Inductors and Inductance
Self-Inductance
RL Circuits – Current Growth
RL Circuits – Current Decay
Energy Stored in a Magnetic Field
Energy Density of a Magnetic Field
Mutual Inductance
Summary
Induction: basics

- Magnetic Flux:
  \[ d\Phi_B = \vec{B} \cdot d\vec{A} = \vec{B} \cdot \hat{n} dA \]

- Faraday’s Law: A changing magnetic flux through a coil of wire induces an EMF in the wire, proportional to the number of turns, \( N \).
  \[ \varepsilon_{\text{ind}} = -N \frac{d\Phi_B}{dt} \]

- Lenz’s Law: The current driven by an induced EMF creates an induced magnetic field that opposes the flux change.
  \[ B_{\text{ind}} \text{ and } i_{\text{ind}} \text{ oppose changes in } \Phi_B \]

- Induction and energy transfer: The forces on the loop oppose the motion of the loop, and the power required to sustain motion provides electrical power to the loop.
  \[ P = \vec{F} \cdot \vec{v} = Fv \quad P = i \varepsilon \quad \varepsilon = -Blv \]

- Transformer principle: changing current \( i_1 \) in primary induces EMF and current \( i_2 \) in secondary coil.

- A changing magnetic flux creates a non-conservative electric field.
  \[ \varepsilon = \int \vec{E} \cdot d\vec{s} = -N \frac{d\Phi_B}{dt} \]

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Changing magnetic flux induces electric fields: trivial transformer

A thin solenoid, cross section A, n turns/unit length
• zero field outside solenoid
• inside solenoid: \( B = \mu_0 in \)

Flux through a conducting loop: \( \Phi = BA = \mu_0 inA \)

Current i varies with time, so flux varies and an EMF is induced in loop “A”:

\[
\varepsilon_{\text{ind}} = -\frac{d\Phi}{dt} = -\mu_0 nA \frac{di}{dt}
\]

Current induced in the loop is:

\[
i_{\text{ind}} = \frac{\varepsilon_{\text{ind}}}{R}
\]

If \( \frac{di}{dt} \) is positive, B is growing, then \( B_{\text{ind}} \) opposes change and \( i_{\text{ind}} \) is Counter-clockwise.

What makes the current \( i_{\text{ind}} \) flow?

• \( B = 0 \) there so it’s not the Lorentz force
• An induced electric field \( E_{\text{ind}} \) along the loop causes current to flow
• It is caused directly by \( dF/dt \)
• Electric field lines are loops that don’t terminate on charge.
• \( E \)-field is there even without the conductor (no current flowing)
• \( E \)-field is non-conservative (not electrostatic) as the line integral around a closed path is not zero

\[ \therefore \varepsilon_{\text{ind}} = \oint_{\text{loop}} E_{\text{ind}} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \]

Generalized Faradays’ Law
Path must be constant
Example: Find the induced electric field

\[ \varepsilon_{\text{ind}} = \oint_{\text{loop}} \mathbf{E}_{\text{ind}} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \]

In the right figure, dB/dt = constant, find the expression for the magnitude E of the induced electric field at points within and outside the magnetic field.

Due to symmetry:

For \( r < R \):

\[ \Phi_B = BA = B(\pi r^2) \]

So

\[ E(2\pi r) = \pi r^2 \frac{dB}{dt} \]

For \( r > R \):

\[ \Phi_B = BA = B(\pi R^2) \]

So

\[ E(2\pi r) = \pi R^2 \frac{dB}{dt} \]

The magnitude of induced electric field grows linearly with \( r \), then falls off as \( 1/r \) for \( r > R \).
Self-Inductance: Analogous to inertia

ANY magnetic flux change is resisted.

Changing current in a single coil induces a “back EMF” $\mathcal{E}_{\text{ind}}$ in the same coil opposing the current change, an induced current $i_{\text{ind}}$, and a consistent induced field $B_{\text{ind}}$.

DISTINGUISH:

- **Mutual-induction**: $di_1/dt$ in “transformer primary” also induces EMF and current $i_2$ in “linked” secondary coil (transformer principle).  

- **Self-induction** in a single Coil: $di/dt$ produces “back EMF” due to Lenz & Faraday Laws: $\Phi_{\text{ind}}$ opposes $d\Phi/dt$ due to current change. $\mathcal{E}_{\text{ind}}$ opposes $di/dt$.
  - Changing current in a single coil causes magnetic field and flux created by this current to change in the same sense
  - Flux Change induces flux opposing the change, along with opposing EMF and current.
  - This back emf limits the rate of current (flux) change in the circuit

  - For increasing current, back EMF limits the rate of increase
  - For decreasing current, back EMF sustains the current

Inductance measures opposition to the rate of change of current
Definition of Self-inductance

Recall capacitance: depends only on geometry
It measures energy stored in the E field

Self-inductance depends only on coil geometry
It measures energy stored in the B field

\[ L \equiv \frac{\text{linked flux}}{\text{unit current}} \]

\[ L \equiv \frac{N \Phi_B}{i} \]

Number of turns

Self-inductance

Flux through one turn depends on current & all \( N \) turns
cancels current dependence in flux above

SI unit of inductance:

\( 1 \text{ Henry} \equiv 1 \text{ H} \equiv 1 \text{T.m}^2 / \text{Ampere} = 1 \text{ Weber} / \text{Ampere} \)
\( = 1 \text{ Volt.sec} / \text{Ampere} \) (\( \Omega \text{.sec} \))

Why choose this definition?

Cross-multiply

\[ Li = N \Phi_B \]

Take time derivative

\[ L \frac{di}{dt} = N \frac{d\Phi_B}{dt} = -\mathcal{E}_L \]

\[ \mathcal{E}_L \equiv -L \frac{di}{dt} \]

- \( L \) contains all the geometry
- \( \mathcal{E}_L \) is the “back EMF”

Another form of Faraday’s Law!
Example: Find the Self-Inductance of a solenoid

Field: \( B = \mu_0 i_n \) where \( n = \frac{N}{\ell} = \) \# turns per unit length

Flux in just one turn:

\[ \Phi_B \equiv BA = \mu_0 i_n \frac{NA}{\ell} \]

Apply definition of self-inductance:

\[ L \equiv \frac{N\Phi_B}{i_n} = \mu_0 \frac{N^2 A}{\ell} = \mu_0 n^2 V \]

Check: Same \( L \) if you start with Faraday’s Law for \( F_B \):

\[ \varepsilon_{\text{ind}} = -N \frac{d\Phi_B}{dt} \]

for solenoid use \( \Phi_B \) above

\[ \varepsilon_{\text{ind}} = -N. \left( \mu_0 \frac{NA}{\ell} \frac{di}{dt} \right) = -\mu_0 N^2 A \frac{di}{dt} = -L \frac{di}{dt} \]

Note: Inductance per unit length has same dimensions as \( \mu_0 \)

\[ \frac{L}{\ell} = \mu_0 N^2 \frac{A}{\ell^2} \]

\[ [\mu_0] = \frac{Tm}{A} = \frac{H}{m} \]
Example: calculate self-inductance $L$ for an ideal solenoid

\[ \ell = 0.2 \text{ m}, \quad r = 0.5 \text{ m}, \quad N = 1000 \text{ turns} \]

\[ L = \frac{\mu_0 N^2 A}{\ell} = \frac{4\pi \times 10^{-7} \times 10^6 \times \pi \times (5 \times 10^{-2})^2}{0.2} \]

\[ \therefore L = 49.4 \times 10^{-3} \text{ Henrys} = 49.4 \text{ milli - Henrys} \]

Ideal inductor (abstraction):
- Internal resistance $r = 0$ (recall ideal battery)
- $B = 0$ outside
- $B = \mu_0$ in inside (ideal solenoid)

Non-ideal inductors have internal resistance:
- $V_{\text{ind}} = \varepsilon_L - \text{ir} = \text{measured voltage}$
- Direction of $\text{ir}$ depends on current
- Direction of $\varepsilon_L$ depends on $\text{di/dt}$
- If current $i$ is constant, then induced $\varepsilon_L = 0$

Inductor behaves like a wire with resistance $r$
Induced EMF in an Inductor

12 – 1: Which statement describes the current through the inductor below, if the induced EMF is as shown?

A. Rightward and constant.
B. Leftward and constant.
C. Rightward and increasing.
D. Leftward and decreasing.
E. Leftward and increasing.

\[ \mathcal{E}_L \equiv -L \frac{di}{dt} \]
Lenz’s Law applied to Back EMF

If \( i \) is increasing: \( \frac{d\Phi_B}{dt} > 0 \)

\[ \therefore \varepsilon_L \text{ opposes increase in } i \]

Power is being stored in B field of inductor

If \( i \) is decreasing: \( \frac{d\Phi_B}{dt} < 0 \)

\[ \therefore \varepsilon_L \text{ opposes decrease in } i \]

Power is being tapped from B field of inductor

What if CURRENT \( i \) is constant?
Example: Current I increases uniformly from 0 to 1 A. in 0.1 seconds. Find the induced voltage (back EMF) across a 50 mH (milli-Henry) inductance.

\[ \varepsilon_L = -L \frac{di}{dt} \]

\( i \) defines positive direction

\( \frac{di}{dt} > 0 \) means that current \( i \) is increasing and toward the right

Apply: \( \varepsilon_L = -L \frac{di}{dt} \)

Substitute: \( \frac{\Delta i}{\Delta t} = \frac{+1 \text{ Amp}}{0.1 \text{ sec}} = 10 \frac{\text{Amp}}{\text{sec}} \)

\[ \varepsilon_L = -50 \text{ mH} \cdot 10 \frac{\text{Amp}}{\text{sec}} = -0.5 \text{ Volts} \]

Negative result means that induced EMF is opposed to both \( di/dt \) and \( i \).
Inductors in Circuits—The RL Circuit

- Inductors, sometimes called “coils”, are common circuit components.
- Insulated wire is wrapped around a core.
- They are mainly used in AC filters and tuned (resonant) circuits.

Analysis of series RL circuits:

- A battery with EMF $E$ drives a current around the loop, producing a back EMF $E_L$ in the inductor.
- Derive circuit equations: apply Kirchoff’s loop rule, convert to differential equations (as for RC circuits) and solve.

New rule: when traversing an inductor in the same direction as the assumed current, insert:

$$\mathcal{E}_L \equiv -L \frac{di}{dt}$$
Series LR circuits

- Inductance & resistance + EMF
- Find time dependent behavior
- Use Loop Rule & Junction Rule
- Treat $\mathcal{E}_L$ as an EMF along current

\[ \mathcal{E}_L \equiv -L \frac{di}{dt} \]

ALWAYS

Given $\mathcal{E}$, $R$, $L$: Find $i$, $\mathcal{E}_L$, $U_L$ for inductor as functions of time

**Growth phase, switch to “a”. Loop equation:**

\[ \mathcal{E} - iR - L \frac{di}{dt} = 0 \]

- $i$ through $R$ is clockwise and growing: $\mathcal{E}_L$ opposes $\mathcal{E}$
- At $t = 0$, rapidly growing current but $i = 0$, $\mathcal{E}_L = \mathcal{E}$
  - $L$ acts like a broken wire
- As $t \to \infty$, large stable current, $di/dt \to 0$
  - $\mathcal{E}_L \to 0$, $i \to \mathcal{E}/R$,
  - $L$ acts like an ordinary wire
- Energy is stored in $L$ & dissipated in $R$

**Decay phase, switch to “b”, exclude $\mathcal{E}$, Loop equation:**

\[ -iR - L \frac{di}{dt} = 0 \]

- Energy stored in $L$ now dissipated in $R$
- Current through $R$ is still clockwise, but collapsing
- $\mathcal{E}_L$ now acts like a battery maintaining current
- Current $i$ at $t = 0$ equals $\mathcal{E}/R$
- Current $\to 0$ as $t \to \infty$ – energy depleted
LR circuit: decay phase solution

- After growth phase equilibrium, switch from a to b, battery out
- Current $i_0 = \varepsilon / R$ initially still flowing CW through $R$
- Inductance tries to maintain current using stored energy
- Polarity of $\varepsilon_L$ reverses versus growth. Eventually $\varepsilon_L \to 0$

Loop Equation is: $-iR + \varepsilon_L = 0$

Substitute: $\varepsilon_L(t) = -L \frac{di}{dt}$

Circuit Equation: $\frac{di}{dt} = -\frac{R}{L} i$

- First order differential equation with simple exponential solution
- At $t = 0$: large current implies large $di / dt$, so $\varepsilon_L$ is large
- As $t \to \infty$: current stabilizes, $di / dt$ and current $i$ both $\to 0$

Current decays exponentially:

$$i(t) = i_0 e^{-t/\tau_L} \quad i_0 \equiv \frac{\varepsilon}{R}$$

$$\tau_L = L/R \equiv \text{inductive time constant}$$

Back EMF $\varepsilon_L$ and $V_R$ decay exponentially:

$$\frac{di}{dt} = -\frac{\varepsilon}{R\tau_L} e^{-t/\tau_L} \Rightarrow \frac{\varepsilon_L}{\tau_L} = +\varepsilon e^{-t/\tau_L}$$

$$V_R = -\varepsilon_L = -iR$$

Compare to RC circuit, decay:

$$Q(t) = C\varepsilon e^{-t/RC}$$

$RC \equiv \text{capacitive time constant}$
LR circuit: growth phase solution

Loop Equation is: \[ E - iR + E_L = 0 \]

Substitute: \[ E_L(t) = -L \frac{di}{dt} \]

Circuit Equation:
\[ i + \frac{L}{R} \frac{di}{dt} = \frac{E}{R} \]

- First order differential equation again - saturating exponential solutions
- As \( t \to \infty \), \( \frac{di}{dt} \) approaches zero, current stabilizes at \( i_{\text{inf}} = \frac{E}{R} \)
- At \( t = 0 \): current is small, \( \frac{di}{dt} \) is large, back EMF opposes battery.

Current starts from zero, grows as a saturating exponential.

\[ i(t) = i_{\text{inf}} \left( 1 - e^{-t/\tau_L} \right) \]
\[ i_{\text{inf}} \equiv \frac{E}{R} \]
\[ \tau_L \equiv \frac{L}{R} = \text{inductive time constant} \]

- \( i = 0 \) at \( t = 0 \) in above equation \( \to \) \( \frac{di}{dt} = \frac{E}{L} \)
  fastest rate of change, largest back EMF

Back EMF \( E_L \) decays exponentially

\[ \frac{di}{dt} = \frac{E}{R \tau_L} e^{-t/\tau_L} \Rightarrow E_L = -E e^{-t/\tau_L} \]

Voltage drop across resistor \( V_R = -iR \)

Compare to RC circuit, charging
\[ Q(t) = C E \left( 1 - e^{-t/RC} \right) \]
\( RC \equiv \text{capacitive time constant} \)
Example: For growth phase find back EMF $\mathcal{E}_L$ as a function of time

Use growth phase solution \( i(t) = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau_L}) \)

At \( t = 0 \): current = 0

\[
 i(t=0) = \frac{\mathcal{E}}{R} (1 - e^0) = 0
\]

Back EMF is $\sim$ to rate of change of current

\[
 \text{Derivative: } \frac{di}{dt} = \frac{\mathcal{E}}{R} (-)(-) e^{-t/\tau_L} \quad \text{where} \quad \tau_L = \frac{L}{R}
\]

\[
 \mathcal{E}_L = -L \frac{di}{dt} = -\mathcal{E} e^{-t/\tau_L} ; \quad \text{At } t = 0: \quad \mathcal{E}_L = -\mathcal{E}
\]

- Back EMF $\mathcal{E}_L$ equals the battery potential causing current $i$ to be 0 at $t = 0$
- $iR$ drop across $R = 0$
- $L$ acts like a broken wire at $t = 0$

After a very long (infinite) time:
- Current stabilizes, back EMF=0

\[
 i_\infty = \frac{\mathcal{E}}{R} = 5A
\]

- $L$ acts like an ordinary wire at $t = \infty$
Current through the battery - 1

12 – 2: The three loops below have identical inductors, resistors, and batteries. Rank them in terms of current through the battery just after the switch is closed, greatest first.

A.I, II, III.
B.II, I, III.
C.III, I, II.
D.III, II, I.
E.II, III, I.

Hint: what kind of wire does L act like?

\[ i(t) = i_{\text{inf}} \left(1 - e^{-t/\tau_L}\right) \quad i_{\text{inf}} = \frac{\mathcal{E}}{R_{\text{eq}}} \quad \tau_L = L/R_{\text{eq}} = \text{inductive time constant} \]
Current through the battery - 2

12 – 3: The three loops below have identical inductors, resistors, and batteries. Rank them in terms of current through the battery a long time after the switch is closed, greatest first.

A. I, II, III.
B. II, I, III.
C. III, I, II.
D. III, II, I.
E. II, III, I.

Hint: what kind of wire does L act like?

\[ i(t) = i_{\text{inf}} \left( 1 - e^{-t/\tau_L} \right) \]
\[ i_{\text{inf}} = \frac{E}{R_{\text{eq}}} \]
\[ \tau_L = L/R_{\text{eq}} = \text{inductive time constant} \]
Summarizing RL circuits growth phase

- When $t$ is large: \[ i = \frac{\varepsilon}{R} \quad \text{Inductor acts like a wire.} \]
  \[ i = \frac{\varepsilon}{R} (1 - e^{-\frac{Rt}{L}}) \]

- When $t$ is small (zero), $i = 0$. \quad \text{Inductor acts like an open circuit.}

- The current starts from zero and increases up to a maximum of $i = \frac{\varepsilon}{R}$ with a time constant given by
  \[ \tau_L = \frac{L}{R} \quad \text{Inductive time constant} \]
  
  Compare: $\tau_C = RC$ Capacitive time constant

- The voltage across the resistor is
  \[ V_R = -iR = -\varepsilon (1 - e^{-\frac{Rt}{L}}) \]

- The voltage across the inductor is
  \[ V_L = -\varepsilon - V_R = -\varepsilon + \varepsilon (1 - e^{-\frac{Rt}{L}}) = -\varepsilon e^{-\frac{Rt}{L}} \]
Summarizing RL circuits decay phase

The switch is thrown from a to b

- Kirchoff’s Loop Rule for growth was:
  \[ \varepsilon - iR - L \frac{di}{dt} = 0 \]

- Now it is:
  \[ iR + L \frac{di}{dt} = 0 \]

- The current decays exponentially:
  \[ i = \frac{\varepsilon}{R} e^{-\frac{Rt}{L}} \]

- Voltage across resistor also decays:
  \[ V_R = -iR = -\varepsilon e^{-\frac{Rt}{L}} \]

- Voltage across inductor:
  \[ V_L = +L \frac{di}{dt} = +L \frac{\varepsilon}{R} \frac{d}{dt} e^{-\frac{Rt}{L}} = -\varepsilon e^{-\frac{Rt}{L}} \]
Energy stored in inductors

Recall: Capacitors store energy in their electric fields

\[ U_E \equiv \text{electrostatic potential energy} \]
\[ U_E \equiv \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 \]

\[ u_E \equiv \text{electrostatic energy density} \]
\[ u_E \equiv \frac{U_E}{\text{Volume}} = \frac{1}{2} \varepsilon_0 E^2 \]

Inductors also store energy, but in their magnetic fields

Magnetic PE Derivation – consider power into or from inductor

\[ \text{Power} \equiv \frac{dU_B}{dt} = \mathcal{L} i = Li \frac{di}{dt} \Rightarrow U_B = \int dU_B = L \int i\,di = \frac{1}{2} Li^2 \]

\[ U_B \equiv \text{magnetic potential energy} \]
\[ U_B = \frac{1}{2} Li^2 \]

\[ u_B \equiv \text{magnetic energy density} \]
\[ u_B \equiv \frac{U_B}{\text{Volume}} = \frac{B^2}{2\mu_0} \]

- \( U_B \) grows as current increases, absorbing energy
- When current is stable, \( U_B \) and \( u_B \) are constant
- \( U_B \) diminishes when current decreases. It powers the persistent EMF during the decay phase for the inductor
Sample problem: energy storage in magnetic field of an inductor during growth phase

a) At equilibrium (infinite time) how much energy is stored in the coil?

\[ i_\infty = \frac{\mathcal{E}}{R} \quad \text{(Coil acts like a wire)} = \frac{12}{0.35} = 34.3 \text{ A} \]

\[ U_\infty = \frac{1}{2} L \cdot i_\infty^2 = \frac{1}{2} \times 53 \times 10^{-3} \times (34.3)^2 \]

\[ U_\infty = 31 \text{ J} \]

b) How long \( t_{1/2} \) does it take to store half of this energy?

At \( t_{1/2} \):

\[ U_B = \frac{1}{2} U_\infty \quad \Rightarrow \quad \frac{1}{2} L i_{1/2}^2 = \frac{1}{2} \cdot \frac{1}{2} \cdot L \cdot i_\infty^2 \quad \Rightarrow \quad i_{1/2} = \frac{i_\infty}{\sqrt{2}} \]

\[ i_{1/2} = \frac{i_\infty}{\sqrt{2}} = i_\infty \left(1 - e^{-t_{1/2} / \tau_L}\right) \]

\[ e^{-t_{1/2} / \tau_L} = \frac{\sqrt{2}}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 - 1 / \sqrt{2} \]

Take natural log of both sides

\[ t_{1/2} = -\tau_L \ln(1 - 1 / \sqrt{2}) = 1.23 \tau \]

\[ \tau_L = \frac{L}{R} = \frac{53 \times 10^{-3}}{0.35} = 0.15 \text{ sec.} \]
Mutual Inductance

- Example: a pair of co-axial coils
- \( \frac{di}{dt} \) in the first coil induces current in the second coil, in addition to self-induced effects.
- \( M_{21} \) depends on geometry only, as did \( L \) and \( C \)
- Changing current in primary (\( i_1 \)) creates varying flux through coil 2 \( \rightarrow \) induced EMF in coil 2

Definition:

\[
\begin{align*}
    M_{21} & \equiv \frac{N_2 \Phi_{21}}{i_1} \\
    \text{number of turns} & \quad \text{mutual inductance} \\
    \text{in coil 2} & \quad \text{flux through one turn of coil 2 due to all } N_1 \\
    \text{mutual inductance} & \quad \text{turns of coil 1} \\
    \text{current in coil 1}
\end{align*}
\]

Cross-multiply

\[
M_{21} i_1 = N_2 \Phi_{21}
\]

time derivative

\[
M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{21}}{dt}
\]

Another form of Faraday’s Law!

\[
\mathcal{E}_2 \equiv -M_{21} \frac{di_1}{dt}
\]

\[
\mathcal{E}_1 \equiv -M_{12} \frac{di_2}{dt}
\]

\[
M_{12} = M_{21} \equiv M
\]

The smaller coil radius determines how much flux is linked, so…..

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Calculating the mutual inductance $M$

Let coil 1 (outer) be a short loop of $N_1$ turns, not a long Solenoid

$$B_1 \equiv \text{Field inside loop 1 near center} = \frac{\mu_0 i_1 N_1}{2R_1} \quad \text{(assume uniform)}$$

Flux through Loop 2 – depends on area $A_2$ & $B_1$

$$\Phi_{21} = B_1 A_2 = \frac{\mu_0 i_1 N_1}{2R_1} \pi R_2^2 \quad \text{(for each loop in coil 2)}$$

If current in Loop 1 is changing:

$$\varepsilon_2 = \text{induced voltage in loop 2} = -N_2 \frac{d\Phi_{21}}{dt} = -\frac{\mu_0 N_1 N_2 \pi R_2^2}{2R_1} \frac{di_1}{dt} \equiv -M_{21} \frac{di_1}{dt}$$

$$M_{21} \equiv M = \frac{\mu_0 N_1 N_2}{2R_1} \pi R_2^2$$

smaller radius $(R_2)$ determines the linkage

Summarizing results for mutual inductance:

$$\varepsilon_2 = -M_{21} \frac{di_1}{dt} \quad M_{21} = \frac{N_2 \Phi_{21}}{i_1}$$

$$M_{21} = M = \frac{N_2 \Phi_{21}}{i_1} = \frac{N_1 \Phi_{12}}{i_2}$$
CHAPTER 30  SUMMARY

Mutual inductance: When a changing current $i_1$ in one circuit causes a changing magnetic flux in a second circuit, an emf $\mathcal{E}_2$ is induced in the second circuit. Likewise, a changing current $i_2$ in the second circuit induces an emf $\mathcal{E}_1$ in the first circuit. If the circuits are coils of wire with $N_1$ and $N_2$ turns, the mutual inductance $M$ can be expressed in terms of the average flux $\Phi_{B2}$ through each turn of coil 2 caused by the current $i_1$ in coil 1, or in terms of the average flux $\Phi_{B1}$ through each turn of coil 1 caused by the current $i_2$ in coil 2. The SI unit of mutual inductance is the henry, abbreviated H. (See Examples 30.1 and 30.2.)

\[
\mathcal{E}_2 = -M \frac{d i_1}{d t} \quad \text{and} \quad \mathcal{E}_1 = -M \frac{d i_2}{d t} \quad \text{(30.4)}
\]

\[
M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_1 \Phi_{B1}}{i_2} \quad \text{(30.5)}
\]

Self-inductance: A changing current $i$ in any circuit causes a self-induced emf $\mathcal{E}$. The inductance (or self-inductance) $L$ depends on the geometry of the circuit and the material surrounding it. The inductance of a coil of $N$ turns is related to the average flux $\Phi_B$ through each turn caused by the current $i$ in the coil. An inductor is a circuit device, usually including a coil of wire, intended to have a substantial inductance. (See Examples 30.3 and 30.4.)

\[
\mathcal{E} = -L \frac{d i}{d t} \quad \text{(30.7)}
\]

\[
L = \frac{N \Phi_B}{i} \quad \text{(30.8)}
\]

Magnetic-field energy: An inductor with inductance $L$ carrying current $I$ has energy $U$ associated with the inductor’s magnetic field. The magnetic energy density $u$ (energy per unit volume) is proportional to the square of the magnetic field magnitude. (See Example 30.5.)

\[
U = \frac{1}{2} LI^2 \quad \text{(30.9)}
\]

\[
u = \frac{B^2}{2\mu_0} \quad \text{(in vacuum)} \quad \text{(30.10)}
\]

\[
u = \frac{B^2}{2\mu} \quad \text{(in a material with magnetic permeability $\mu$)} \quad \text{(30.11)}
\]
**R-L circuits:** In a circuit containing a resistor $R$, an inductor $L$, and a source of emf, the growth and decay of current are exponential. The time constant $\tau$ is the time required for the current to approach within a fraction $1/e$ of its final value. (See Examples 30.6 and 30.7.)

$$\tau = \frac{L}{R} \quad (30.16)$$

**L-C circuits:** A circuit that contains inductance $L$ and capacitance $C$ undergoes electrical oscillations with an angular frequency $\omega$ that depends on $L$ and $C$. This is analogous to a mechanical harmonic oscillator, with inductance $L$ analogous to mass $m$, the reciprocal of capacitance $1/C$ to force constant $k$, charge $q$ to displacement $x$, and current $i$ to velocity $v_x$. (See Examples 30.8 and 30.9.)

$$\omega = \sqrt{\frac{1}{LC}} \quad (30.22)$$

**L-R-C series circuits:** A circuit that contains inductance, resistance, and capacitance undergoes damped oscillations for sufficiently small resistance. The frequency $\omega'$ of damped oscillations depends on the values of $L$, $R$, and $C$. As $R$ increases, the damping increases; if $R$ is greater than a certain value, the behavior becomes overdamped and no longer oscillates. (See Example 30.10.)

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad (30.29)$$