Physics 121 - Electricity and Magnetism
Lecture 3 - Electric Field
Y&F Chapter 21 Sec. 4 – 7

- Recap & Definition of Electric Field
- Electric Field Lines
- Charges in External Electric Fields
- Field due to a Point Charge
- Field Lines for Superpositions of Charges
- Field of an Electric Dipole
- Electric Dipole in an External Field: Torque and Potential Energy
- Method for Finding Field due to Charge Distributions
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- Motion of a charged particle in an Electric Field
- CRT example
Recap: Electric charge

**Basics:**
- Positive and negative flavors. Like charges repel, opposites attract.
- Charge is conserved and quantized. \( e = 1.6 \times 10^{-19} \) Coulombs.
- Ordinary matter seeks electrical neutrality - screening.
- In conductors, charges are free to move around:
  - screening and induction.
- In insulators, charges are not free to move around:
  - but materials polarize.

**Coulomb's Law:** forces at a distance enabled by a field

\[
\vec{F}_{12} = k \frac{q_1 q_2}{r_{12}^2} \vec{r}
\]

**Superposition of Forces or Fields**

\[
\vec{F}_{\text{net on } 1} = \sum_{i=2}^{n} \vec{F}_{1,i} = \vec{F}_{1,2} + \vec{F}_{1,3} + \vec{F}_{1,4} + \ldots
\]

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## Fields

### Scalar Field

**Examples:**
- Temperature - $T(r)$
- Pressure - $P(r)$
- Gravitational Potential energy – $U(r)$
- Electrostatic potential – $V(r)$
- Electrostatic potential energy – $U(r)$

### Vector Field

**Examples:**
- Velocity - $v(r)$
- Gravitational field/acceleration - $g(r)$
- Electric field – $E(r)$
- Magnetic field– $B(r)$
- Gradients of scalar fields

---

**Fields “explain” forces at a distance – space altered by source**

### Gravitational Field versus Electrostatic Field

- **force/unit mass**
  
  \[
  \ddot{g}(\vec{r}) = \lim_{m_0 \to 0} \left( \frac{\ddot{F}_g(\vec{r})}{m_0} \right)
  \]

  *$m_0$ is a “test mass”*

- **force/unit charge**
  
  \[
  \ddot{E}(\vec{r}) = \lim_{q_0 \to 0} \left( \frac{\ddot{F}_e(\vec{r})}{q_0} \right)
  \]

  *$q_0$ is a positive “test charge”*
Field due to a charge distribution

**Test charge q₀:**
- small and positive
- does not affect the charge distribution that produces \( \mathbf{E} \).

A charge distribution creates a field:
- Map \( \mathbf{E} \) field by moving \( q₀ \) around and measuring the force \( \mathbf{F} \) at each point
- \( \mathbf{E}(\mathbf{r}) \) is a vector parallel to \( \mathbf{F}(\mathbf{r}) \)
- \( \mathbf{E} \) field exists whether or not the test charge is present
- \( \mathbf{E} \) varies in direction and magnitude

\[
\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q₀}
\]

\[
\mathbf{F}(\mathbf{r}) = q₀ \mathbf{E}(\mathbf{r})
\]

\( \mathbf{E} = \) Force on test charge \( q₀ \) at point \( \mathbf{r} \) due to the charge distribution
\( \mathbf{E} = \) External electric field at point \( \mathbf{r} \) = Force/unit charge

SI Units: Newtons / Coulomb later: V/m
Electrostatic Field Examples

\[ \vec{E} = \frac{\vec{F}}{q_0} \]

- **Magnitude:** \( E = \frac{F}{q_0} \)
- **Direction:** same as the force that acts on the positive test charge
- **SI unit:** N/C

<table>
<thead>
<tr>
<th>Field Location</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inside copper wires in household circuits</td>
<td>( 10^{-2} ) N/C</td>
</tr>
<tr>
<td>Near a charged comb</td>
<td>( 10^3 ) N/C</td>
</tr>
<tr>
<td>Inside a TV picture tube (CRT)</td>
<td>( 10^5 ) N/C</td>
</tr>
<tr>
<td>Near the charged drum of a photocopier</td>
<td>( 10^5 ) N/C</td>
</tr>
<tr>
<td>Breakdown voltage across an air gap (arcing)</td>
<td>( 3 \times 10^6 ) N/C</td>
</tr>
<tr>
<td>E-field at the electron's orbit in a hydrogen atom</td>
<td>( 5 \times 10^{11} ) N/C</td>
</tr>
<tr>
<td>E-field on the surface of a Uranium nucleus</td>
<td>( 3 \times 10^{21} ) N/C</td>
</tr>
</tbody>
</table>
Electric Field due to a point charge Q

Coulomb's Law
\[ \vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{Qq_0}{r^2} \hat{r} \]

Test charge \( q_0 \)

Find the field \( \vec{E} \) due to point charge \( Q \) as a function over all of space

\[ \vec{F} = q_0 \vec{E} \Rightarrow \]

\[ \vec{E} \equiv \frac{\vec{F}}{q_0} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{r} \]

\[ \hat{r} \equiv \frac{\vec{r}}{r} \]

- Magnitude \( E = KQ/r^2 \) is constant on any spherical shell (spherical symmetry)
- Visualize: E field lines are radially out for +|Q|, in for -|Q|
- Flux through any closed (spherical) shell enclosing Q is the same:
  \[ \Phi = EA = Q \cdot \frac{4\pi r^2}{4\pi \varepsilon_0 r^2} = \frac{Q}{\varepsilon_0} \]
  Radius cancels

The closed (Gaussian) surface intercepts all the field lines leaving Q
Use superposition to calculate net electric field at each point due to a group of individual charges.

Example: for point charges at \( r_1, r_2 \ldots \)

\[
\mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n
\]

\[
\mathbf{E}_{\text{net}} = \frac{\mathbf{F}_{\text{tot}}}{q_0} = \frac{\mathbf{F}_1}{q_0} + \frac{\mathbf{F}_2}{q_0} + \cdots + \frac{\mathbf{F}_n}{q_0}
= \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_n
\]

\[
\mathbf{E}_{\text{net at } i} = \frac{1}{4\pi\varepsilon_0} \sum_{j} \frac{q_i}{r_{ij}^2} \hat{r}_{ij}
\]

Do the sum above for every test point \( i \).
Visualization: Electric field lines (Lines of force)

- Map direction of an electric field line by moving a positive test charge around.
- The tangent to a field line at a point shows the field direction there.
- The density of lines crossing a unit area perpendicular to the lines measures the strength of the field. Where lines are dense the field is strong.
- Lines begin on positive charges (or infinity) and end on negative charges (or infinity).
- Field lines cannot cross other field lines.
DETAIL NEAR A POINT CHARGE

NEAR A LARGE, UNIFORM SHEET OF $+$ CHARGE
- No conductor – just an infinitely large charge sheet
- $E$ approximately constant in the “near field” region ($d \ll L$)

The field has uniform intensity & direction everywhere except on sheet

TWO EQUAL $+$ CHARGES (REPEL)  EQUAL $+$ AND $-$ CHARGES ATTRACT
Field lines for a spherical shell or solid sphere of charge

Shell Theorem Conclusions

Outside point:
  Same field as point charge

Inside spherical distribution at distance $r$ from center:
  • $E = 0$ for hollow shell;
  • $E = \frac{kQ_{\text{inside}}}{r^2}$ for solid sphere
Example: Find $E_{\text{net}}$ at a point on the axis of a dipole

- Use superposition
- Symmetry $\Rightarrow$ $E_{\text{net}}$ parallel to z-axis

$$r^+ = z - d/2 \quad \text{and} \quad r^- = z + d/2$$

$$E_{\text{at } O} = E^+ - E^- = \frac{q}{r^2} - \frac{q}{r^-}$$

- Limitation: $z > d/2$ or $z < -d/2$

$$E_{\text{at } O} = kq \left[ \frac{1}{(z - d/2)^2} - \frac{1}{(z + d/2)^2} \right]$$

$$E_{\text{at } O} = 2kqd \left[ \frac{z}{(z^2 - d^2/4)^2} \right]$$

For $z \gg d$: point “O” is “far” from center of dipole

$$\therefore E_{\text{at } O} \approx \frac{q}{2\pi\varepsilon_0} \frac{1}{z^3} = \frac{p}{2\pi\varepsilon_0} \frac{1}{z^3}$$

- Fields cancel as $d \to 0$ so $E \to 0$
- $E$ falls off as $1/z^3$ not $1/z^2$
- $E$ is negative when $z$ is negative
- Does “far field” $E$ look like point charge?

Exercise: Do these formulas describe $E$ at the point midway between the charges

Ans: $E = -4p/2\pi\varepsilon_0 d^3$
3-2: Put the magnitudes of the electric field values at points A, B, and C shown in the figure in decreasing order.

A) $E_C > E_B > E_A$
B) $E_B > E_C > E_A$
C) $E_A > E_C > E_B$
D) $E_B > E_A > E_C$
E) $E_A > E_B > E_C$
A Dipole in a Uniform EXTERNAL Electric Field

Feels torque - Stores potential energy (See Sec 21.7)

**ASSSUME RIGID DIPOLE**

\[ \vec{p} \equiv qd \]

\[ \vec{\tau} = \vec{p} \times \vec{E} \]

- \(|\tau| = 0\) at \(\theta = 0\) or \(\theta = \pi\)
- \(|\tau| = pE\) at \(\theta = +/- \pi/2\)
- RESTORING TORQUE: \(\tau(-\theta) = \tau(+\theta)\)

**OSCILLATOR**

\[ U = -W \]

\[ U = -\int \tau d\theta = +pE\int \sin(\theta) d\theta = -pE \cos(\theta) \]

\[ U_E = -\vec{p} \cdot \vec{E} \]

- \(U = 0\) for \(\theta = +/- \pi/2\)
- \(U = -pE\) for \(\theta = 0\) minimum
- \(U = +pE\) for \(\theta = \pi\) maximum
3-3: In the sketch, a dipole is free to rotate in a uniform external electric field. Which configuration has the smallest potential energy?

3-4: Which configuration has the largest potential energy?
Method for finding the electric field at point P -
- given a known continuous charge distribution

This process is just superposition

\[ \vec{E}_P = \frac{1}{4\pi\varepsilon_0} \lim_{\Delta q \to 0} \sum_i \frac{\Delta q_i}{r_i^2} \hat{r}_i \Rightarrow \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r} \]

1. Find an expression for \( dq \), the “point charge” within a differentially “small” chunk of the distribution

\[ dq = \begin{cases} \lambda dl & \text{for a linear distribution} \\ \sigma dA & \text{for a surface distribution} \\ \rho dV & \text{for a volume distribution} \end{cases} \]

2. Represent field contributions at P due to a point charge \( dq \) located anywhere in the distribution. Use symmetry where possible.

\[ \Delta \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\Delta q}{r^2} \hat{r} \Rightarrow d\vec{E} = \frac{dq}{4\pi\varepsilon_0 r^2} \hat{r} \]

3. Add up (integrate) the contributions \( d\vec{E} \) over the whole distribution, varying the displacement and direction as needed. Use symmetry where possible.

\[ \vec{E}_P = \int_{\text{dist}} d\vec{E} \text{ (line, surface, or volume integral)} \]
Example: Find electric field on the axis of a charged rod

- Rod has length \( L \), uniform positive charge per unit length \( \lambda \), total charge \( Q \).
  \( \lambda = Q/L \).
- Calculate electric field at point \( P \) on the axis of the rod a distance \( a \) from one end. Field points along \( x \)-axis.

\[
\begin{align*}
\lambda &= \frac{Q}{L} \\
\text{Calculate electric field at point } P \text{ on the axis of the rod a distance } a \text{ from one end. Field points along } x \text{-axis.}
\end{align*}
\]

\[
dq = \lambda \, dx
\]

\[
dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{x^2} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, dx}{x^2}
\]

- Add up contributions to the field from all locations of \( dq \) along the rod (\( x \in [a, L + a] \)).

\[
E = \int_a^{L+a} \frac{\lambda}{4\pi\varepsilon_0} \frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0} \int_a^{L+a} \frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0} \left[-\frac{1}{x}\right]_a^{L+a} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{L} \left(\frac{1}{a} - \frac{1}{L+a}\right)
\]

\[
\therefore \ E = \frac{Q}{4\pi\varepsilon_0 a(L+a)}
\]

- Interpret Limiting cases:
  - \( L \to 0 \) rod becomes point charge
  - \( L << a \) same, \( L/a << 1 \)
  - \( L >> a \) \( a/L << 1 \),
Electric field at center of an ARC of charge

- Uniform linear charge density $\lambda = \frac{Q}{L}$
  
  $$dq = \lambda ds = \lambda R d\theta$$

- P on symmetry axis at center of arc
  → Net E is along y axis → need $E_y$ only
  
  $$d\vec{E}_P = \frac{k dq}{R^2} \hat{r} \rightarrow d\vec{E}_{P,y} = -\frac{k dq \cos(\theta)}{R^2} \hat{j}$$

- Angle $\theta$ is between $-\theta_0$ and $+\theta_0$

- Integrate:
  
  $$\vec{E}_{P,y} = \frac{-1}{4\pi \varepsilon_0} \frac{\lambda}{R} R \int_{-\theta_0}^{+\theta_0} \cos(\theta) d\theta = -\frac{\lambda}{4\pi \varepsilon_0} \frac{1}{R} \sin(\theta) \Bigg|_{-\theta_0}^{+\theta_0}$$

  note: $\int \cos(\theta) d\theta = \sin(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

  $$\therefore \vec{E}_{P,y} = -2k \frac{\lambda}{R} \sin(\theta_0) \hat{j}$$

  In the plane of the arc

- For a semi-circle, $\theta_0 = \frac{\pi}{2}$
  
  $$\vec{E}_{P,y} = -2k \frac{\lambda}{R} \hat{j}$$

- For a full circle, $\theta_0 = \pi$
  
  $$\vec{E}_{P,y} = 0$$
Electric field due to a straight LINE of charge
Point P on symmetry axis, a distance y off the line

- uniform linear charge density: $\lambda = Q/L$
- point “P” is at y on symmetry axis
- by symmetry, total E is along y-axis
  - x-components of dE pairs cancel
- solve for line segment, then let $y << L$

\[
\begin{align*}
dE_P &= \frac{k dq \mathbf{\hat{r}}}{r^2} \\
dE_{P,y} &= -\frac{k dq \cos(\theta) \mathbf{\hat{j}}}{y} \\
dE_{y,P} &= -\frac{k \lambda \cos(\theta) d\theta \mathbf{\hat{j}}}{y} \\
\end{align*}
\]

- “1 + tan$^2$(\theta)” cancels in numerator and denominator

\[
\begin{align*}
E_{y,P} &= -\frac{k \lambda}{y} \mathbf{\hat{j}} \int_{-\theta_0}^{+\theta_0} \cos(\theta) d\theta = -\frac{k \lambda}{y} \mathbf{\hat{j}} \sin(\theta) \bigg|_{-\theta_0}^{+\theta_0} \\
E_P &= -\frac{2k \lambda}{y} \mathbf{\hat{j}} \sin(\theta_0) \\
\end{align*}
\]

- For $y << L$ (wire looks infinite) $\theta_0 \rightarrow \pi/2$

\[
\begin{align*}
E_P &= -\frac{2k \lambda}{y} \mathbf{\hat{j}} \sin(\theta_0) \\
\end{align*}
\]

Finite length wire

Falls off as $1/y$ along $-y$ direction

\[
J_{cp} \neq 0
\]
Electric field due to a RING of charge at point P on the symmetry (z) axis

- Uniform linear charge density along circumference: \( \lambda = Q/2\pi R \)
- \( dq = \lambda ds = \) charge on arc segment of length \( ds = R d\phi \)
- P on symmetry axis \( \rightarrow \) xy components of \( \mathbf{E} \) cancel
- Net \( \mathbf{E} \) field is along z only, normal to plane of ring

\[
\mathbf{dE}_P = \frac{k dq}{r^2} \mathbf{r} \rightarrow \mathbf{dE}_{z,P} = \frac{k dq \cos(\theta)}{r^2} \mathbf{\hat{k}}
\]

\[
dq \equiv \lambda \ ds = \lambda \ R \ d\phi \quad \cos(\theta) = z/r \quad r^2 = R^2 + z^2
\]

\[
d\mathbf{E}_{P,z} = \frac{k \lambda R z d\phi}{r^3} \mathbf{\hat{k}}
\]

- Integrate on azimuthal angle \( \phi \) from 0 to \( 2\pi \)

\[
\mathbf{E}_{P,z} = \frac{k \lambda R z}{[ R^2 + z^2 ]^{3/2}} \mathbf{\hat{k}} \int_0^{2\pi} d\phi
\]

\[
2\pi R \lambda \equiv Q \quad \text{total charge on disk}
\]

\[
\mathbf{E}_{P,z} = \frac{k Q z}{[ R^2 + z^2 ]^{3/2}} \mathbf{\hat{k}}
\]

\( E_z \rightarrow 0 \) as \( z \rightarrow 0 \) (see result for arc)

- Limit: For P “far away” use \( z \gg R \)

\[
\mathbf{E}_{P,z} \rightarrow \frac{k Q}{z^2} \quad \text{Ring looks like a point charge if point P is very far away!}
\]

Exercise: Where is \( E_z \) a maximum?

Set \( dE_z/dz = 0 \)

Ans: \( z = R/\sqrt{2} \)
Electric field due to a DISK of charge for point P on z (symmetry) axis

• Uniform surface charge density on disc in x-y plane
  \[ \sigma = \frac{Q}{\pi R^2} \]

• Disc is a set of rings, each of them dr wide in radius

• P on symmetry axis \( \rightarrow \) net E field only along z

• \( dq = \) charge on arc segment \( rd \phi \) with radial extent \( dr \)

\[
\frac{dA}{r} = \frac{dA}{dr} \quad dq = \sigma \frac{dA}{dr} = \sigma r dr d\phi
\]

\[
\cos(\theta) = \frac{z}{s} \quad s^2 = r^2 + z^2
\]

\[
d\vec{E}_z = \frac{k}{s^2} \cos(\theta) \hat{k} = \frac{1}{4\pi \varepsilon_0} \frac{\sigma z r dr d\phi}{\left[ r^2 + z^2 \right]^{3/2}} \hat{k}
\]

• Integrate twice: first on azimuthal angle \( \phi \) from 0 to \( 2\pi \) which yields a factor of \( 2\pi \)
  then on ring radius \( r \) from 0 to \( R \)

\[
\vec{E}_z = \frac{2\pi \sigma}{4\pi \varepsilon_0} z \int_0^R \frac{r \ dr}{\left[ r^2 + z^2 \right]^{3/2}} \hat{k}
\]

\[
\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{\left( z^2 + R^2 \right)^{1/2}} \right] \hat{k}
\]

Note Anti-derivative
\[
\frac{r}{\left[ r^2 + z^2 \right]^{3/2}} = \frac{d}{dr} \left\{ \frac{-1}{\left[ r^2 + z^2 \right]^{1/2}} \right\}
\]

See Y&F Ex 21.11
Electric field due to a DISK of charge, continued

**Exact Solution:**

\[
\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right] \hat{k}
\]

**Near-Field:** \( z << R \): P is close to the disk. Disk looks like infinite sheet.

For \( z/R << 1 \):

\[
\vec{E}_{disk} \approx \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{R} \right] \hat{k} \approx \frac{\sigma}{2\varepsilon_0}
\]

"near field" is constant - disk approximates an infinite sheet of charge

**Far-Field:** \( R << z \): P is far from the disk. Disk looks like a point charge.

For \( R/z << 1 \):

\[
\vec{E}_{disk} = \frac{\sigma}{2\varepsilon_0} \left[ 1 - \frac{z}{z\left(1 + (R/z)^2\right)^{1/2}} \right] \hat{k} \quad \text{Recall} \quad \sigma = \frac{Q}{\pi R^2}
\]

Series Expansion

\[(1 + s)^n = 1 + ns / 1! + n(n - 1)s^2 / 2! + ...\]

converges quickly for \( s << 1 \)

\[
\therefore \quad \vec{E}_{disk} \approx \frac{\sigma}{2\varepsilon_0} \left[ 1 - 1 + \frac{1}{2} \frac{R^2}{z^2} \right] \hat{k} = \frac{\sigma}{4\varepsilon_0} \frac{R^2}{z^2} \hat{k}
\]

\[
\vec{E}_{disk} \approx \frac{Q}{4\pi\varepsilon_0} \frac{1}{z^2} \hat{k}
\]

Point charge formula
Infinite (i.e. "large") uniformly charged sheet

Non-conductor, fixed surface charge density $\sigma$

Infinite sheet $\rightarrow$ $d << L$ $\rightarrow$ "near field" $\rightarrow$ uniform field

$$E = \frac{\sigma}{2\varepsilon_0}$$ for infinite, non-conducting charged sheet

Method: solve non-conducting disc of charge for point on z-axis then approximate $z << R$
Motion of a Charged Particle in a Uniform Electric Field

\[ \vec{F} = q \vec{E} \]

\[ \vec{F}_{\text{net}} = m \vec{a} \]

- Stationary charges produce an electric field at the location of charge \( q \).
- Acceleration \( \vec{a} \) is parallel or anti-parallel to \( \vec{E} \).
- Acceleration is \( \vec{F}/m \) not \( \vec{F}/q = \vec{E} \).
- Acceleration is the same everywhere in a uniform field.

Example: Early CRT tube with electron gun and electrostatic deflector.

- Electrons are negative so acceleration \( \vec{a} \) and electric force \( \vec{F} \) are in the direction opposite the electric field \( \vec{E} \).

Heated cathode (- pole) “boils off” electrons from the metal (thermionic emission). The electron gun controls intensity. The electrostatic deflector plates control the direction of the electron beam.
Motion of a Charged Particle in a Uniform Electric Field

\[ \vec{F} = q\vec{E} \quad \vec{F}_{\text{net}} = m\vec{a} \]

Electrons are negative so acceleration \( a \) and electric force \( F \) are in the direction opposite the electric field \( E \).

\( \Delta y \) is the DEFLECTION of the electron as it crosses the field.

Acceleration has only a constant \( y \) component. \( v_x \) is constant, \( a_x = 0 \)

\[ a_y = -\frac{eE}{m} \]

\( v_x \) yields time of flight \( \Delta t \)

\[ v_x = \frac{L}{\Delta t} \]

Kinematics: ballistic trajectory

Measure deflection, find \( \Delta t \) via kinematics. Evaluate \( v_y \) & \( v_x \)

\[ \Delta y = \frac{1}{2} a_y \Delta t^2 = -\frac{1}{2} \frac{eE}{m} \Delta t^2 \]

\[ v_y = a_y \Delta t \]

\[ v = \left( v_x^2 + v_y^2 \right)^{1/2} \]

Use to find \( e/m \)

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