Chapter 3

Motion in Two or Three Dimensions

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Goals for Chapter 3

• To use vectors to represent the position of a body
• To determine the velocity vector using the path of a body
• To investigate the acceleration vector of a body
• To describe the curved path of a projectile
• To investigate circular motion
The position vector from the origin to point $P$ has components $x$, $y$, and $z$. 

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \]
Position and Displacement

The position of an object is described by its position vector, \( \vec{r} \).

The **displacement** of the object is defined as the **change in its position**

\[ \Delta \vec{r} \equiv \vec{r}_f - \vec{r}_i \]

Where

\[ \vec{r} = x\hat{i} + y\hat{j} \]
Average velocity

- The average velocity between two points is the displacement divided by the time interval between the two points, and it has the same direction as the displacement.

\[ \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} \]

Displacement vector \( \Delta \vec{r} \) points from \( P_1 \) to \( P_2 \).
Average Velocity

The average velocity is the ratio of the displacement to the time interval for the displacement

\[ \vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta \vec{x}}{\Delta t} \hat{i} + \frac{\Delta \vec{y}}{\Delta t} \hat{j} \]

The direction of the average velocity is the direction of the displacement vector

The average velocity between points is independent of the path taken

- This is because it is dependent on the displacement, also independent of the path
Instantaneous velocity

- The *instantaneous velocity* is the instantaneous rate of change of position vector with respect to time.

- The components of the instantaneous velocity are $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$.

- The instantaneous velocity of a particle is always tangent to its path.
The instantaneous velocity is the limit of the average velocity as $\Delta t$ approaches zero

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d\vec{x}}{dt} \hat{i} + \frac{d\vec{y}}{dt} \hat{j} = v_x \hat{i} + v_y \hat{j}$$

- As the time interval becomes smaller, the direction of the displacement approaches that of the line tangent to the curve
Instantaneous acceleration

- The *instantaneous acceleration* is the instantaneous rate of change of the velocity with respect to time.

- Any particle following a curved path is accelerating, even if it has constant speed.

- The components of the instantaneous acceleration are \( a_x = \frac{dv_x}{dt}, \) \( a_y = \frac{dv_y}{dt}, \) and \( a_z = \frac{dv_z}{dt}. \)
The instantaneous acceleration is the limiting value of the ratio $\Delta \vec{v}/\Delta t$ as $\Delta t$ approaches zero

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}_x}{dt} \hat{i} + \frac{d\vec{v}_y}{dt} \hat{j} = a_x \hat{i} + a_y \hat{j}$$

- The instantaneous acceleration equals the derivative of the velocity vector with respect to time
Kinematic Equations for Two-Dimensional Motion

When the two-dimensional motion has a constant acceleration, a series of equations can be developed that describe the motion.

These equations will be similar to those of one-dimensional kinematics.

Motion in two dimensions can be modeled as two independent motions in each of the two perpendicular directions associated with the x and y axes.

- Any influence in the y direction does not affect the motion in the x direction.
Kinematic Equations

Position vector for a particle moving in the xy plane
\[ \vec{r} = x\hat{i} + y\hat{j} \]

The velocity vector can be found from the position vector
\[ \vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{x}}{dt}\hat{i} + \frac{d\vec{y}}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j} \]

- Since acceleration is constant, we can also find an expression for the velocity and position as a function of time:
  \[ \vec{v}_f = \vec{v}_i + \vec{a}t \]
  \[ \vec{a} = a_x\hat{i} + a_y\hat{j} \]

\[ \vec{v}_f = (v_{ix} + a_x t)\hat{i} + (v_{iy} + a_y t)\hat{j} \]

\[ \vec{r}_f = (v_{ix} t + \frac{1}{2}a_x t^2)\hat{i} + (v_{iy} t + \frac{1}{2}a_y t^2)\hat{j} \]
Object is thrown with initial velocity $v_0$ at an angle $\theta$.

Acceleration: $a_x = 0$  $a_y = -g$  It is directed downward

The effect of air friction is negligible

With these assumptions, an object in projectile motion will follow a parabolic path

- This path is called the *trajectory*
**Projectile motion**

- A projectile is any body given an initial velocity that then follows a path determined by the effects of gravity and air resistance.

- Begin by neglecting resistance and the curvature and rotation of the earth.

  - A projectile moves in a vertical plane that contains the initial velocity vector $\vec{v}_0$.
  - Its trajectory depends only on $\vec{v}_0$ and on the downward acceleration due to gravity.

\[
a_x = 0, \quad a_y = -g
\]
The $x$ and $y$ motion are separable—Figure 3.16

- The red ball is dropped at the same time that the yellow ball is fired horizontally.
- The strobe marks equal time intervals.
- We can analyze projectile motion as horizontal motion with constant velocity and vertical motion with constant acceleration: $a_x = 0$ and $a_y = -g$. 
Analyzing Projectile Motion

Consider the motion as the superposition of the motions in the x- and y-directions.

The actual position and velocity at any time is given by:

\[ r_f = (v_{ox} \ t + \frac{1}{2}a_x \ t^2) \ \hat{i} + (v_{oy} \ t + \frac{1}{2}a_y \ t^2) \ \hat{j} \]

\[ v_f = (v_{ix} + a_x \ t) \ \hat{i} + (v_{iy} + a_y \ t) \ \hat{j} \]

- The initial velocity can be expressed in terms of its components
  \[ v_{xi} = v_i \cos \ \theta \ \text{and} \ v_{yi} = v_i \sin \ \theta \]

In x-direction the motion is motion with constant velocity

\[ a_x = 0 \quad x = v_{ox} \ t \quad v_x = v_{ox} \]

In y-direction the motion is motion with constant acceleration

\[ a_y = -g \]

\[ y = (v_{oy} \ t - \frac{1}{2}g \ t^2) \quad v_y = v_{oy} - g \ t \]
Projectile Motion Diagram

A projectile is launched at an angle \( \theta \) with an initial velocity \( v_{xi} \) and \( v_{yi} \). The projectile moves in a parabolic path, with its maximum height being reached at point C, where the vertical component of velocity \( v_y \) is zero. The projectile then falls under the influence of gravity, represented by the vector \( \vec{g} \), returning to the ground at point E with a horizontal velocity \( v_{xi} \) and a vertical velocity \( v_y \) equal in magnitude but opposite in direction to its initial vertical velocity.
When analyzing projectile motion, three characteristics are of special interest:

- **The range**, $R$, is the horizontal distance of the projectile.
- **The maximum height** the projectile reaches is $h$.
- **The total time** the projectile spends in air $t_{tot}$.
The range of a projectile can be expressed in terms of the initial velocity vector:

\[ R = \frac{v_i^2 \sin 2\theta_i}{g} \]

This is valid only for symmetric trajectory

The \( y \)-component of the velocity is zero at the maximum height of the trajectory

\[ y_{\text{max}} = \frac{0 - v_{oy}^2}{-2g} = \frac{v_{oy}^2}{2g} \]

The total time is:

\[ t_{\text{tot}} = \frac{2v_{oy}}{g} \]
More About the Range of a Projectile

$v_i = 50 \text{ m/s}$
Trajectory Equation $y$ vs $x$

$$x = (v_0 \cos \theta) t \quad (1) \quad y = (v_0 \sin \theta) t - \frac{1}{2} gt^2 \quad (2)$$

By substituting value of $t$ from Equation 1 into Equation 2, the trajectory equation is obtained:

$$y = \tan \theta \cdot x - \left(\frac{g}{2v_0^2} \cdot \cos^2 \theta\right)x^2$$
The equations for projectile motion

- If we set \( x_0 = y_0 = 0 \), the equations describing projectile motion are shown at the right.

- The trajectory is a parabola.

\[
\begin{align*}
x &= (v_0 \cos \alpha_0) t \\
y &= (v_0 \sin \alpha_0) t - \frac{1}{2} gt^2 \\
v_x &= v_0 \cos \alpha_0 \\
v_y &= v_0 \sin \alpha_0 - gt
\end{align*}
\]
Analysis Model, Summary

\[ v_{ox} = v_o \cos \theta \]
\[ x = v_{ox} t \]
\[ v_x = v_{ox} \]
\[ t_{tot} = \frac{2v_{oy}}{g} \]
\[ R = \frac{v_o^2 \sin(2\theta)}{g} \]

\[ v_{oy} = v_o \sin \theta \]
\[ y = v_{oy} t - \frac{1}{2}gt^2 \]
\[ v_y = v_{oy} - gt \]
\[ y = \frac{v_y^2 - v_{oy}^2}{-2g} \]
\[ y_{max} = \frac{v_{oy}^2}{2g} \]

\[ y = \tan \theta \cdot x - (g/2v_o^2 \cdot \cos^2 \theta)x^2 \]
The effects of air resistance—Figure 3.20

- Calculations become more complicated.
- Acceleration is not constant.
- Effects can be very large.
- Maximum height and range decrease.
- Trajectory is no longer a parabola.

Baseball’s initial velocity: $v_0 = 50 \text{ m/s, } \alpha_0 = 53.1^\circ$
Height and range of a projectile - Example

A batter hits a baseball so that leaves the bat at a speed of 37.0 m/s at an angle 53.10. (a) Find the position of the ball and its speed at t = 4 s. (b) Find the time where the ball reaches the highest point of its flight, and its height at this time. (c) find the range of the motion.

(a) \( v_{0x} = (37\text{m/s})\cos53.1^0 = 22.2\text{ m/s} \)
\( v_{0y} = (37\text{m/s})\sin53.1^0 = 29.6\text{ m/s} \)
\( x = v_{0x} t = 22.2\times4 = 88.8\text{m} \)
\( y = v_{0y} t - \frac{1}{2}gt^2 = 29.6\times4 - 4.9\times4^2 = 40\text{m} \)
\( v_y = v_{0y} - gt = 29.6 - 9.8\times4 = -9.6\text{m/s} \)
\( v = \sqrt{22.2^2 + (-9.6)^2} = 24.2\text{m/s} \)

(b) \( v_y = v_{0y} - gt; v_y = 0 \) \( t = 29.6/9.8 = 3.0\text{s} \)
\( y_{max} = v_{0y} \times 3.0 - \frac{1}{2}g3.0^2 = 44.7\text{m} \)

(c)
\[
R = \frac{v_o^2\sin(2\theta)}{g}
\]
\[
R = \frac{(37^2\sin(53.1^0)^2)}{9.8} = 134.1\text{m}
\]
or \( R = 22.2\text{m/s}\times t_{tot} = 22.2\times(3.0\times2) = 133.2\text{m} \)
A body projected horizontally - Example

A motorcycle rider rides off the edge of a cliff at horizontal velocity of 9 m/s. (a) Find the motorcycle’s position and velocity 0.50 s after it leaves the edge of the cliff. (b) If the height of the cliff is 10 m, how long will it take for rider to reach the ground?

$V_{ox} = 9 \text{ m/s} \quad v_{0y} = 0 \quad t = 0.50 \text{s}$

$y = y_0 - \frac{1}{2}gt^2 = 0 - \frac{1}{2}(9.8 \text{ m/s}^2)(0.50 \text{s})^2 = -1.2 \text{ m}$

$X = v_{0x}t = (9 \text{ m/s})(0.50 \text{s}) = 4.5 \text{ m}$

$V_x = 9 \text{ m/s} \quad v_y = -gt = -(9.8 \text{ m/s})(0.50 \text{s}) = -4.9 \text{ m/s}$

$V = \sqrt{9^2 + 4.9^2} = 10.2 \text{ m/s}$

(b) $-y = -\frac{1}{2}gt^2 \quad t = \sqrt{\frac{2y}{g}} = 1.42 \text{ s}$
Different initial and final heights- Example

You throw a ball from your window 8.0 m above the ground with an initial velocity of 10 m/s at an angle of 20° below the horizontal. How far horizontally from your window will the ball hit the ground? Ignore air resistance.

\[ v_{ox} = (10 \text{ m/s}) \cos(-20^\circ) = 9.4 \text{ m/s} \quad v_{oy} = (10 \text{ m/s}) \sin(-20^\circ) = -3.42 \text{ m/s} \]

Find time: \[ y = v_{oy} t - \frac{1}{2}gt^2 - 8 = -3.43t - 4.9t^2 \quad 4.9t^2 + 3.43t - 8 = 0 \]

\[ t = 0.98 \text{ s} \quad \text{then} \quad x = v_{ox} t \]

\[ x = (9.4 \text{ m/s}) \times 0.98 \text{ s} = 9.2 \text{ m} \]
Tranquilizing a falling monkey

• Where should the zookeeper aim?

• Follow Example 3.10.

Without gravity
• The monkey remains in its initial position.
• The dart travels straight to the monkey.
• Therefore, the dart hits the monkey.

With gravity
• The monkey falls straight down.
• At any time $t$, the dart has fallen by the same amount as the monkey relative to where either would be in the absence of gravity: $\Delta y_{\text{dart}} = \Delta y_{\text{monkey}} = -\frac{1}{2}gt^2$.
• Therefore, the dart always hits the monkey.
For *uniform circular motion*, the speed is constant and the acceleration is perpendicular to the velocity.

(a) Car speeding up along a circular path

Component of acceleration parallel to velocity: Changes car’s speed

Component of acceleration perpendicular to velocity: Changes car’s direction

(b) Car slowing down along a circular path

Component of acceleration parallel to velocity: Changes car’s speed

Component of acceleration perpendicular to velocity: Changes car’s direction

(c) Uniform circular motion: Constant speed along a circular path

Acceleration is exactly perpendicular to velocity; no parallel component

To center of circle
Acceleration for uniform circular motion

- For uniform circular motion, the instantaneous acceleration always points toward the center of the circle and is called the centripetal acceleration.

- The magnitude of the acceleration is \( a_{\text{rad}} = \frac{v^2}{R} \).

- The period \( T \) is the time for one revolution, and \( a_{\text{rad}} = 4\pi^2 \frac{R}{T^2} \).
Centripetal acceleration on a curved road

• A sports car has a lateral acceleration as it rounds a curve in the road.

• Follow Example 3.11.
Centripetal acceleration on a carnival ride

- Passengers move horizontally at constant speed with a known period.
- Follow Example 3.12.
Nonuniform circular motion—Figure 3.30

- If the speed varies, the motion is *nonuniform circular motion*.
- The radial acceleration component is still \( a_{\text{rad}} = \frac{v^2}{R} \), but there is also a tangential acceleration component \( a_{\text{tan}} \) that is *parallel* to the instantaneous velocity.
The velocity of a moving body seen by a particular observer is called the velocity *relative* to that observer, or simply the *relative velocity*.

A *frame of reference* is a coordinate system plus a time scale.
Relative velocity in one dimension

- If point $P$ is moving relative to reference frame $A$, we denote the velocity of $P$ relative to frame $A$ as $v_{P/A}$.

- If $P$ is moving relative to frame $B$ and frame $B$ is moving relative to frame $A$, then the $x$-velocity of $P$ relative to frame $A$ is $v_{P/A-x} = v_{P/B-x} + v_{B/A-x}$.
Relative velocity on a straight road

- Motion along a straight road is a case of one-dimensional motion.
- Follow Example 3.13 and Figure 3.33.
- Refer to Problem-Solving Strategy 3.2.
Relative velocity in two or three dimensions

- We extend relative velocity to two or three dimensions by using vector addition to combine velocities.

- In Figure 3.34, a passenger’s motion is viewed in the frame of the train and the cyclist.
Flying in a crosswind

- A crosswind affects the motion of an airplane.
- Follow Examples 3.14 and 3.15.
- Refer to Figures 3.35 and 3.36.