Linear stability analysis of thin viscoelastic liquid of Jeffreys type with van der Waals interaction

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Abstract

We study the linear instability of a thin viscoelastic liquid film under the influence of van der Waals interaction. The Jeffreys model is used to describe the viscoelasticity with a relaxation time and a retardation time. We use the thin film equation that governs the nonlinear evolution of the interface and study the linear stability of the interface in the long-wave limit. We include the dewetting effect through the van der Waals attractive-repulsive force. The model is simplified considering the fluid in regime of weak-slip. The role of the liquid viscoelasticity as well as the contact angle are studied. We also study the influence of the slippage on the length scale and time scale of the instability.

Introduction

We simplify the generalized Maxwell model of Jeffreys type for the moving interface of viscoelastic liquids in the $2D$ lubrication approximation. We study the effects of the perturbation of a thin film of fluid in the presence of van der Waals forces. Our investigations are motivated by applications of thin polymer films as in semi-conductors, solar cells, etc. We drive our analysis in the case of weak-slip regime and see how the slippage together with the viscoelasticity affect the instability. A thin film of fluid of constant initial thickness h_0 is perturbed and the linear stability analysis on the governing equation describes whether the film breaks up into separate rims (instability) or return to its initial configuration (stability). We use numerical simulations to confirm the theoretical analysis in the case of absence of slippage and of retardation time and relaxation time due to viscoelasticity.

Governing Equations

The equation governing the hydrodinamics for the fluid interface of viscoelastic media is derived as a long-wave approximation of the conservation laws. The liquid is considered incompressible, with mass density ρ. The equation of conservation of mass and continuity of momentum are:

$$\nabla \cdot \mathbf{u} = 0\,,$$
$$\rho\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p_R + \nabla\cdot\tau\,,$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector field, and p_R the reduced pressure such that $p_R = p - \Pi$, where p is the hydrostatic pressure, while Π is the pressure induced by body forces of van der Waals type (attractive or repulsive). The stress tensor τ follows the Jeffreys model for viscoelastic fluids, which describes the nonlinear relation $\tau(\dot\gamma)$ between the stress tensor τ and the *strain rate* $\dot\gamma$. This dependence in the linear Jeffreys model interpolates between a purely elastic and a purely viscous behavior (respectively in the two members of the equation):

$$\tau + \lambda_1\partial_t\tau = \eta(\dot\gamma + \lambda_2\partial_t\dot\gamma) \qquad (1)$$

in which η is the shear viscosity coefficient and $\lambda_1,\ \lambda_2$ are the two relaxation times of the liquid when it shrinks back to its original shape after deformation, with generally $\lambda_1 > \lambda_2$. λ_1 is called *relaxation time* and λ_2 *retardation time*. In figure 1 we can see a scheme of the fluid's interface. At the solid substrate we have Navier boundary conditions where $b \geq 0$ is the *slip length* ($b = 0$ means no slip, and $b \gg 1$ means strong-slip).

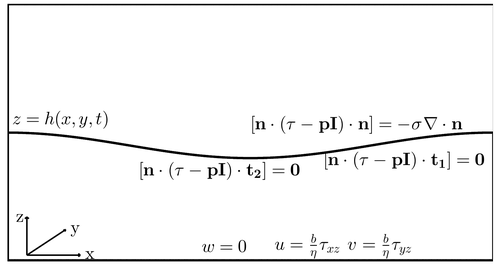

Figure 1: *Scheme of the fluid interface and boundary conditions.*

We nondimensionalize these equations, using:

$$(x, y) = L(x^*, y^*),\ z = Hz^*,\ (u, v) = U(u^*, v^*),\ w = \varepsilon U w^*$$
$$t = Tt^*, \text{ with } T = \frac{L}{U} \text{ and } \frac{H}{L} = \varepsilon\,,$$

where ε is the small parameter. In the weak-slip regime the slip length $b = O(1)$ and the pressure is scaled as $PH/\eta U \sim \varepsilon^{-1}$ [4]. Keeping only $O(1)$ terms in the boundary conditions we obtain $p_R = -\nabla^2 h - \Pi$ and using this together with the kinematic boundary condition into the governing equations (dropping the *) leads to the closed form equation for the fluid's interface:

$$(1+\lambda_2\partial_t)h_t + (\lambda_2-\lambda_1)\nabla\cdot\left[\left(\frac{h^2}{2}\mathbf{Q} - h\mathbf{R}\right)h_t\right] = \nabla\cdot\left\{\left[(+\lambda_1\partial_t)\frac{h^3}{3}\nabla p_R + (1+\lambda_2\partial_t)bh^2\nabla p_R\right]\right\} \quad (2)$$

where $\mathbf{Q}$ and $\mathbf{R}$ satisfy $\mathbf{Q} + \lambda_2\mathbf{Q}_t = \nabla p_R$, $\mathbf{R} + \lambda_2\mathbf{R}_t = h\nabla p_R$ and the van der Waals potential is defined by: $\Pi(h) = \dfrac{\sigma(1-cos\theta)}{Mh_\star}\left[\left(\dfrac{h_\star}{h}\right)^n - \left(\dfrac{h_\star}{h}\right)^m\right]$, with θ the contact angle, $M = (n-m)/[(m-1)(n-1)]$ (generally $n > m$) [2], and σ the surface tension.

Linear Stability Analysis

To study the film's response to a perturbation we consider $h = h_0 + \delta h_0 e^{ikx+\omega t}$, $Q = \delta Q_1$, $R = \delta R_1$, where h_0 is the flat initial thickness, k the wave number $k = 2\pi/\lambda$, and ω the growth rate. Using these into equation (2) and keeping only terms up to $O(\delta)$, we obtain the following disperion/dissipation relation:

$$\lambda_2\omega^2 + \left[1 + (k^4 - k^2\Pi'(h_0))\left(\lambda_1\frac{h_0^3}{3} + \lambda_2 bh_0^2\right)\right]\omega + (k^4 - k^2\Pi'(h_0))\left(\frac{h_0^3}{3} + bh_0^2\right) = 0\,. \quad (3)$$

Solving for the two roots of this quadratic equation we obtain one root strictly negative, let us say ω_2, and one root with varying sign, call it ω_1. The latter one is positive (unstable) for $-\sqrt{\Pi'(h_0)} < \omega_1 < \sqrt{\Pi'(h_0)}$. The most unstable mode is given by $k_m = \pm\sqrt{\Pi'(h_0)/2}$. Therefore from the definition above we can see that both $k_c = \pm\sqrt{\Pi'(h_0)}$ and k_m do not depend on the viscoelasticity times λ_1 and λ_2 and neither on the slip length b.

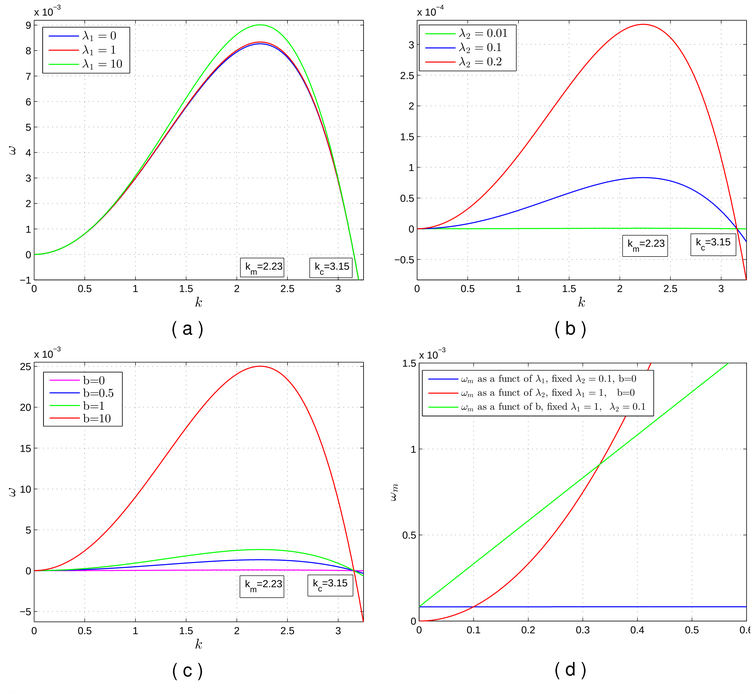

Figure 2: *(a) (b) and (c): Influence of the dispersion curve $\omega_1(k)$ on its parameters; (d): The direct dependence of $\omega_1(k)$ on λ_1, λ_2 and b respectively.*

In figure 2(a) and 2(b) we have plotted the dispersion curve $\omega_1(k)$ for fixed values of $h_0 = 0.1$, $h_\star = 0.01$, $b = 0$ and $\lambda_2 = 0$ (for 2(a)) and $\lambda_1 = 1$ for 2(b) respectively. In 2(c) we see the effect of slippage for fixed values of λ_1 and λ_2: the film breaks up faster with a stronger slip and slower in absence of slip. The difference of behaviors are so marked that we need a new formulation of our governing equation in the case of $b \gg O(1)$. In figure 2(d) we fixed $k_m = 2.23$, $h_0 = 0.1$, and $h_\star = 0.01$ and see how ω_m varies depending on λ_1, λ_2 and b respectively. The linear dependence on λ_1 is less pronounced that the one on b, suggesting that slippage has a stronger effect on the time of the break up of the interface than the elasticity of the fluid itself.

Numerical Results

We developed simulations of the evolution of the film in the $1D$ case of absence of viscoelasticity and slippage, where the governing equation (2) reduces to: $h_t = -\dfrac{\partial}{\partial x}\left[\dfrac{h^3}{3}\left(h_{xxx} + \Pi'(h)\right)\right]$. The numerical method uses Newton linearization of the nonlinear term and (implicit) Crank-Nicolson and central finite differences for the time and spacial derivatives respectively [3]. In figure 3(a) we see the evolution in time of a film of initial thickness $h_0 = 0.2$: the liquid's interface is perturbed and it does not returns to its flat profile, but it breaks up into two separate rims. The instability is due to van der Waals forces' interaction with a precursor film $h_\star = 0.05$ and contact angle $\theta = 45°$.

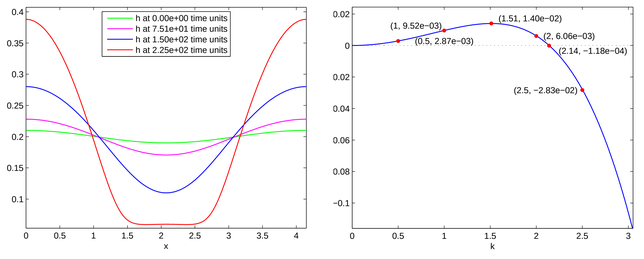

Figure 3: *(a): Profile evolution in time; (b): Comparison of growth rates given by numerical results with the ones derived in the linear stability analysis.*

While in figure 3(b) we compare the growth rates ω of the amplitude of the interface of the film obtained by numerical simulations (red dots) with the theoretical growth rates given by the LSA analysis (blue solid line) for selected stable and unstable modes. We see how the most unstable mode has a higher growth rate (shorter time of break up) respect to the stable mode (longer time for stability).

Conclusions and Future Work

The numerical results of our simulations are in agreement with the linear stability analysis. In our future work we will implement the full nonlinear equation (2) in the weak-slip regime and investigate how the transition from weak to moderate to strong-slip regimes affects the instability together with the viscoelastic effects.

References

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