These notes are intended as an addition to the lectures given in class. They are NOT designed to replace the actual lectures. Some of the notes will contain less information then in the actual lecture, and some will have extra info; some of the graphics is deliberately unfinished, so that we have what to do in class.. Not all formulas which will be needed for exams are contained in these notes. Also, these notes will NOT contain any up to date organizational or administrative information (changes in schedule, assignments, etc.) but only physics. If you notice any typos - let me know at vitaly@oak.njit.edu. I will keep all notes in a single file - each time you can print out only the added part. Make sure the file is indeed updated, there is a date indicating the latest modification. There is also a Table of Contents, which is automatically updated.

Lecture Notes for Phys 106 " Mechanics B"

Dr. Vitaly A. Shneidman
New Jersey Institute of Technology

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Contents

Chapter 1

Review: Mechanics A

1.1 Vectors

A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("rule of parallelogram").

This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are time, distance, mass, kinetic energy. Examples of vectors are the displacement, velocity and force.

1.1.1 Single vector

Consider a vector $\vec{a}$ with components a_x and a_y (let's talk 2D for a while). There is an associated scalar, namely the magnitude (or length) given by the Pythagoras theorem

$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2} \tag{1.1}$$

Note that for a different system of coordinates with axes x', y' the components $a_{x'}$ and $a_{y'}$ can be very different, but the length in eq. (1.1) , obviously, will not change, which just means that it is a scalar.

Another operation allowed on a single vector is multiplication by a scalar. Note that the physical dimension (”units”) of the resulting vector can be different from the original, as in $\vec{F} = m\vec{a}$.

1.1.2 Two vectors: addition

For two vectors, $\vec{a}$ and $\vec{b}$ one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x \ , \ \ c_y = a_y + b_y \tag{1.2}$$

The magnitude of $\vec{c}$ then follows from eq. (1.1). Note that physical dimensions of $\vec{a}$ and $\vec{b}$ must be identical.

1.1.3 Two vectors: scalar (dot) product

If $\vec{a}$ and $\vec{b}$ make an angle ϕ with each other, their scalar (dot) product is defined as $\vec{a} \cdot \vec{b} = ab \cos(\phi)$, or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{1.3}$$

A different system of coordinates can be used, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$. The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta\vec{r}$, with $\Delta\vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!
Example: Prove the Pythagoras theorem $c^2 = a^2 + b^2$.

1.2 Kinematics and Dynamics

Point mass (”point”) - a body whose size is insignificant in a given problem. (Can be as large as a planet). Position is given by $\vec{r}(t)$ with

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) \tag{1.4}$$

known as displacement.

1.2.1 Kinematics

Velocity

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{1.5}$$

Speed: $v = |\vec{v}|$

Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} \tag{1.6}$$

Examples:

- 1 dimensional motion with constant acceleration (in x-direction)

$$v = v_0 + at \tag{1.7}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{1.8}$$

$$x - x_0 = \frac{v^2 - v_0^2}{2a} \tag{1.9}$$

- circular motion with constant speed

$$|\vec{a}| = const = \frac{v^2}{r} \tag{1.10}$$

1.2.2 Dynamics

2d Law:

$$\vec{F} = m\vec{a} \tag{1.11}$$

3d Law:

$$\vec{F}_{12} = -\vec{F}_{21} \tag{1.12}$$

1.2.3 Work and potential energy

$$W = \int_1^2 \vec{F} \cdot d\vec{r} \tag{1.13}$$

Fo special ("conservative") forces the shape of the path does not matter, and we can introduce potential energy $U(\vec{r})$

$$W = -\Delta U = U_1 - U_2 \tag{1.14}$$

Examples:

- Gravitational (with $\vec{g} \simeq const$)

$$U = mgh \tag{1.15}$$

- Elastic (with $F = -kx$)

$$U = \frac{1}{2}kx^2 \tag{1.16}$$

1.2.4 Kinetic energy, work-energy theorem and energy conservation

Kinetic energy:

$$K = \frac{1}{2}mv^2 \tag{1.17}$$

work-energy theorem:

$$\Delta K = W \tag{1.18}$$

(any force). For conservative force:

$$E = K + U = const \tag{1.19}$$

the energy conservation.

1.3 Momentum

$$\vec{p} = m\vec{v} \tag{1.20}$$

2d Law:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{1.21}$$

System of particles

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots \tag{1.22}$$

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext} \tag{1.23}$$

(external forces only!). If $F_{ext} = 0$:

$$\vec{P} = const \tag{1.24}$$

i.e. conservation of momentum.

1.4 Center of mass

$$\vec{R}_{cm} = \frac{1}{M}\sum_i \vec{r}_i m_i \ , \quad M = \sum_i m_i \tag{1.25}$$

$$\vec{V}_{cm} = \vec{P}/M \tag{1.26}$$

which is constant if $F_{ext} = 0$.

Chapter 2

Kinematics of rotation

2.1 Radian measure of an angle

see Fig. 2.1. Arc length

$$l = r\theta \tag{2.1}$$

if θ is measured in radians.

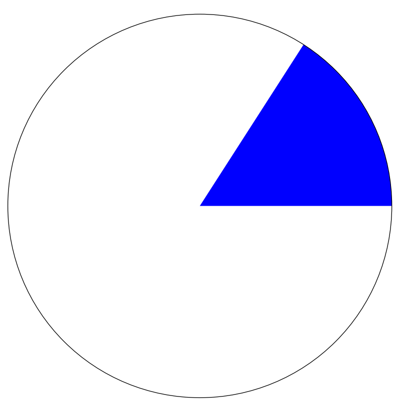

Figure 2.1: Angle of $1\,rad \approx 57.3^o$. For this angle the length of the circular arc exactly equals the radius. The full angle, 360^o, is 2π radians.

2.2 Angular velocity

Notations: ω (omega)

Units: rad/s

Definition:

$$\omega = \frac{d\theta}{dt} \approx \frac{\Delta\theta}{\Delta t} \tag{2.2}$$

Conversion from revolution frequency ($\omega = const$):
Example: find ω for $45\, rev/min$

$$45\,\frac{rev}{min} = 45\,\frac{2\pi\, rad}{60\, s} \approx 4.7\frac{rad}{s}$$

2.2.1 Connection with linear velocity and centripetal acceleration for circular motion

$$v = \frac{dl}{dt} = \frac{d(r\theta)}{dt} = \omega r \tag{2.3}$$

$$a_c = v^2/r = \omega^2 r \tag{2.4}$$

2.3 Angular acceleration

Definition:

$$\alpha = \frac{d\omega}{dt} \tag{2.5}$$

Units:

$$[\alpha] = rad/s^2$$

2.3.1 Connection with tangential acceleration

$$a_\tau = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{2.6}$$

(very important for rolling problems!)

2.4 Rotation with $\alpha = const$

Direct analogy with linear motion:

$$x \;\rightarrow\; \theta \;,\quad v \;\rightarrow\; \omega \;,\quad a \;\rightarrow\; \alpha$$

$$\omega = \omega_0 + \alpha t \tag{2.7}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{2.8}$$

$$\theta - \theta_0 = \frac{\omega^2 - \omega_0^2}{2\alpha} \tag{2.9}$$

Example: a free spinning wheel makes N revolutions in t seconds, and stops. Find α. *Solution*: in formulas

$$\omega = 0\,,\ \theta - \theta_0 = 2\pi N$$

Thus,

$$\omega_0 = -\alpha t$$

and

$$2\pi N = -\frac{1}{2}\alpha t^2\,, \ldots$$

Chapter 3

Kinetic Energy of Rotation and Rotational Inertia

3.1 The formula $K = 1/2\, I\omega^2$

For any point mass

$$K_i = \frac{1}{2} m_i v_i^2 \tag{3.1}$$

For a solid rotating about an axis

$$v_i = \omega r_i \tag{3.2}$$

with r_i being the distance from the axis and ω, the angular velocity being *the same* for every point. Thus, the full kinetic energy is

$$K = \sum_i K_i = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \equiv \frac{1}{2} I\omega^2 \tag{3.3}$$

Here I, the rotational inertia, is the property of a body, independent of ω (but sensitive to selection of the rotational axis):

$$I = \sum_i m_i r_i^2 \tag{3.4}$$

Or, for continuos distribution of masses:

$$I = \int dl\, \lambda r^2 \tag{3.5}$$

for a linear object with linear density λ (in kg/m);

or

$$I = \int dS\, \sigma r^2 \tag{3.6}$$

for a flat object (S-area) with planar density σ (in kg/m^2),

or

$$I = \int dV\, \rho r^2 \tag{3.7}$$

for a 3D object (V-volume) with density ρ (in kg/m^3)

3.2 Rotational Inertia: Examples

3.2.1 Dumbell

Two identical masses m at $x = \pm a/2$. Rotation in the xy plane about the z-axis

$$I = 2 \cdot m(a/2)^2 = ma^2/2 = \frac{1}{4}Ma^2\ , \quad M = m + m \tag{3.8}$$

3.2.2 Hoop

Hoop of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis.

Linear density

$$\lambda = M/(2\pi R)$$

$$I_{hoop} = \int_0^{2\pi R} dl\, \lambda R^2 = MR^2 \tag{3.9}$$

(the same for hollow cylinder about the axis)

3.2.3 Rod

Uniform rod of mass M between at $x = \pm l/2$. Rotation in the xy plane about the z-axis through the center of mass

Linear density

$$\lambda = M/l$$

Thus,

$$I = \int_{-l/2}^{l/2} dx\, \lambda x^2 = 2\lambda \cdot \int_0^{l/2} dx\, x^2 = 2\lambda \frac{1}{3}(l/2)^3$$

Or,

$$I_{rod} = \frac{Ml^2}{12} \tag{3.10}$$

3.2.4 Disk

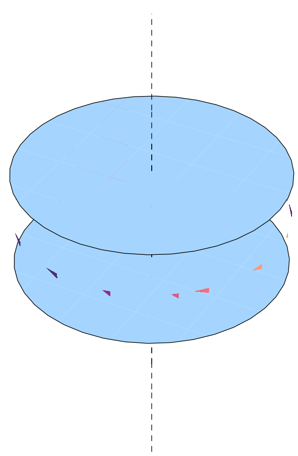

Figure 3.1: Uniform disk (short cylinder) rotating about the z-axis.

Uniform disk of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis - see fig. 3.1.

Planar density

$$\sigma = M/\left[\pi R^2\right]$$

Elementary area

$$dS = 2\pi r\, dr$$

$$I_{disk} = \int_0^R dr\, 2\pi r \sigma r^2 = \frac{1}{2} M R^2 \tag{3.11}$$

(the same for solid cylinder about the axis).

3.3 Parallel axis theorem

$$I = I_{cm} + MD^2 \tag{3.12}$$

with I_{cm} being rotational inertia about a parallel axis passing through the center of mass and D - distance to that axis.

Proof:

$$\vec{R}_{cm} = \frac{1}{M} \sum_i \vec{r}_i m_i$$

Introduce

$$\vec{r}_i' = \vec{r}_i - \vec{R}_{cm}$$

with

$$\sum_i \vec{r}_i' m_i = 0$$

and

$$I_{cm} = \sum_i m_i \left(r_i'\right)^2$$

Now

$$I = \sum_i m_i \left(\vec{r}_i' + \vec{D}\right)^2 = I_{cm} + MD^2 + 2\vec{D} \cdot \sum_i \vec{r}_i' m_i$$

where the last sum is zero, which completes the proof.

3.4 Rotational Inertia: standard Summary and more complicated objects

graphics in a separate file graphics3.4

Chapter 4

Conservation of energy

$$K + U = const \tag{4.1}$$

where K is the *total* kinetic energy (translational and rotational for all bodies) and U is total potential energy.

4.1 Atwood machine

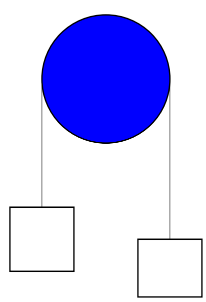

Figure 4.1: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

Suppose larger mass goes distance h down starting from rest. Find final velocity v.

- energy conservation

$$\frac{1}{2}(M+m)v^2 + \frac{1}{2}I\omega^2 = (M-m)gh$$

- constrains

$$v = \omega R$$

Thus

$$v^2 = 2gh\frac{M-m}{M+m+I/R^2}$$

Acceleration from

$$h = v^2/2a$$

which gives

$$a = g\frac{M-m}{M+m+I/R^2}$$

Note the limit: $m = 0$, $I = 0$ gives $a = g$ (free fall).

4.2 Rolling

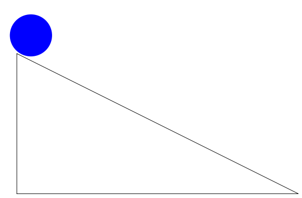

Figure 4.2: Rolling down of a body with mass m, rotational inertia I and radius R.

Suppose the body rolls vertical distance h starting from rest. Find final velocity v.

- energy conservation

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

- constrains

  $$v = \omega R$$

  Thus

  $$v^2 = 2gh \frac{1}{1 + I/\left(mR^2\right)}$$

  Acceleration from

  $$h = v^2/2l \,, \; l = h/\sin\theta$$

  which gives

  $$a = g\sin\theta \frac{1}{1 + I/\left(mR^2\right)}$$

  Note the limit: $I = 0$ gives $a = g\sin\theta$.

Chapter 5

Torque

5.1 Definition

Consider a point mass m at a fixed distance r from the axis of rotation. Only motion in tangential direction is possible. Let F_t be the tangential component of force. The *torque* is defined as

$$\tau = F_t r = F r \sin\phi \tag{5.1}$$

with ϕ being the angle between the force and the radial direction. $r\sin\phi$ is the "lever arm". Counterclockwise torque is positive, and for several forces torques add up.

Units:

$$[\tau] = N \cdot m$$

5.2 2d Law for rotation

Start with a single point mass. Consider the tangential projection of the 2d Law

$$F_t = ma_t$$

Now multiply both sides by r and use $a_t = \alpha r$ with α the angular acceleration.

$$\tau = mr^2\alpha$$

For a system of particles m_i each at a distance r_i and the same α

$$\sum \tau = I\alpha \tag{5.2}$$

Chapter 6

Application of $\tau = I\alpha$

Example. *How long will it take to open a heavy, freely revolving door by 90 degrees starting from rest, if a constant force F , which is perpendicular to the door, is applied at a distance r away from the hinges? Make some reasonable approximations about parameters of the door, F and r.*
Solution:
From

$$\theta = 1/2\,\alpha t^2$$

$$t = \sqrt{2\theta/\alpha} = \sqrt{\pi \frac{I}{Fr}}$$

For I can use $1/3\,Ml^2$, with l being the horizontal dimension. Using, e.g. $M = 30\,kg$, $F = 30\,N$, $r = l = 1\,m$ one gets $t \sim 1\,s$, which is reasonable.

6.1 Rotating rod

(from textbook).

$$\alpha = \tau/I = (1/2\,Mgl) \Big/ \left(\frac{1}{3}Ml^2\right) = \frac{3}{2}g/l$$

Linear acceleration of the end:

$$a = \alpha l = \frac{3}{2}g \;>\; g \text{ (!)}$$

6.1.1 Rotating rod with a point mass m at the end.

Will it go faster or slower?

Solution:

same as above, but

$$I \;\rightarrow\; 1/3\,Ml^2 + ml^2\,, \;\; \tau \;\rightarrow\; 1/2\,Mgl + mgl$$

Thus,

$$\alpha = \frac{3}{2}\frac{g}{l}\,\frac{1+2m/M}{1+3m/M}$$

Linear acceleration of the end:

$$\alpha = \frac{3}{2}g\frac{1+2m/M}{1+3m/M}$$

(which is smaller than before, but still larger than g)

6.2 Atwood machine revisited.

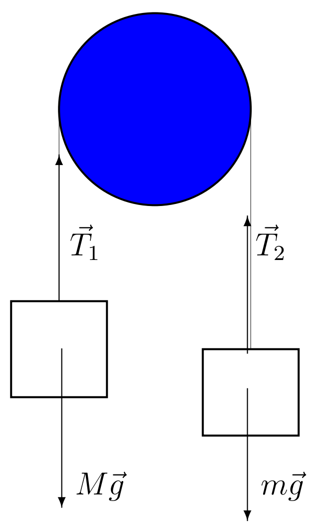

Figure 6.1: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

Let T_1 and T_2 be tensions in left string (connected to larger mass M) and in the right string, respectively.

- 2d Law(s) for each body (and for the pulley with $\tau = (T_1 - T_2)\,R$)
- constrains $a = \alpha R$

From 2d Law(s):

$$Mg - T_1 = Ma\,,\ T_2 - mg = ma\,,\ (T_1 - T_2) = I\alpha/R$$

Add all 3 together to get (with constrains)

$$(M - m)g = (M + m)a + I\alpha/R = (M + m + I/R^2)a$$

which gives

$$a = g\frac{M - m}{M + m + I/R^2}$$

and $\alpha = a/R$. What if need tension?

$$T_1 = Mg - ma\ <\ Mg\ ,\ \ T_2 = mg + ma\ >\ mg$$

6.3 Rolling down incline revisited.

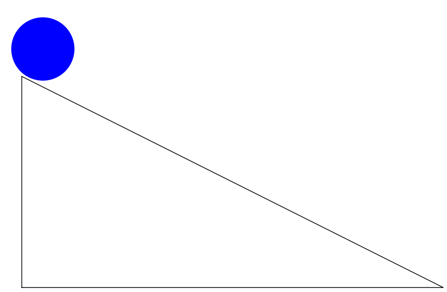

Figure 6.2: Rolling down of a body with mass m, rotational inertia I and radius R. Three forces act on the body: $\vec{f}$ - static friction at the point of contact, up the plane; $\vec{N}$ - normal reaction, perpendicular to the plain at the point of contact and $m\vec{g}$ is applied to CM. Note that onlly friction has a torque with respect to CM.

- 2d Law(s) for linear and for rotational accelerations with torque $\tau = fR$ (f - static friction)

- constrains $a = \alpha R$

2d Law (linear)

$$\vec{f} + \vec{N} + m\vec{g} = m\vec{a}$$

or with x-axis down the incline

$$-f + mg\sin\theta = ma$$

2d Law (rotation)

$$fR = I\alpha$$

or with constrain

$$f = aI/R^2$$

Thus,

$$-aI/R^2 + mg\sin\theta = ma$$

or

$$a = g\sin\theta\,\frac{1}{1 + I/mR^2}$$

the same as from energy conservation.

Chapter 7

Vectors and angular momentum

7.1 Vector (cross) product

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 7.1. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands). If $\vec{a}$ and $\vec{b}$ make an angle ϕ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude $c = ab\sin(\phi)$. The direction is defined as perpendicular to both $\vec{a}$ and $\vec{b}$ using the following rule: curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ in the shortest direction (i.e., the angle must be smaller than 180^o). Then the thumb points in the $\vec{c}$ direction. Check with Fig. 7.2.

Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

7.1.1 Unit-vector representation

Suppose now a system of coordinates is introduced with unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ pointing in the x, y and z directions, respectively. First of all, if $\hat{\imath}$, $\hat{\jmath}, \hat{k}$ are written "in a ring", the cross product of any two of them equals the third

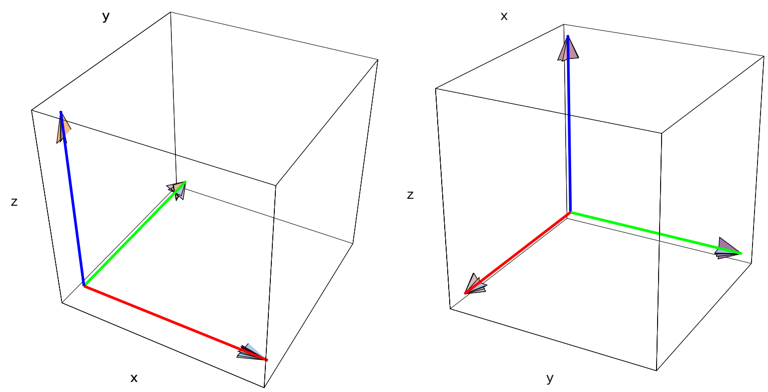

Figure 7.1: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x (red) to y (green), then the thumb should point into the z direction (blue). (Note that axes labeling of the figures is outside of the boxes, not necessarily near the corresponding axes; also, for the figure on the right the origin of coordinates is at the *far* end of the box, if it is hard to see in your printout).

one in clockwise direction, i.e.

$$\hat{\imath} \times \hat{\jmath} = \hat{k}\,,\ \hat{\jmath} \times \hat{k} = \hat{\imath}\,,\ \hat{k} \times \hat{\imath} = \hat{\jmath}$$

, etc. (check this for Fig. 7.1 !). Together with

$$\hat{\imath} \times \hat{\imath} = \hat{\imath} \times \hat{\imath} = \hat{\imath} \times \hat{\imath} = 0$$

and distributive law, this allows to find any vector product.

Example. $\vec{a} = 3\hat{\imath} - 4\hat{\jmath}\,,\ \vec{b} = \hat{\imath} + 2\hat{\jmath}$

$$\vec{a} \times \vec{b} = \left(3\hat{\imath} - 4\hat{\jmath}\right) \times \left(\hat{\imath} + 2\hat{\jmath}\right) = 3 \cdot 2\,\hat{\imath} \times \hat{\jmath} - 4\hat{\jmath} \times \hat{\imath}$$

Note that self-products like $\hat{\imath} \times \hat{\imath}$ are ignored. The 1st term gives $6\hat{k}$, the 2d gives $-4 \cdot \left(-\hat{k}\right) = 4\hat{k}$. Thus,

$$\vec{a} \times \vec{b} = 10\hat{k}$$

Note that if both vectors are in the $x - y$ plane their cross product is in z-direction.

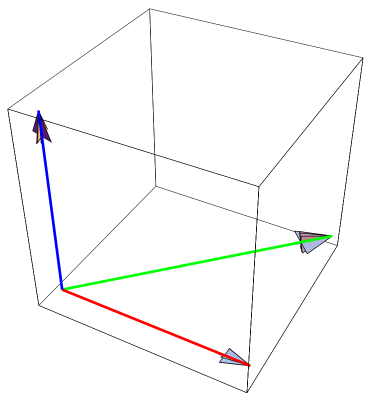

Figure 7.2: Example of a cross product $\vec{c}$ (blue) = $\vec{a}$ (red) × $\vec{b}$ (green). (If you have no colors, $\vec{c}$ is vertical in the example, $\vec{a}$ is along the front edge to lower right, $\vec{b}$ is diagonal).

More generally, the cross product is now expressed as a 3-by-3 determinant

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \quad (7.1)$$

The two-by-two determinants can be easily expanded.

7.2 Angular velocity as a vector

Direct $\vec{\omega}$ allong the axis of rotation using the right-hand rule.

Example. Find $\vec{\omega}$ for the spinning Earth.
Solution: Direction - from South to North pole. Magnitude:

$$\omega = \frac{2\pi\, rad}{24\cdot 3600\, s} \simeq \ldots \frac{rad}{s}$$

Now for any point of a rotating solid

$$\vec{v} = \vec{\omega}\times\vec{r}$$

7.3 Torque as a vector

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{7.2}$$

7.4 Angular momentum

7.4.1 Single point mass

$$\vec{L} = \vec{r} \times \vec{p} \tag{7.3}$$

with $\vec{p} = m\vec{v}$, the momentum.

Example. Find $\vec{L}$ for circular motion.

Solution: Direction - along the axis of rotation (as $\vec{\omega}$!). Magnitude:

$$L = mvr \sin 90^o = mr^2\omega$$

or

$$\vec{L} = mr^2\vec{\omega} \tag{7.4}$$

7.4.2 System of particles

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \tag{7.5}$$

7.4.3 Rotating symmetric solid

If axis if rotation is also an axis of symmetry for the body

$$L = \sum_i m_i r_i^2 \omega = I\omega \tag{7.6}$$

or

$$\vec{L} = I\vec{\omega}$$

7.5 2d Law for rotation in terms of $\vec{L}$

Start with

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Then

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{7.7}$$

Dr. Vitaly A. Shneidman, Lectures on Mechanics, NJIT. Start with

Chapter 8

Conservation of angular momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

If $\vec{\tau} = 0$ (no net external torque)

$$\vec{L} = const \tag{8.1}$$

valid everywhere (from molecules and below, to stars and beyond).

For a closed *mechanical* system, thus

$$E = const\,,\ \vec{P} = const\,,\ \vec{L} = const$$

For *any* closed system (with friction, inelastic collisions, break up of material, chemical or nuclear reactions, etc.)

$$E \neq const\,,\ \vec{P} = const\,,\ \vec{L} = const$$

8.1 Examples

8.1.1 Free particle

$$\vec{L}(t) = \vec{r}(t) \times m\vec{v}$$

with $\vec{v} = const$ and

$$\vec{r}(t) = \vec{r}_0 + \vec{v}t$$

Thus, from $\vec{v} \times \vec{v} = 0$

$$\vec{L}(t) = (\vec{r}_0 + \vec{v}t) \times m\vec{v} = \vec{r}_0 \times m\vec{v} = \vec{L}(0)$$

8.1.2 Student on a rotating platform

(in class demo) Let I be rotational inertia of student+platform, and

$$I' \simeq I + 2Mr^2$$

the rotational inertia of student+platform+extended arms with dumbbels (r is about the length of an arm). Then,

$$L = I\omega \;=\; I'\omega'$$

or

$$\omega' = \omega\frac{I}{I'} = \omega\frac{1}{1 + 2Mr^2/I}$$

8.1.3 Chewing gum on a disk

An $m = 5\,g$ object is dropped onto a uniform disc of rotational inertia $I = 2 \cdot 10^{-4}\,kg \cdot m^2$ rotating freely at 33.3 revolutions per minute. The object adheres to the surface of the disc at distance $r = 5\,cm$ from its center. What is the final angular velocity of the disc?

Solution. Similarly to previous example

$$L = I\omega \;=\; I'\omega'$$

or

$$\omega' = \omega\frac{I}{I'} = \omega\frac{I}{I + mr^2} = \omega\frac{1}{1 + mr^2/I}$$

8.1.4 Measuring speed of a bullet

To measure the speed of a fast bullet a mass-less thin rod with the length L with two wood blocks with the masses M at each end, is used. The whole system can rotate in a horizontal plane about a vertical axis through its center. The rod is at rest when a small bullet of mass m and velocity v is fired into one of the blocks. The bullet remains stuck in the block after it hits. Immediately after the collision, the whole system rotates with angular velocity ω. Find v.

Solution - in class

8.1.5 Rotating star (white dwarf)

A uniform spherical star collapses to 0.3% of its former radius. If the star initially rotates with the frequency 1 rev/day what would the new rotation frequency be?

Solution - in class

Chapter 9

Equilibrium

9.1 General conditions of equilibrium

$$\sum \vec{F}_i = 0 \tag{9.1}$$

$$\sum \vec{\tau}_i = 0 \tag{9.2}$$

Theorem. In equilibrium, torque can be calculated about *any* point.

Proof. Let $\vec{r}_i$ determine positions of particles in the system with respect to point O. Selecting another point as a reference is equivalent to a shift of every $\vec{r}_i$ by the same $\vec{r}_o$. Then,

$$\vec{\tau}_{new} = \sum_i (\vec{r}_i + \vec{r}_o) \times \vec{F}_i = \vec{\tau} + \vec{r}_o \times \sum_i \vec{F}_i = \vec{\tau}$$

9.2 Center of gravity

Theorem. For a *uniform* field $\vec{g}$ the center of gravity coinsides with the COM.

Proof. Torque due to gravity is

$$\tau_g = \sum_i \vec{r}_i \times m_i \vec{g} = \left(\sum_i m_i \vec{r}_i \right) \times \vec{g} = M \vec{R}_{CM} \times \vec{g}$$

9.3 Examples

9.3.1 Seesaw

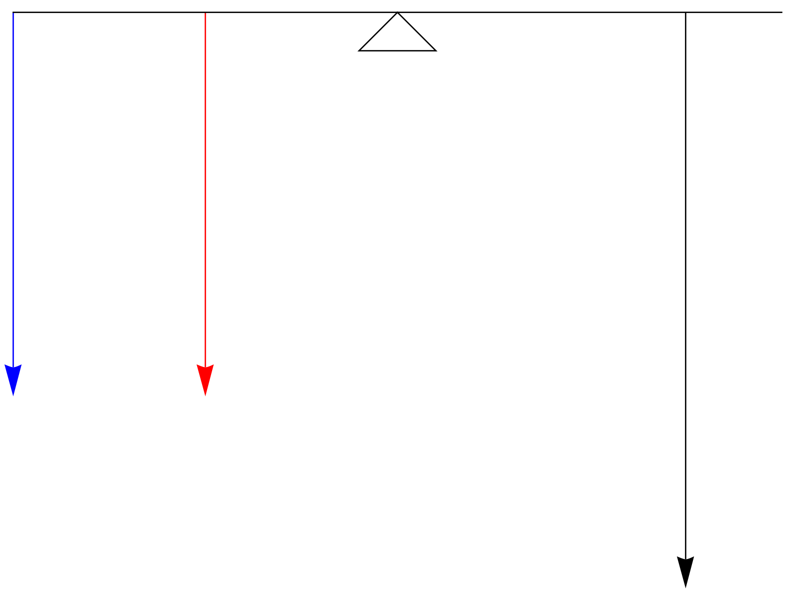

Figure 9.1: Two twins, masses m and m (left) against their dad with mass M. Force of gravity on the seesaw and reaction of the fulcrum are not shown since they produce no torque.

If $2d$, d and D are distances from the fulcrum for each of the twins and the father,

$$mg \cdot 2d + mgd = MgD$$

or

$$3md = MD$$

Note that used only torque condition of equilibrium. If need reaction from the fulcrum $\vec{N}$ use the force condition

$$\vec{N} + (2m + M + M_{seesaw})\,\vec{g} = 0$$

9.3.2 Horizontal beam

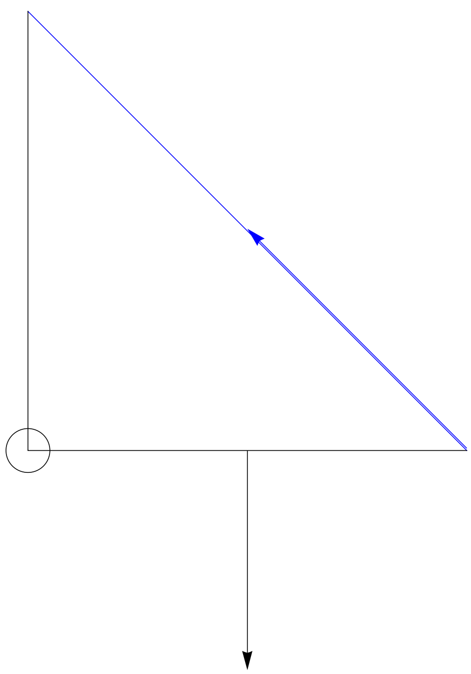

Figure 9.2: Horizontal beam of mass M and length L supported by a blue cord making angle θ with horizontal. Only forces with non-zero torque about the pivot (tension $\vec{T}$ -blue- and gravity $M\vec{g}$ -black) are shown.

Cancellation of torques gives

$$TL\sin\theta = Mg\frac{L}{2}$$

or

$$T = Mg/2\sin\theta$$

Note: if you try to make the cord horizontal, it will snap ($T \to \infty$). For $\theta \to \pi/2$ one has $T \to Mg/2$, as expected. The force condition will allow to find reaction from the pivot.

9.3.3 Ladder against a wall

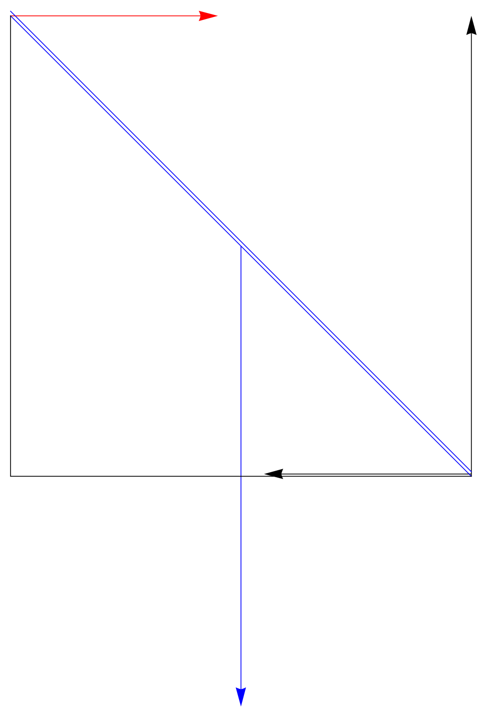

Figure 9.3: Blue ladder of mass M and length L making angle θ with horizontal. Forces: $M\vec{g}$ (blue), wall reaction $\vec{F}$ (red), floor reaction $\vec{N}$ (vertical), friction $\vec{f}$ (horizontal).

Torques about the upper point:

$$NL\sin\left(\frac{\pi}{2}-\theta\right) - fL\sin\theta - Mg\frac{L}{2}\sin\left(\frac{\pi}{2}-\theta\right) = 0$$

But from force equilibrium (vertical) $N = Mg$ and (on the verge) $f = \mu N$. Thus,

$$\frac{1}{2}\cos\theta - \mu\sin\theta = 0$$

Chapter 10

Gravitation

10.1 Solar system

$1\,AU \simeq 150 \cdot 10^6\,km$, about the average distance between Earth and Sun.
Mer - about $1/3\,AU$ (0.39)
V - about $3/4\,AU$ (0.73)
Mars - about $1.5\,AU$ (1.53)
J - about $5\,AU$ (5.2)
... Solar radius - about $0.5\%\,AU$

10.2 The Law

$$F = G\frac{Mm}{r^2} \tag{10.1}$$

or in vector form

$$\vec{F} = -G\frac{Mm\vec{r}}{r^3} \tag{10.2}$$

with $G \approx 6.7 \cdot 10^{-11}\,N \cdot m^2/kg^2$.

10.2.1 Gravitational acceleration

$$g = F/m = G\frac{M}{r^2} = g_s\left(\frac{R}{r}\right)^2 \tag{10.3}$$

with g_s acceleration on the surface and R - radius of the planet. In vector form

$$\vec{g} = -G\frac{M\vec{r}}{r^3} = -\hat{r}g_s\left(\frac{R}{r}\right)^2 \ , \ \ \hat{r} = \frac{\vec{r}}{r} \tag{10.4}$$

10.2.2 Satellite

$$\frac{v^2}{r} = g(r) = g_s\left(\frac{R}{r}\right)^2$$

for $r \simeq R$

$$v \simeq \sqrt{g_s R} \tag{10.5}$$

10.3 Energy

$$U = -G\frac{Mm}{r} \tag{10.6}$$

10.4 Kepler's Laws

in class

10.4.1 Deviations from Kepler's and Newton's laws

in class

Chapter 11

Oscillations

11.1 Introduction: Math

11.1.1 $\sin(x)$, $\cos(x)$ for small x

$$\sin x \simeq x \tag{11.1}$$

error is about $-x^3/6$ and can be neglected for $x \ll 1$ [we will need this for a pendulum]

$$\cos x \simeq 1 - \frac{x^2}{2} \tag{11.2}$$

error is tiny for small x, about $x^4/24$.

11.1.2 Differential equation $\ddot{x} + x = 0$

The equation

$$\ddot{x}(t) + x(t) = 0 \tag{11.3}$$

has a general solution

$$x(t) = B\cos t + C\sin t$$

with arbitrary constants B, C. Can be checked by direct verification (note that $\ddot{x} = -x$). The values of B, C are determined by *initial conditions* $x(0)$ and $\dot{x}(0)$. Alternatively, one can combine sin and cos:

$$x(t) = A\cos(t + \phi)$$

with two constants $A = \sqrt{B^2 + C^2}$, and ϕ.

The equation

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{11.4}$$

is reduced to the above by replacing t with ωt. Thus,

$$x(t) = B\cos(\omega t) + C\sin(\omega t) \tag{11.5}$$

with

$$B = x(0)\,,\ C = \dot{x}(0)/\omega \tag{11.6}$$

Or,

$$x(t) = A\cos(\omega t + \phi)\ ,\ A = \sqrt{B^2 + C^2} \tag{11.7}$$

with A known as the *amplitude* and ϕ the initial *phase*.

11.2 Spring pendulum

Hook's law:

$$F = -kx \tag{11.8}$$

and 2nd Newton's law

$$F = m\ddot{x} \tag{11.9}$$

give eq. (11.4) with

$$\omega = \sqrt{\frac{k}{m}} \tag{11.10}$$

(in radians per second). The oscillation frequency

$$f = \omega/2\pi \tag{11.11}$$

with units $1/s$, or Hz. Period of oscillations

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{11.12}$$

11.2.1 Energy

Kinetic:

$$K(t) = \frac{1}{2}m\left[\dot{x}(t)\right]^2$$

Potential:

$$U(t) = \frac{1}{2}kx(t)^2$$

Total:

$$E = K + U = const$$

11.3 Simple pendulum

Restoring force:

$$F = -mg\sin\theta \simeq -mg\theta$$

Tangential acceleration:

$$a = L\ddot{\theta}$$

Thus

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad (11.13)$$

exactly like eq. (11.4). Thus, the same solution with $x \to \theta$ and

$$\omega^2 = \frac{g}{L} \qquad (11.14)$$

or

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad (11.15)$$

11.4 Physical pendulum

Rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

If l - distance from CM:

$$\tau = -mgl\sin\theta \approx -mgl\theta$$

for small amplitudes.

Thus,

$$\ddot{\theta} + \theta\frac{mgl}{I} = 0 \tag{11.16}$$

which is the same differential equation as before with $x(t) \rightarrow \theta(t)$. Thus, the same trigonometric solution with

$$\omega^2 = \frac{mgl}{I}\,,\ T = 2\pi\sqrt{\frac{I}{mgl}} \tag{11.17}$$

Example: uniform rod of length L, pivoted at a distance l from the center.

Solution: from the rotational inertia of a rod about the CM, $I_0 = \frac{1}{12}ML^2$ and the parallel axis theorem

$$I = I_0 + Ml^2 = M\left(L^2/12 + l^2\right)$$

Thus,

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{L^2/12 + l^2}{gl}}$$

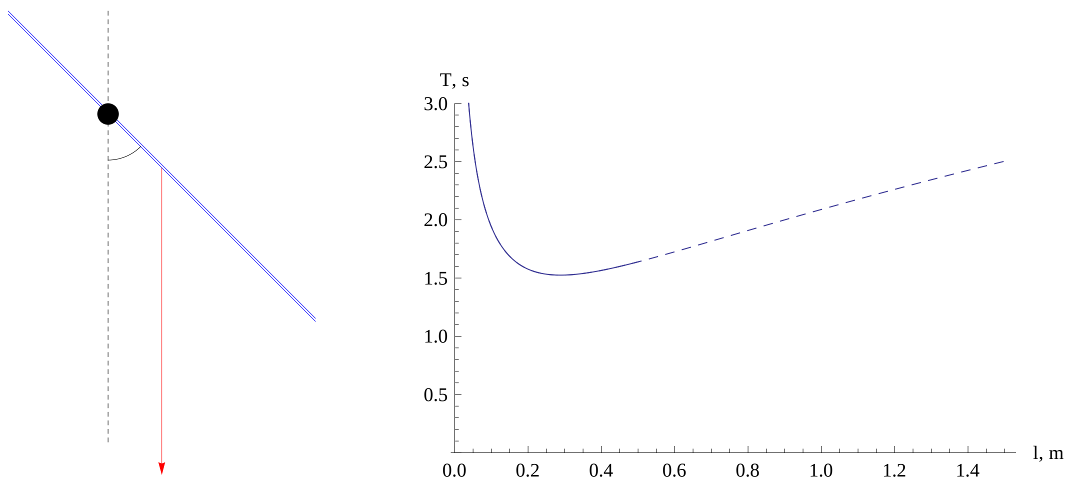

Figure 11.1: Physical pendulum of mass M and length L making angle θ with vertical. Forces: $M\vec{g}$ (red), with torque $\tau = -Mgl\sin\theta$. Right: period of small oscillations T for $L = 1\,m$ as a function of the off-center distance l, with a minimum at $l \approx 29\,cm$; dashed line corresponds to the pivot outside of the rod (on a massless extension).

11.5 Torsional pendulum

Consider

$$\tau = -\kappa\theta$$

which is a torsional "Hook's law". Then, from rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

and

$$\ddot{\theta} + \theta\frac{\kappa}{I} = 0 \tag{11.18}$$

Thus,

$$\omega^2 = \frac{\kappa}{I}\,,\ T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{11.19}$$

11.6 Why are small oscillations so universal?

in class

11.7 Resonance

Add external driving to a spring pendulum:

$$F = -kx + F_0 \cos(\omega_d t)$$

Then

$$m\ddot{x} + kx = F_0 \cos(\omega_d t) \tag{11.20}$$

Look for a solution

$$x(t) = A \cos(\omega_d t)$$

where the amplitude A has to be found. Using

$$\ddot{x}(t) = -\omega_d^2 x(t)$$

one obtains

$$A\left(-m\omega_d^2 + k\right) = F_0$$

or, with $\omega_0 = \sqrt{k/m}$, the natural frequency

$$|A| = \frac{F_0/m}{|\omega_0^2 - \omega_d^2|} \tag{11.21}$$

Note INFINITY when $\omega_d = \omega_0$. This is the resonance!

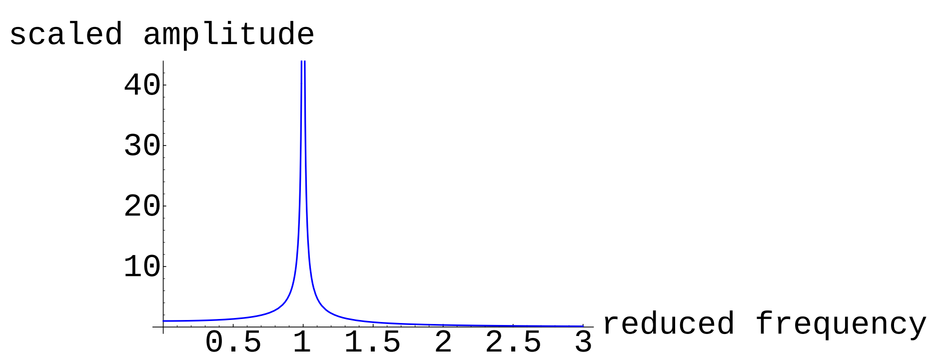

Figure 11.2: Resonance. When the driving frequency ω_d is close to the natural frequency $\omega = \sqrt{k/m}$ there is an enormous increase in the amplitude.