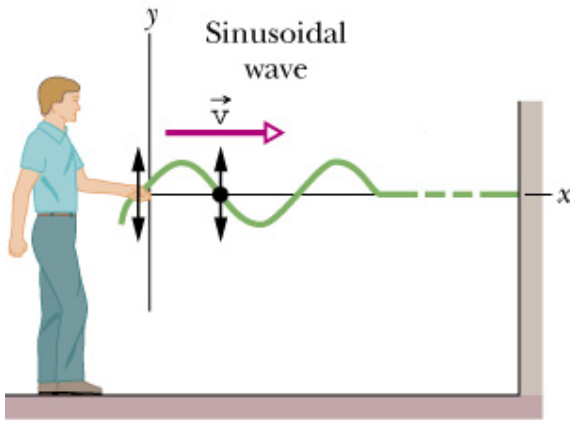
Transverse and Longitudinal waves (16 – 3)

Waves can be divided into the following two categories depending on the orientation of the disturbance with respect to the wave propagation velocity $\vec{v}$.

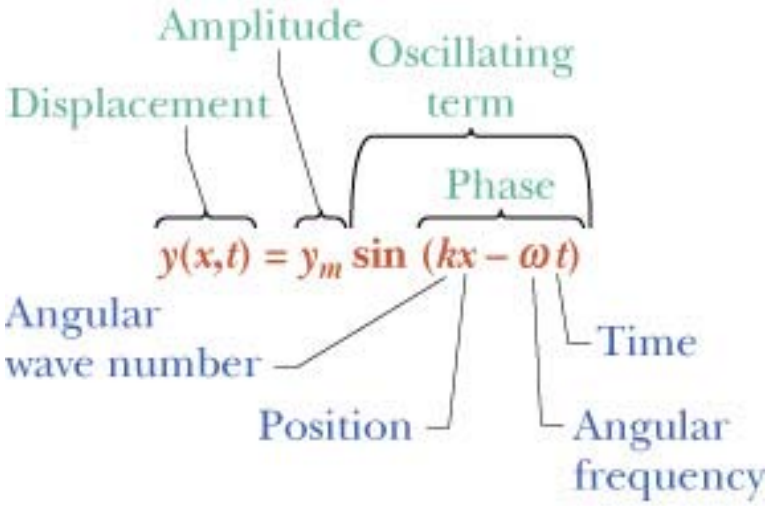
If the disturbance associated with a particular wave is perpendicular to the wave propagation velocity, this wave is called "***transverse***". An example is given in the upper figure which depicts a mechanical wave that propagates along a string. The movement of each particle on the string is along the *y*-axis; the wave itself propagates along the *x*-axis.

A wave in which the associated disturbance is parallel to the wave propagation velocity is known as a "***lonitudinal wave***". An example of such a wave is given in the lower figure. It is produced by a piston oscillating in a tube filled with air. The resulting wave involves movement of the air molecules along the axis of the tube which is also the direction of the wave propagation velocity $\vec{v}$.

(16 – 4)



(b)



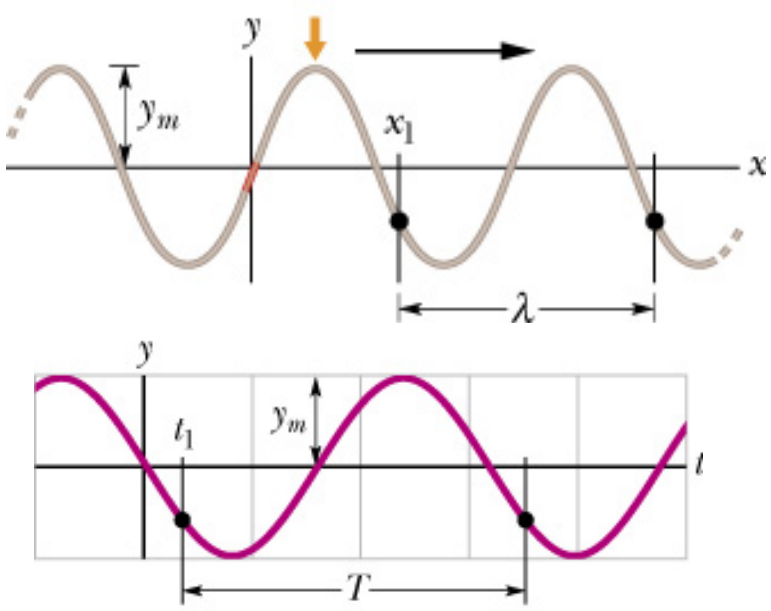
Consider the transverse wave propagating along the string as shown in the figure. The position of any point on the string can be described by a function $y = h(x,t)$. Further along the chapter we shall see that function h has to have a specific form to describe a wave. Once such suitable function is: $y(x,t) = y_m \sin\left(kx - \omega t\right)$

Such a wave which is described by a sine (or a cosine) function is known as "***harmonic wave***".

The various terms that appear in the expression for a harmonic waveare identified in the lower figure

Function $y(x,t)$ depends on x and t. There are two ways to visualize it. The first is to "freeze" time (i.e. set $t = t_o$). This is like taking a snapshot of the wave at $t = t_o$. $y = y\left(x, t_o\right)$ The second is to set $x = x_o$. In this case $y = y\left(x_o, t.\right)$

(16 – 5)



$y(x,t) = y_m \sin(kx - \omega t)$

The **amplitude** y_m is the absolute value of the maximum displacement from the equilibrium position.

The **phase** is defined as the argument $(kx - \omega t)$ of the sine function.

The **wavelength** λ is the shortest distance between two repetitions of the wave at a fixed time.

We fix t at $t = 0$. We have the condition: $y(x_1, 0) = y(x_1 + \lambda, 0) \rightarrow$

$y_m \sin(kx_1) = y_m \sin[k(x_1 + \lambda)] = y_m \sin(kx_1 + k\lambda)$

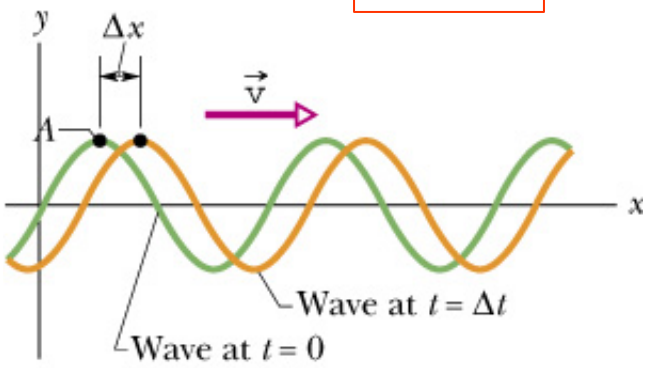
Since the sine function is periodic with period $2\pi \rightarrow k\lambda = 2\pi \rightarrow k = \dfrac{2\pi}{\lambda}$

A **period** T is the time it takes (with fixed x) tthe sine function to complete one oscillation. We take $x = 0 \rightarrow y(0,t) = y(0,t+T) \rightarrow$

$-y_m \sin(\omega t) = -y_m \sin[\omega(t+T)] = -y_m \sin(\omega t + \omega T) \rightarrow \omega T = 2\pi \rightarrow \omega = \dfrac{2\pi}{T}$

(16 – 6)

$$v = \frac{\omega}{k}$$



The speed of a traveling wave

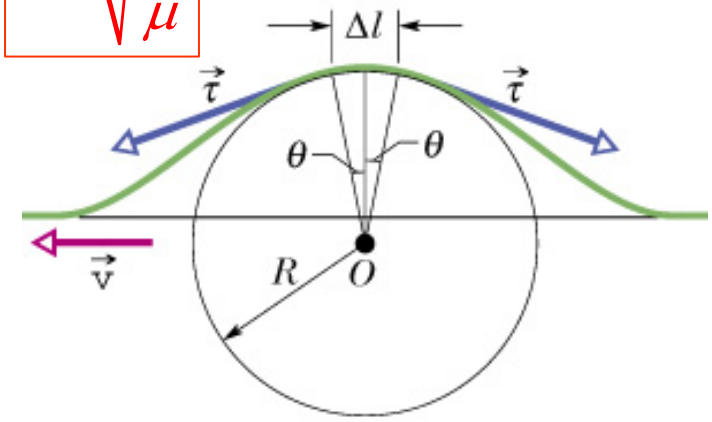
In the figure we show two snapshots of a harmonic wave taken at times t and $t + \Delta t$. During the time interval Δt the wave has traveled a distance Δx. The wave speed $v = \frac{\Delta x}{\Delta t}$. One method of finding v is to imagine that we move with the same speed along the x-axis. In this case the wave will seem to us that it does not change.

Since $y(x,t) = y_m \sin(kx - \omega t)$ this means that the argument of the sine function is constant. $kx - \omega t = \text{constant}$. We take the derivative with respect to t.

$k\frac{dx}{dt} - \omega = 0 \rightarrow \frac{dx}{dt} = \frac{\omega}{k}$ The speed $v = \frac{dx}{dt} = \frac{\omega}{k}$

A harmonic wave that propagates along the **negative** x-axis is described by the equation: $y(x,t) = y_m \sin(kx + \omega t)$. The function $y(x,t) = h(kx - \omega t)$ describes a general wave that propagates along the positive x-axis. A general wave that propagates along the **negative** x-axis is described by the equation: $y(x,t) = h(kx + \omega t)$

$$v = \sqrt{\frac{\tau}{\mu}}$$



Wave speed on a stretched string

Below we will determine the speed of a wave that propagates along a string whose linear mass density is μ. The tension on the string is equal to τ.

Consider a small section of the string of length Δl.

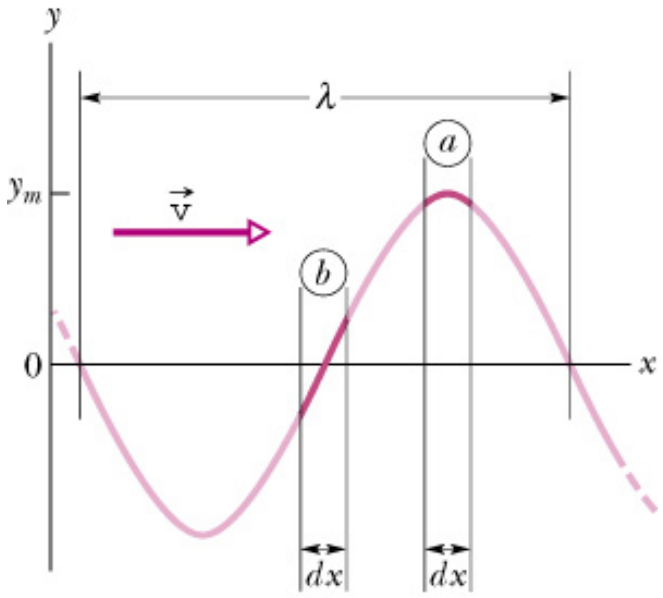
The shape of the element can be approximated to be an arc of a circle of radius R whose center is at O. The net force in the direction of O is $F = 2(\tau \sin\theta)$.

Here we assume that $\theta = 1 \rightarrow \sin\theta$; $\theta = \dfrac{\Delta l}{2R} \rightarrow F = \tau \dfrac{\Delta l}{R}$ **(eqs.1)**

The force is also given by Newton's second law: $F = \Delta m \dfrac{v^2}{R} = (\mu \Delta l)\dfrac{v^2}{R}$ **(eqs.2)**

If we compare equations 1 and 2 we get: $(\mu \Delta l)\dfrac{v^2}{R} = \tau \dfrac{\Delta l}{R} \rightarrow v = \sqrt{\dfrac{\tau}{\mu}}$

Note : The speed v depends on the tension τ and the mass density μ but not on the wave frequency f.



Rate of energy transmission

Consider a transverse wave propagating along a string which is described by the equation: $y(x,t) = y_m \sin(kx - \omega t)$. The transverse velocity $u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t)$ At point a both y and u are equal to zero. At point b both y and u have maxima.

In general the kinetic energy of an element of mass dm is given by: $dK = \frac{1}{2} dm v^2$

$dK = \frac{1}{2}\left(\frac{1}{2} dx\right)\left[-\omega y_m \cos(kx - \omega t)\right]^2$ The rate at which konetic energy propagates along the string is equal to $\frac{dK}{dt} = \frac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$ The average rate

$\left(\frac{dK}{dt}\right)_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2 \left[\cos^2(kx - \omega t)\right]_{avg} = \frac{1}{4}\mu v \omega^2 y_m^2$ As in the case of the oscillating spring-mass system $\left(\frac{dU}{dt}\right)_{avg} = \left(\frac{dK}{dt}\right)_{avg} \rightarrow P_{avg} = \left(\frac{dU}{dt}\right)_{avg} + \left(\frac{dK}{dt}\right)_{avg} = \frac{1}{2}\mu v \omega^2 y_m^2$

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

The wave equation

Consider a string of mass density μ and tension τ

A transverse wave propagates along the string.

The transverse motion is described by $y(x,t)$

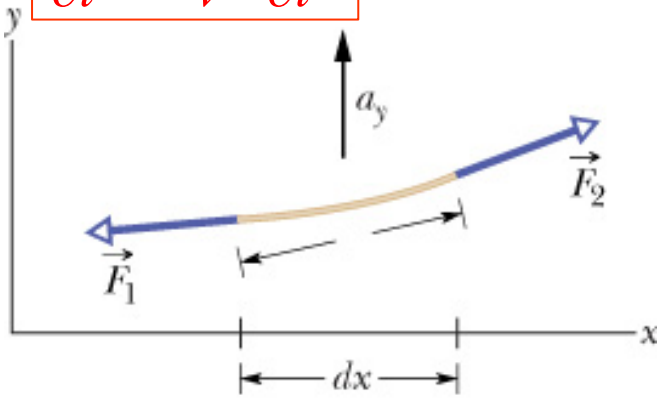
Consider an element of length dx and mass $dm = \mu dx$

The forces $F_1 = F_2 = \tau$

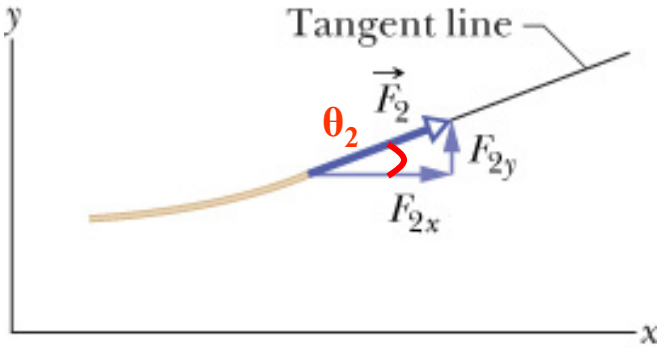
the net force along the y-axis is given by the equation:

$F_{ynet} = F_2 \sin\theta_2 - F_1 \sin\theta_1 = \tau\left(\sin\theta_2 - \sin\theta_1\right)$ Here we assume that $\theta_1 = 1$ and $\theta_2 = 1 \rightarrow \sin\theta_1$; $\tan\theta_1 = \left(\frac{\partial y}{\partial x}\right)_1$

and $\sin\theta_2$; $\tan\theta_2 = \left(\frac{\partial y}{\partial x}\right)_2 \rightarrow F_{ynet} = \tau\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right]$



(a)

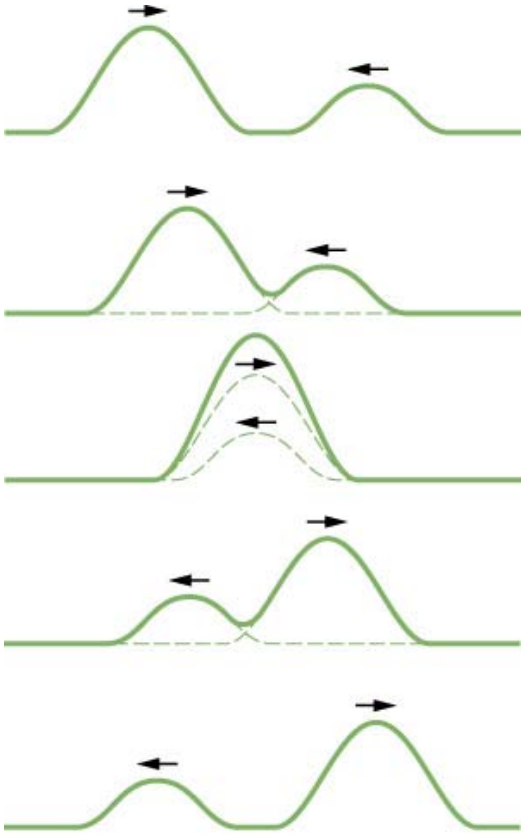


(b)

From Newton's second law we have: $F_{ynet} = dma_y = \mu dx\frac{\partial^2 y}{\partial t^2} = \tau\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] \rightarrow$

$$\tau\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \mu dx\frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1}{dx} = \frac{\partial^2 y}{\partial t^2} = \frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$y'(x,t) = y_1(x,t) + y_2(x,t)$$

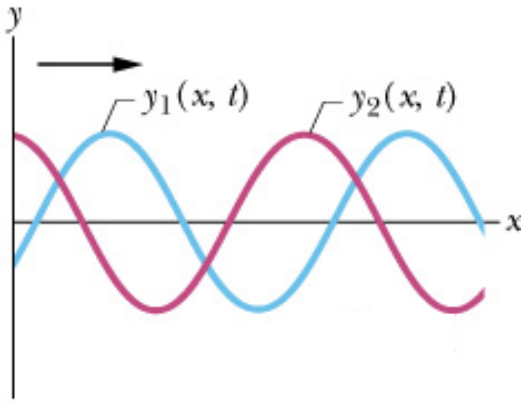


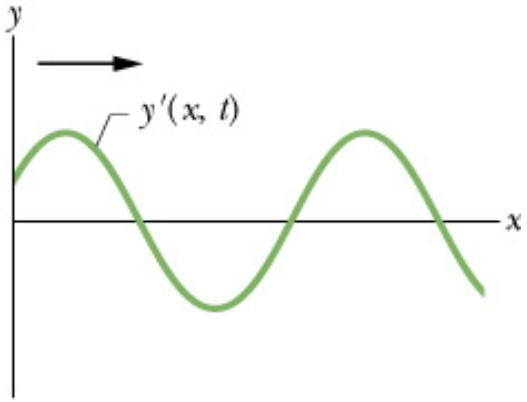
The principle of superposition for waves

The wave equation $\frac{\partial^2 y}{\partial t^2} = \frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ even though it was derived for a transverse wave propagating along a string under tension, is true for all types of waves. This equation is "linear" which means that if y_1 and y_2 are solutions of the wave equation, the function $c_1 y_1 + c_2 y_2$ is also a solution. Here c_1 and c_2 are constants. The principle of superposition is a direct consequence of the linearity of the wave equation. This principle can be expressed as follows:

Consider two waves of the same type that overlap at some point P in space. Assume that the functions $y_1(x,t)$ and $y_2(x,t)$ describe the displacements if the wave arrived at P alone. The displacement at P when both waves are present is given by: $y'(x,t) = y_1(x,t) + y_2(x,t)$

Note : Overlapping waves do not in any way alter the travel of each other





Displacement

$$y'(x,t) = [2y_m \cos \tfrac{1}{2}\phi] \sin(kx - \omega t + \tfrac{1}{2}\phi)$$

Magnitude gives amplitude

Oscillating term

Interference of waves

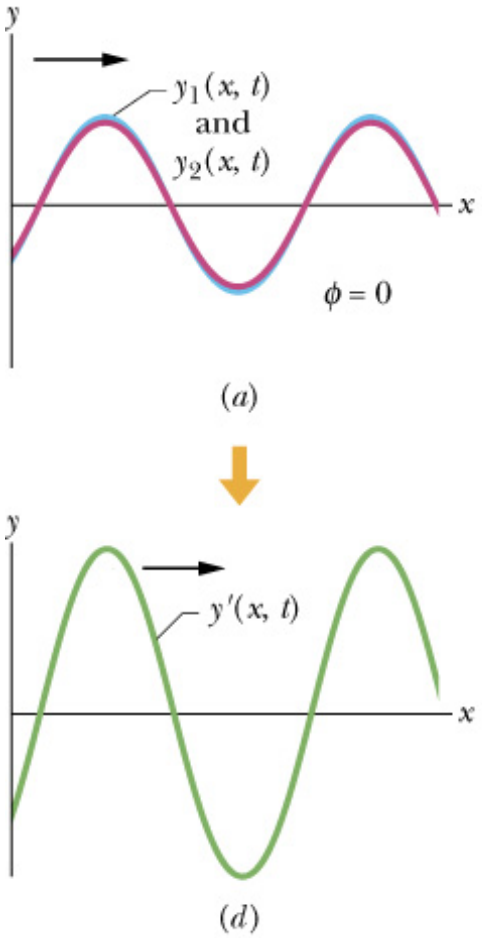
Consider two harmonic waves of the same amplitude and frequency which propagate along the x-axis. The two waves have a phase difference ϕ. We will combine these waves using the principle of superposition. The phenomenon of combing waves is knwon as **interference** and the two waves are said to **interfere**. The displacement of the two waves are given by the functions: $y_1(x,t) = y_m \sin(kx - \omega t)$ and $y_2(x,t) = y_m \sin(kx - \omega t + \phi)$. $y' = y_1 + y_2$

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$$

$$y'(x,t) = \left[2y_m \cos\frac{\phi}{2}\right] \sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

The resulting wave has the same frequency as the original waves, and its amplitude

$y'_m = \left|2y_m \cos\frac{\phi}{2}\right|$ Its phase is equal to $\frac{\phi}{2}$



Constructive interference

The amplitude of two interefering waves is given by:

$$y'_m = \left| 2y_m \cos\frac{\phi}{2} \right|$$ It has its maximum value if $\phi = 0$

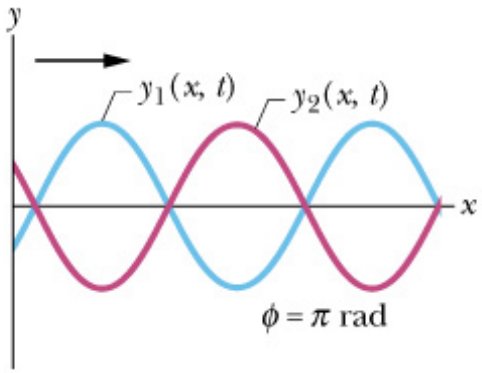
In this case $y'_m = 2y_m$

The displacement of the resulting wave is:

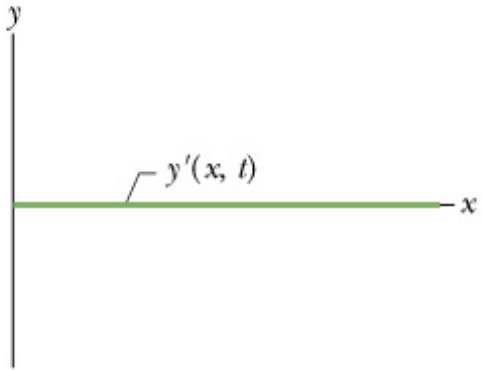
$$y'(x,t) = [2y_m]\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

This phenomenon is known as

fully constructive interference



(b)



(e)

Destructive interference

The amplitude of two interefering waves is given by:

$y'_m = \left| 2y_m \cos\frac{\phi}{2} \right|$ It has its minimum value if $\phi = \pi$

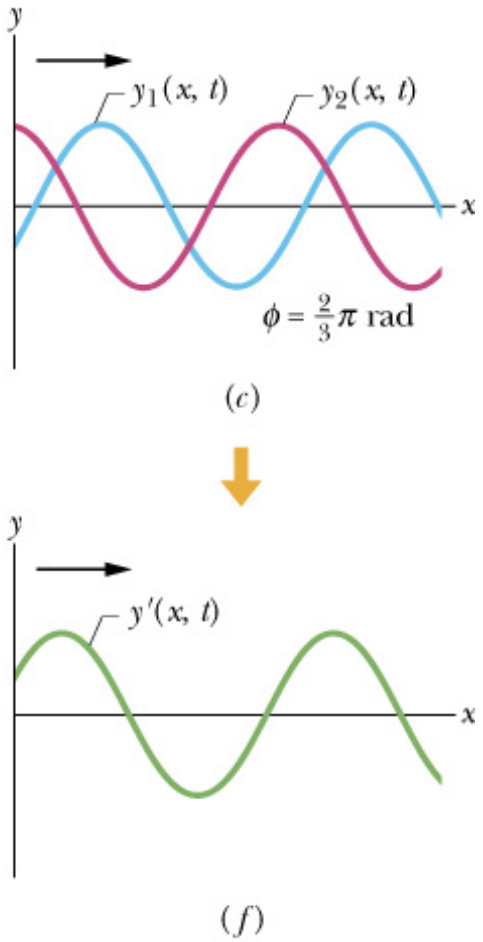
In this case $y'_m = 0$

The displacement of the resulting wave is:

$y'(x,t) = 0$

This phenomenon is known as

fully destructive interference



Intermediate interference

The amplitude of two interefering waves is given by:

$$y'_m = \left| 2y_m \cos\frac{\phi}{2} \right|$$ When interference is neither fully constructive nor fully destructive it is called ***intermediate interference***

An example is given in the figure for $\phi = \dfrac{2\pi}{3}$

In this case $y'_m = y_m$

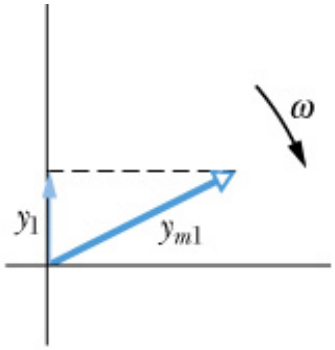
The displacement of the resulting wave is:

$$y'(x,t) = [y_m]\sin\left(kx - \omega t + \frac{\pi}{3}\right)$$

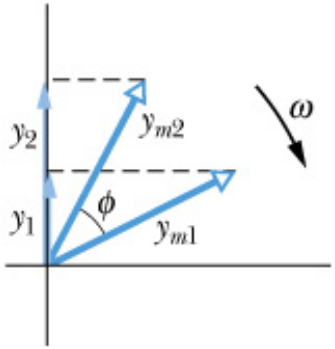
Note : Sometimes the phase difference is expressed as a difference in wavelength λ

In this case remembre that:

2π radians $\leftrightarrow$ 1λ



(a)



(b)

(16 – 15)

Phasors

A phasor is a method for representing a wave whose diasplacement is: $y_1(x,t) = y_{m1}\sin(kx-\omega t)$

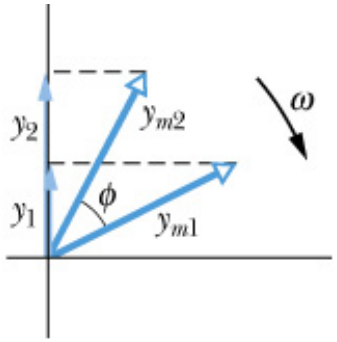
The phasor is defined as a vector with the following properties:

1. Its magnitude is equal to the wave's amplitude y_{m1}

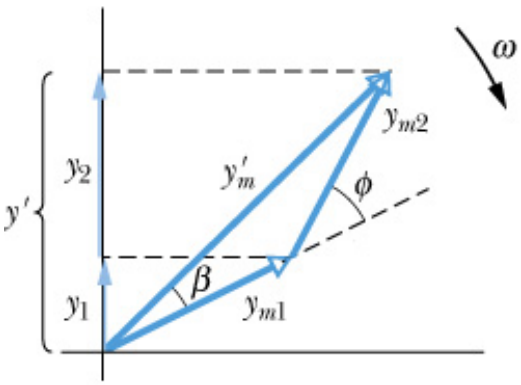
2. The phasor has its tail at the origin O and rotates in the clockwise direction about an axis through O with angular speed ω.

Thus defined, the projection of the phasor on the y-axis (i.e. its y-component) is equal to $y_{m1}\sin(kx-\omega t)$

A phasor diagram can be used to represent more than one waves. (see fig.b). The displacement of the second wave is: $y_2(x,t) = y_{m2}\sin(kx-\omega t+\phi)$ The phasor of the second wave forms an angle ϕ with the phasor of the first wave indicating that it lags behind wave 1 by a phase angle ϕ.



(b)



(c)

(16 – 16)

Wave addition using phasors

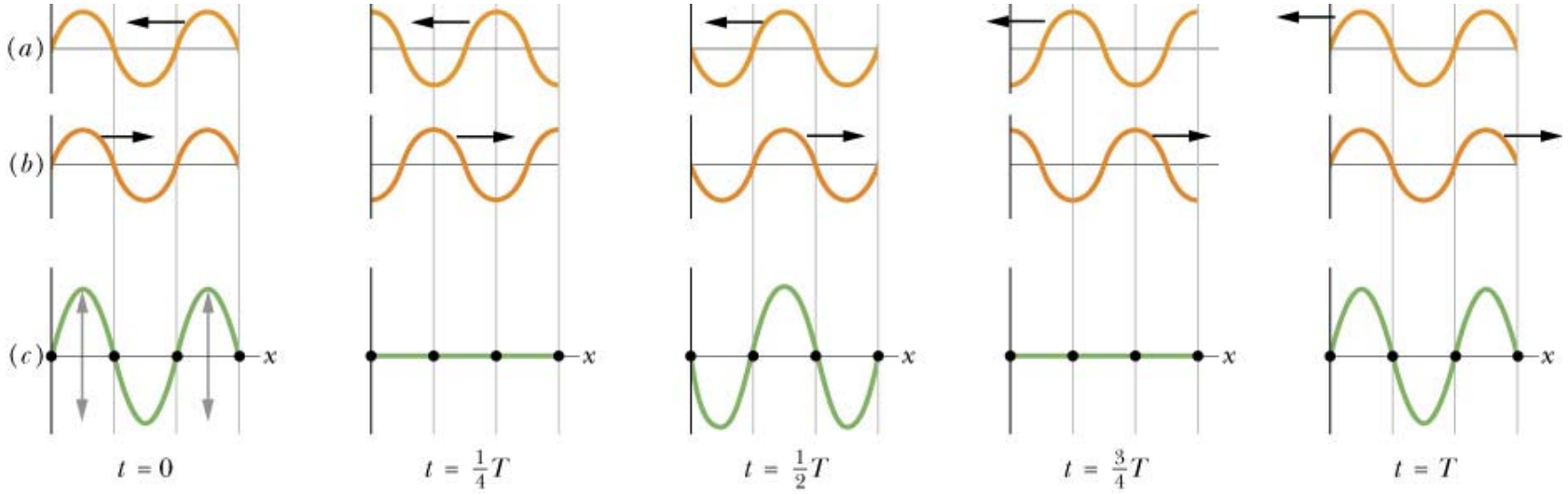
Consider two waves that have the same frequency but different amplitudes. They also have a phase difference ϕ. The displacements of the two waves are: $y_1(x,t) = y_{m1}\sin(kx-\omega t)$ and $y_2(x,t) = y_{m2}\sin(kx-\omega t+\phi)$. The superposition of the two waves yields a wave that has the sameangular frequency ω and is described by: $y' = y'_m \sin(kx-\omega t+\beta)$ Here y'_m is the wave amplitude and β is the phase angle.

To determine y'_m and β

we add the two phasors representing the waves as vectors (see fig.c).

Note: The phasor mathod can be used to add vectors that have different amplitudes.

$$y'(x,t) = [2y_m \sin kx]\cos \omega t$$

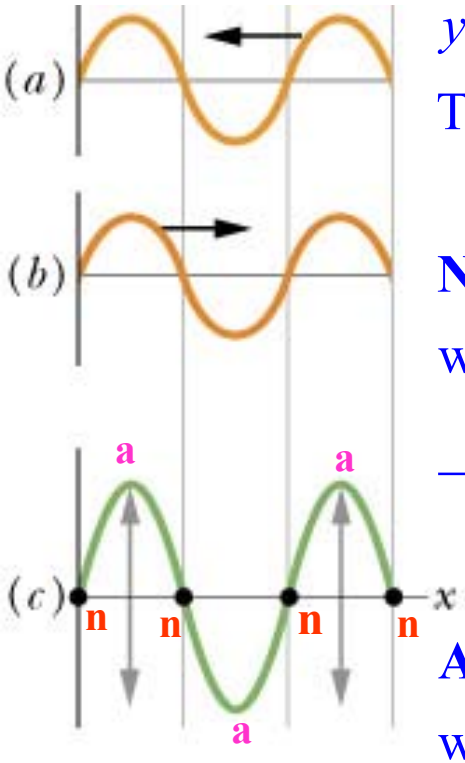


Standing Waves : Consider the superposition of two waves that have the same frequency and amplitude but travel in opposite directions. The displacements of two waves are: $y_1(x,t) = y_m \sin(kx - \omega t)$, $y_2(x,t) = y_m \sin(kx + \omega t)$

The displacement of the resulting wave $y'(x,t) = y_1(x,t) + y_2(x,t)$

$$y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) = [2y_m \sin kx]\cos \omega t$$

This is not a traveling wave but an oscillation that has a position dependent amplitude. It is known as a ***standing wave.***



The displacement of a standing wave is given by the equation:

$$y'(x,t) = [2y_m \sin kx]\cos \omega t$$

The position dependant amplitude is equal to $2y_m \sin kx$

Nodes : These are defined as positions where the standing wave amplitude vanishes. They occur when $kx = n\pi \quad n = 0,1,2,$

$$\rightarrow \frac{2\pi}{\lambda}x = n\pi \rightarrow x_n = n\frac{\lambda}{2} \quad n = 0,1,2,\ldots$$

Antinodes : These are defined as positions where the standing wave amplitude is maximum.

They occur when $kx = \left(n + \frac{1}{2}\right)\pi \qquad n = 0,1,2,\ldots$

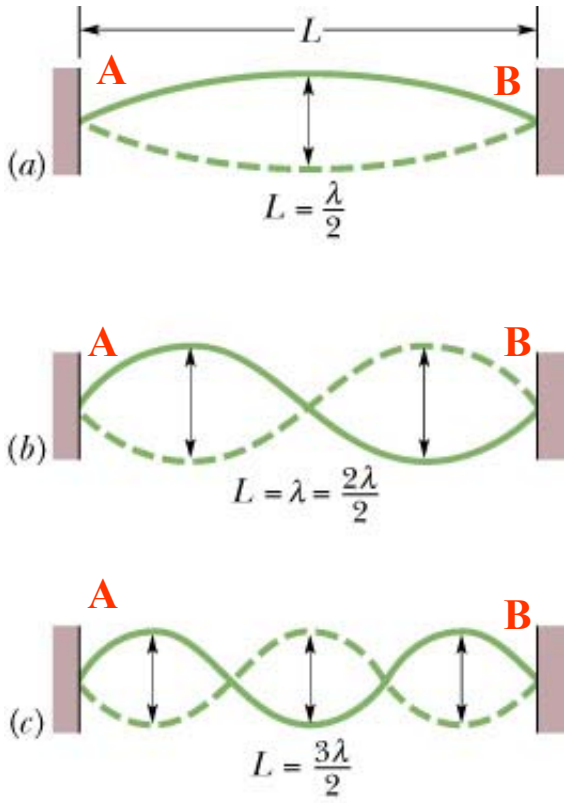
$$\rightarrow \frac{2\pi}{\lambda}x = \left(n + \frac{1}{2}\right)\pi \rightarrow x'_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad n = 0,1,2,\ldots$$

Note 1 : The distance between ajacent nodes and antinodes is $\lambda/2$

Note 2 : The distance between a node and an ajacent antinode is $\lambda/4$

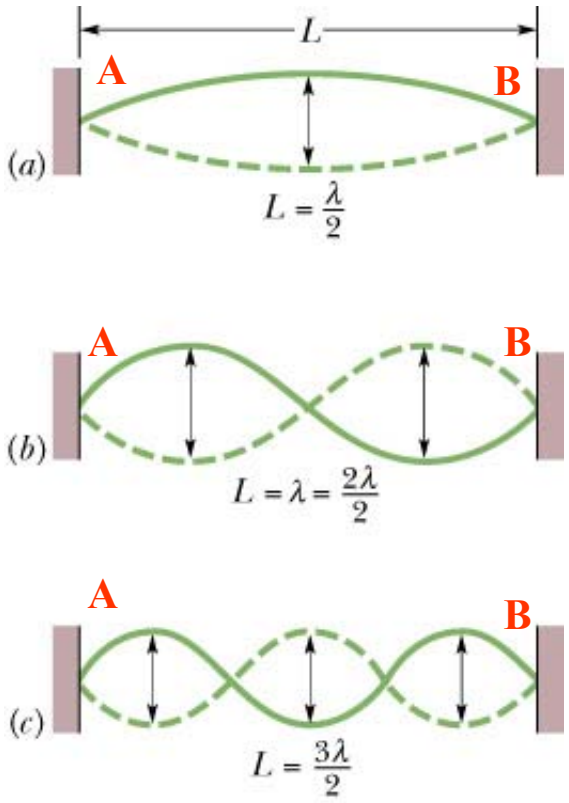
(16 – 19)

Standing waves and resonance



Consider a string under tension on which is clamped at points A and B separated by a sistance *L*. We send a harmonic wave traveling the thr right. the wave is reflected at point B and the reflected wave travels to the left. The left going wave reflects back at point A and creates a thrird wave traveling to the right. Thus we have a large number of overlapping waves half of which travel to the right and the rest to the left.

For certain frequencies the interference produces a standing wave. Such a standing wave is said to be **at resonance**. The frequencis at which the standing wave occurs are known as the **resonant frequencies** of the system.



Resonances occur when the resulting standing wave satisfies the boundary condition of the problem. These are that the Amplitude must be zero at point A and point B and arise from the fact that the string is clamped at both points and therefore cannot move. The first resonance is shown in fig.a. The standing wave has two nodes at points A and B. Thus $L = \frac{\lambda_1}{2}$ $\rightarrow \lambda_1 = 2L$. The second standing wave is shown in fig.b. It has three nodes (two of them at A and B)

In this case $L = 2\left(\frac{\lambda}{2}\right) = \lambda \rightarrow \lambda_2 = L$

The third standing wave is shown in fig.c. It has four nodes (two of them at A and B)

In this case $L = 3\left(\frac{\lambda}{2}\right) = \lambda \rightarrow \lambda_3 = \frac{2}{3}L$ The general expression for the resonant wavelengths is: $\lambda_n = \frac{2L}{n}$ $n = 1, 2, 3, \ldots$ the resonant frequencies $f_n = \frac{v}{\lambda_n} = n\frac{v}{2L}$