MECHANICS

Lecture notes for Phys 111

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Abstract

These notes are intended as an addition to the lectures given in class. They are NOT designed to replace the actual lectures. Some of the notes will contain less information then in the actual lecture, and some will have extra info. Not all formulas which will be needed for exams are contained in these notes. Also, these notes will NOT contain any up to date organizational or administrative information (changes in schedule, assignments, etc.) but only physics. If you notice any typos - let me know at vitaly@njit.edu. For convenience, I will keep all notes in a single file - each time you can print out only the added part. A few other things:

Graphics: Some of the graphics is deliberately unfinished, so that we have what to do in class.

Advanced topics: these will not be represented on the exams. Read them only if you are really interested in the material.

Contents

I. INTRODUCTION

A. Physics and other sciences

in class

B. Point mass

The art physics is the art of idealization. One of the central concepts in mechanics is a

"particle" or "point mass"

i.e. a body the size or structure of which are irrelevant in a given problem. Examples: electron, planet, etc.

C. Units

1. *Standard units*

In SI system the **basic** units are:

$$\boxed{\text{m (meter), kg (kilogram) and s (second)}}$$

Everything else in mechanics is derived. Examples of derived units (may or may not have a special name): m/s, m/s^2 (no name), kg $\cdot$ m/s^2 (Newton), kg $\cdot$ m^2/s^2 (Joule), etc.

Major variables, their typical notations and units:

variable	units	name
speed, v	m/s	-
acceleration (a or g)	m/s^2	-
force (F, f, N, T)	N=kg m/s^2	Newton
work (W), energy (K, U, E)	J=kg m^2/s^2	Joule

Standard path: all units are converted to SI. E.g., length:

$$1\,\mathrm{in} = 0.0254\,\mathrm{m}\ ,\quad 1\,\mathrm{ft} = 0.3048\,\mathrm{m}\ ,\quad 1\,\mathrm{mi} \simeq 1609\,\mathrm{m}$$

Examples:

$$\text{speed:}\ \ 70\,\frac{\mathrm{mi}}{\mathrm{h}} = 70\frac{1609\,\mathrm{m}}{3600\,\mathrm{s}} \simeq 31.3\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\text{area:}\ \ 3\,\mathrm{cm}^2 = 3\,\left(10^{-2}\,\mathrm{m}\right)^2 = 3\cdot 10^{-4}\,\mathrm{m}^2$$

$$\text{volume:}\ \ 1\,\mathrm{mm}^3 = 1(10^{-3}\,\mathrm{m})^3 = 10^{-9}\,\mathrm{m}^3$$

$$\text{density:}\ \ 1\frac{\mathrm{gram}}{\mathrm{cm}^3} = 1\frac{10^{-3}\mathrm{kg}}{(0.01\,\mathrm{m})^3} \approx 1000\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(water)}$$

$$218\frac{\mathrm{gram}}{\mathrm{in}^3} = 218\frac{10^{-3}\mathrm{kg}}{(0.0254\,\mathrm{m})^3} \approx 13.6\cdot 10^3\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(mercury)}$$

$$315\frac{\mathrm{gram}}{\mathrm{in}^3} = 315\frac{10^{-3}\mathrm{kg}}{(0.0254\,\mathrm{m})^3} \approx 19.2\cdot 10^3\frac{\mathrm{kg}}{\mathrm{m}^3}\ \ \text{(gold)}$$

Oil is spilling from a pipe at a rate of $0.2\,ft^3/min$. Express this in SI units.

$$0.2\,\frac{ft^3}{min} = 0.2\cdot\frac{(0.305\,m)^3}{60\,s} \simeq 9.4\cdot 10^{-5}\frac{m^3}{s}$$

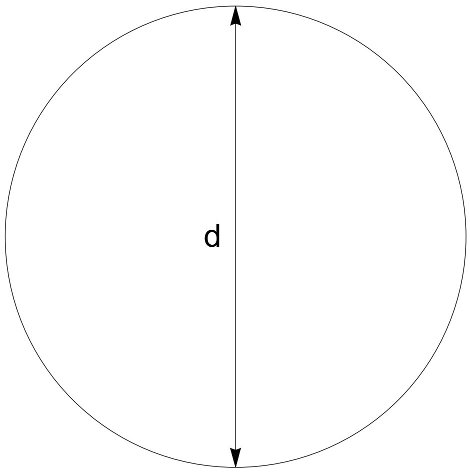
Non-standard: Convert the previous in $L/hour$:

$$1\,m^3 = 1000\,L\,,\ 1\,s = \frac{1}{3600}hour\ \Rightarrow\ 9.4\cdot 10^{-5}\frac{m^3}{s} = 9.4\cdot 10^{-5}\frac{1000\,L}{hour/3600} \approx 340\frac{L}{hour}$$

Earth is $150\cdot 10^6\,km$ from Sun; find speed in km/s if 1 year $\simeq$ 365 days

$$2\pi\frac{150\cdot 10^6\,km}{365\cdot 24\cdot 3600\,s} \simeq 30\,km/s$$

3. Significant figures



A solid disk has a diameter of 1 cm. Find the circumference.

$$s = \pi d = 3.141592654\ldots \text{ cm. Wrong!! Why?}$$

Advanced.

The period of small oscillations of a pendulum is independent of its amplitude (Galileo). Use this to find the dependence of the period T on the length of the pendulum L, gravitational acceleration g and, possibly, mass M. Namely, look for

$$T \sim L^{\alpha} g^{\beta} M^{\gamma}$$

and find α, β and γ.

$$[T] = \mathrm{s}\,,\ [L] = \mathrm{m}\,,\ [g] = \frac{\mathrm{m}}{\mathrm{s}^2}\,,\ [M] = \mathrm{kg}$$

$$s = \mathrm{m}^{\alpha}\left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)^{\beta} kg^{\gamma} = \mathrm{kg}^{\gamma} m^{\alpha+\beta} s^{-2\beta}$$

$$\gamma = 0\,,\ \alpha = -\beta\,,\ \beta = -1/2\ \Rightarrow\ T \sim \sqrt{\frac{L}{g}}$$

Advanced. Less trivial example: gravitational waves. What is the speed? Can depend on $g, [m/s^2]$ on $\lambda, [m]$ and on $\rho, [kg/m^3]$

$$v \sim g^{\alpha}\lambda^{\beta}\rho^{\gamma}\ or\ [m/s] = [m/s^2]^{\alpha}[m]^{\beta}[kg/m^3]^{\gamma}$$

From dimensions,

$$\alpha = \beta = 1/2\,, \gamma = 0(!)$$

$$v \sim \sqrt{g\lambda}$$

What is neglected? Depth of the ocean, H. Thus,

$$v_{\max} \sim \sqrt{gH} \sim \sqrt{10 \cdot 4 \cdot 10^3} \sim 200\,\mathrm{m/s}$$

(the longest and fastest gravitational wave is tsunami). Note that we know very little about the precise physics, and especially the precise math of the wave, but from dimensional analysis could get a reasonable estimation.

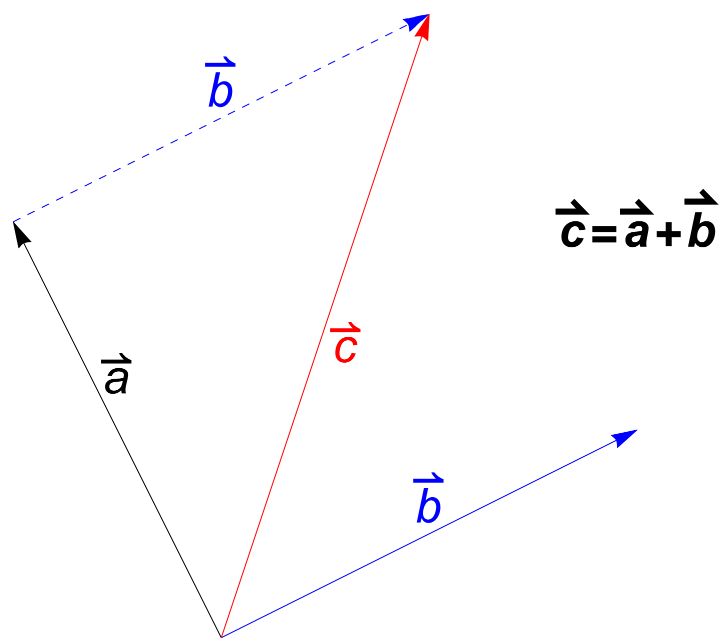
II. VECTORS

A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("Tail-to-Head" or, equivalently "rule of parallelogram").

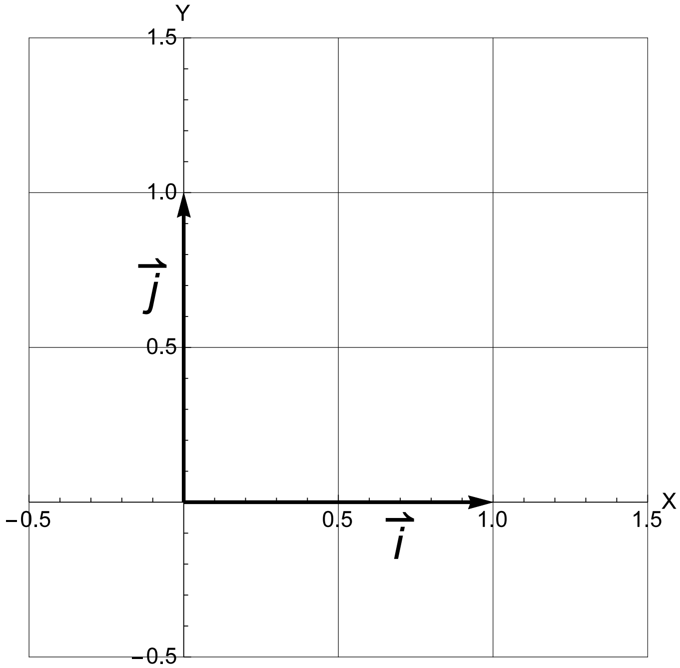
This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are mass, time, kinetic energy. Examples of vectors are the displacement, velocity and force.



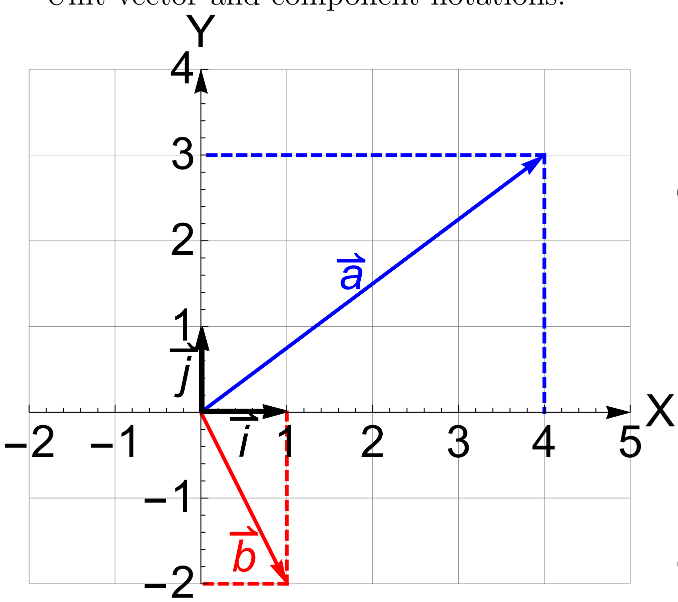
Tail-to-Head addition rule.

A. System of coordinates



Unit vectors $\vec{i}$, $\vec{j}$

Unit vector and component notations:



$$\vec{a} = 4\hat{i} + 3\hat{j} = (4, 3)$$

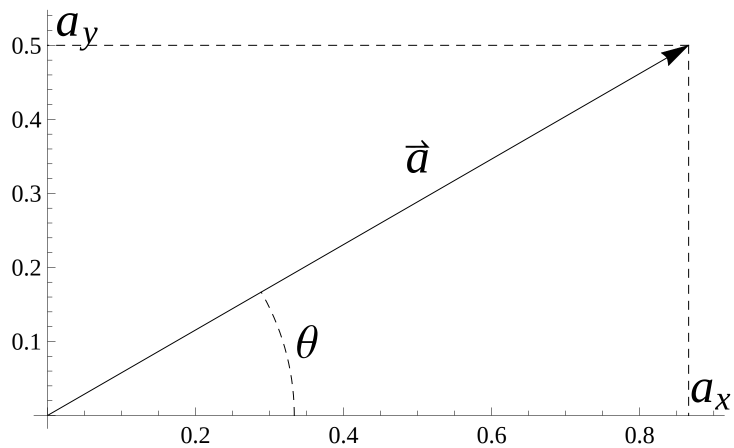
$$\vec{b} = \hat{i} - 2\hat{j} = (1, -2)$$

$$\text{Length: } a = |\vec{a}| = \sqrt{4^2 + 3^2} = 5$$

$$b = |\vec{b}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

B. Operations with vectors

1. Single vector



Consider a vector $\vec{a}$ with components a_x and a_y (let's talk 2D for a while). The magnitude (or length) is given by the Pythagorean theorem

$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2} \tag{1}$$

Note that for a different system of coordinates with axes x', y' the components $a_{x'}$ and $a_{y'}$ can be different, but the length in eq. (1) , obviously, will not change, which just means that length is a *scalar*.

Primary example: position vector (note two equivalent forms of notation)

$$\vec{r} = (x, y) = x\vec{i} + y\vec{j}$$

(sometimes, $\hat{\imath}$ and $\hat{\jmath}$ is used) with $|\vec{i}| = |\vec{j}| = 1$.

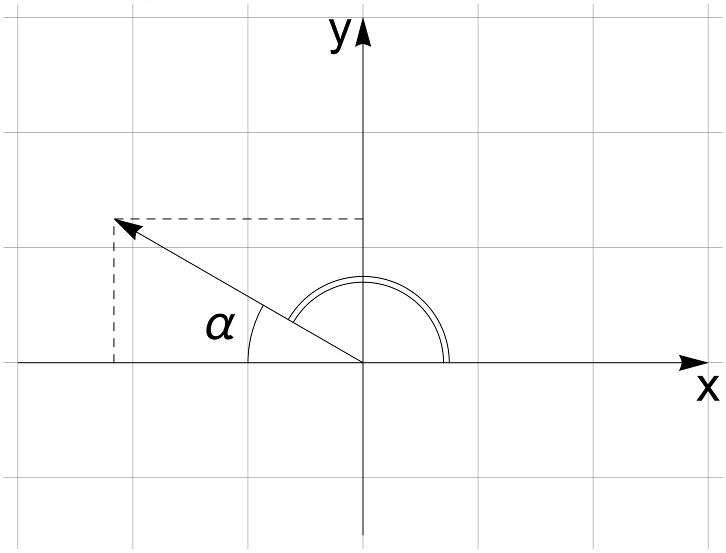
Polar coordinates:

$$r = \sqrt{x^2 + y^2}\ , \quad \theta = \arctan(y/x)$$

$$x = r\cos\theta\ , \quad y = r\sin\theta$$

Note: arctan might require adding 180^o - always check with a picture!

Example. Find the components of the vector $\vec{r}$ in the figure if its length is $r = 2.5$ units and $\alpha = 20^o$.



Solution 1 (from picture). The (x,y) components are "cut off" by the dashed lines. Thus, $x = -r\cos\alpha = -2.5\cos 20^o = -\ldots$ (note the minus!) and $y = 2.5\sin 20^o = \ldots$.

Solution 2 (from formulas). One has $x = r\cos\theta$, $y = r\sin\theta$, but θ is the angle with *positive* x-direction (double arc in the figure), or $\theta = 180^o - \alpha = 160^o$. Thus,

$$x = 2.5\cos 160^o = \cdots < 0\,,\; y = 2.5\sin 160^o > 0$$

(Check that you get the same numbers!)

Another operation allowed on a single vector is multiplication by a scalar. Note that the physical dimension ("units") of the resulting vector can be different from the original, as in $\vec{F} = m\vec{a}$.

2. *Two vectors: addition*

For two vectors, $\vec{a}$ and $\vec{b}$ one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x\;,\quad c_y = a_y + b_y \tag{2}$$

The magnitude of $\vec{c}$ then follows from eq. (1). Note that physical dimensions of $\vec{a}$ and $\vec{b}$ must be identical.

Note: for most problems (except rotation!) it is allowed to carry a vector parallel to itself. Thus, we usually assume that every vector starts at the origin, $(0,0)$.

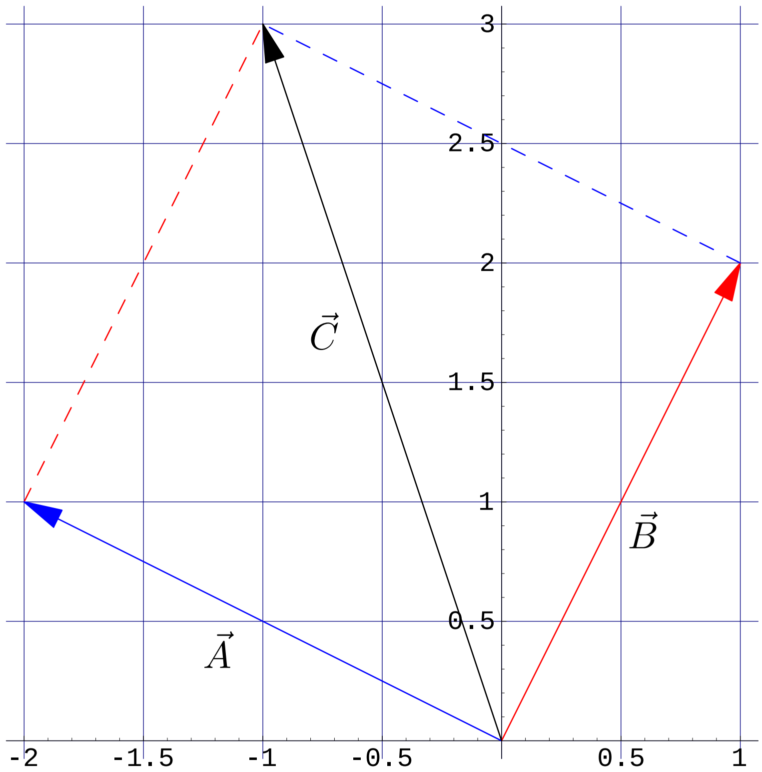


FIG. 1: Adding two vectors: $\vec{C} = \vec{A} + \vec{B}$. Note the use of rule of parallelogram (equivalently, tail-to-head addition rule). Alternatively, vectors can be added by components: $\vec{A} = (-2, 1)$, $\vec{B} = (1, 2)$ and $\vec{C} = (-2 + 1,\ 1 + 2) = (-1, 3)$.

Example Displacement $\vec{A} = -2\hat{\imath} + \hat{\jmath}$ is followed by $\vec{B} = \hat{\imath} + 2\hat{\jmath}$. Find magnitude and direction of the resultant displacement.

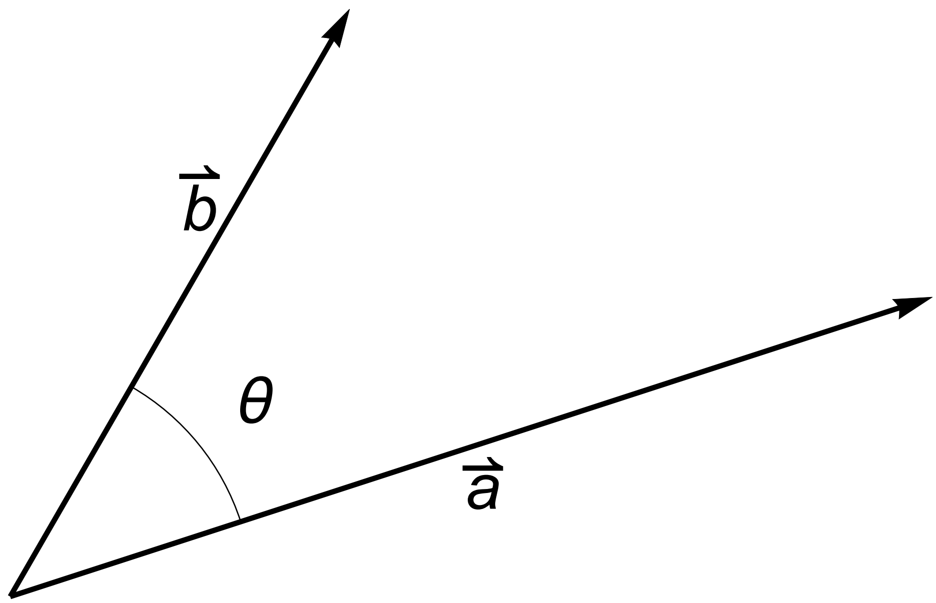
$$\vec{C} = \vec{A} + \vec{B} = -\hat{\imath} + 3\hat{\jmath}$$

$$C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}\,,$$
$$\cos\theta = \frac{C_x}{C} = \frac{-1}{\sqrt{10}} = -\ldots\,,\ \ \theta = \ldots > 90^o$$

Example (3D). For $\vec{A} = (1, 2, 3)$ and $\vec{B} = (-1, 1, 7)$ find $3\vec{A} + 4\vec{B}$

$$3\vec{A} + 4\vec{B} = (3 \cdot 1 + 4 \cdot (-1),\ 3 \cdot 2 + 4 \cdot 1,\ 3 \cdot 3 + 4 \cdot 7) = (-1, 10, 37)$$

3. Two vectors: dot product - need for "work"



If $\vec{a}$ and $\vec{b}$ make an angle θ with each other, their scalar (dot) product is defined as

$$\vec{a} \cdot \vec{b} = ab\cos(\theta)$$

or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{3}$$

A different system of coordinates can be used, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$. *Preview.* The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta\vec{r}$, with $\Delta\vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!

Example. See Fig. 1. $\vec{A} = -2\hat{i} + \hat{j}$, $\vec{B} = \hat{i} + 2\hat{j}$

$$\vec{A} \cdot \vec{B} = (-2)1 + 1 \cdot 2 = 0$$

(thus angle is 90^o).

Example Find angle between 2 vectors $\vec{B}$ and $\vec{C}$ in Fig. 1.

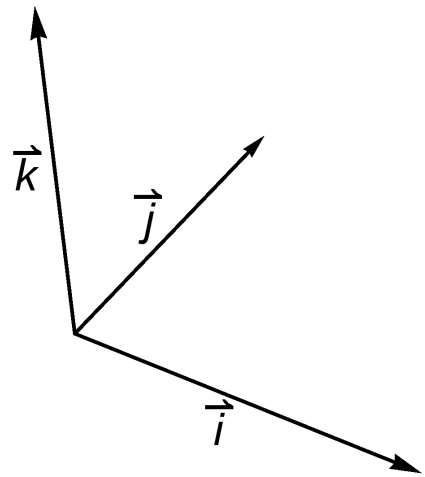
General:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab}$$

$$\text{In Fig. 1: } B = \sqrt{1^2 + 2^2} = \sqrt{5}\,,\ C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\cos\theta = \frac{(-1) \cdot 1 + 3 \cdot 2}{5\sqrt{2}} = \frac{1}{\sqrt{2}}\,,\ \theta = 45^o$$

Add unit vector $\vec{k}$ in the z-direction.



$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = (A_x\ ,\ A_y\ ,\ A_z)\ ,\ A = \left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB}$$

Example. $\vec{A} = (1, 2, 3)$, $\vec{B} = (1, -1, 0)$. Find $\vec{A}\cdot\vec{B}$ and the angle between them.

$$\vec{A}\cdot\vec{B} = 1 * 1 + 2 * (-1) + 3 * 0 = -1$$

$$A = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}\,,\ B = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\cos\theta = \frac{-1}{\sqrt{14}\sqrt{2}} = -\frac{1}{2\sqrt{7}}\ ,\ \theta \simeq 101^o$$

4. Two vectors: vector product - need for "torque"

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 2. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands).

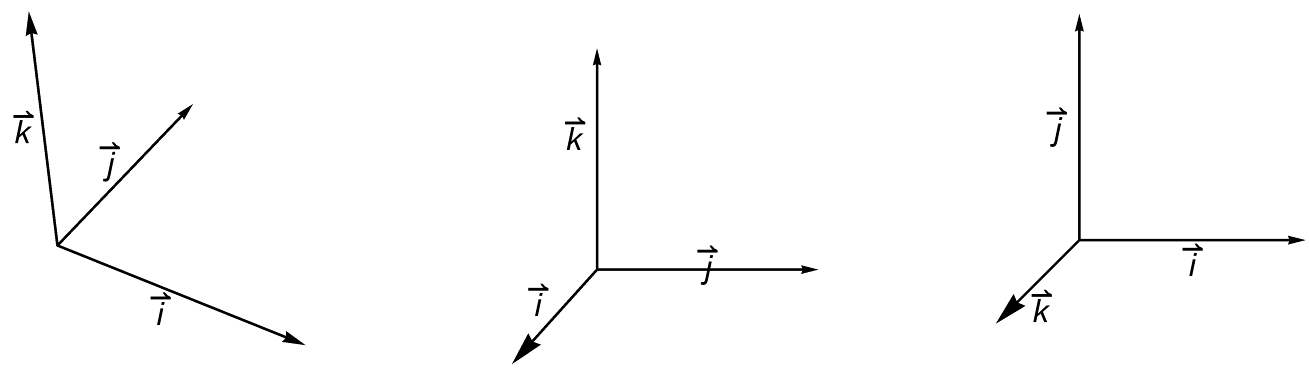


FIG. 2: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x-direction ($\vec{i}$) to y-direction ($\vec{j}$), then the thumb should point into the z-direction ($\vec{k}$).

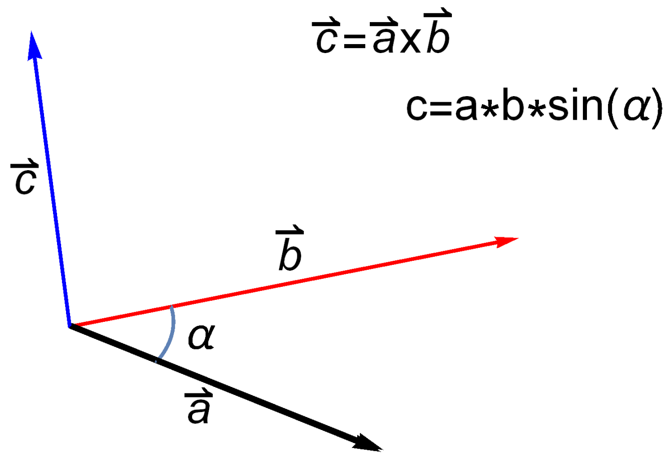


FIG. 3: Example of a cross product $\vec{c} = \vec{a} \times \vec{b}$. Direction: perpendicular to it both $\vec{a}$ and $\vec{b}$ ('right hand rule'). Magnitude - as indicated.

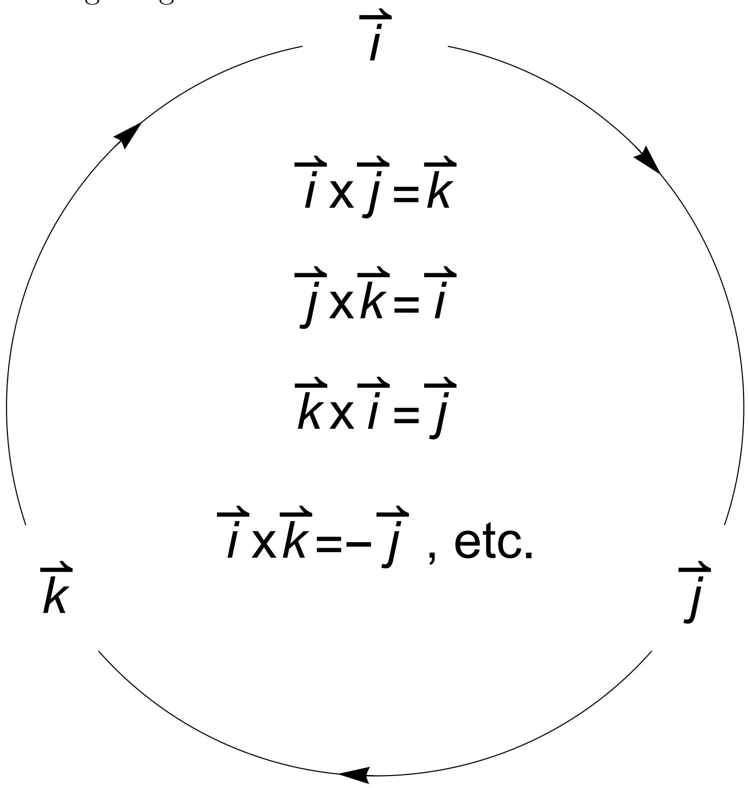
If $\vec{a}$ and $\vec{b}$ make an angle $\alpha \leq 180^o$ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude $c = ab \sin(\alpha)$. The direction is defined as perpendicular to both $\vec{a}$ and $\vec{b}$ using the following rule: curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ in the shortest direction (i.e., the angle must be smaller than 180^o). Then the thumb points in the $\vec{c}$ direction. Check with Fig. 3. Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

Suppose now a system of coordinates is introduced with unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ pointing in the x, y and z directions, respectively. First of all, if $\hat{\imath}$, $\hat{\jmath}, \hat{k}$ are written "in a ring", the cross product of any two of them equals the third one in clockwise direction, i.e.

$$\hat{\imath} \times \hat{\jmath} = \hat{k}\,,\; \hat{\jmath} \times \hat{k} = \hat{\imath}\,,\; \hat{k} \times \hat{\imath} = \hat{\jmath}$$

, etc.

Ring Diagram:



Example. Fig. 1:

$$\vec{A} = -2\hat{\imath} + \hat{\jmath} \, , \, \vec{B} = \hat{\imath} + 2\hat{\jmath}$$

$$\vec{A} \times \vec{B} = (-2\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 2\hat{\jmath}) = (-2) \cdot 2\hat{\imath} \times \hat{\jmath} + \hat{\jmath} \times \hat{\imath} =$$

$$= -4\hat{k} - \hat{k} = -5\hat{k}$$

(Note: in Fig. 1 $\hat{k}$ goes out of the page; the cross product $\vec{A} \times \vec{B}$ goes into the page, as indicated by "-".)

More generally, the cross product is now expressed as a 3-by-3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \tag{4}$$

The two-by-two determinants can be easily expanded. In practice, there will be many zeros, so calculations are not too hard.

Preview. Vector product is most relevant to rotation.

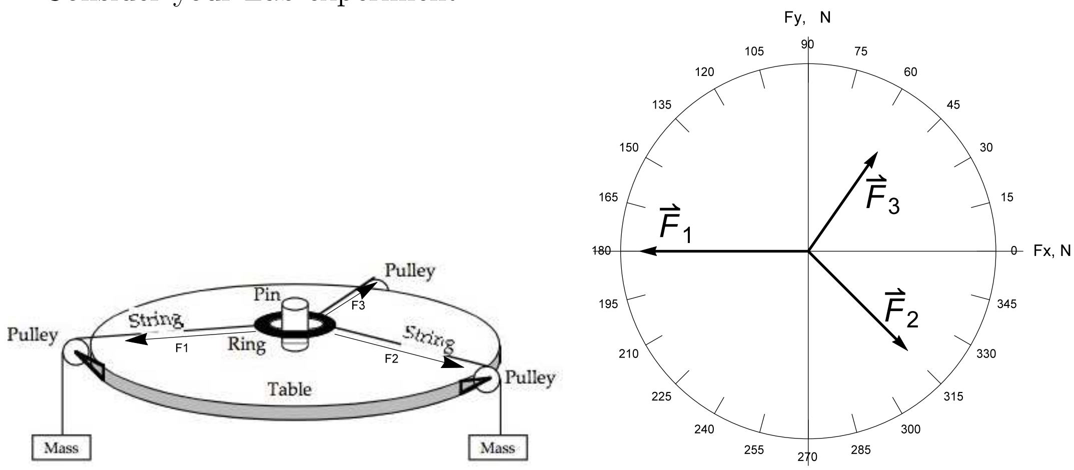
Example. See Fig. 1.

$$\vec{A} \times \vec{B} = \hat{k}((-2)2 - 1 \cdot 1) = -5\hat{k}$$

C. Preview. Forces as vectors.

Besides displacement $\vec{r}$, and velocity $\vec{v}$, forces represent another example of a vector. Note that they are measured in different units, N (*newtons*), i.e. each component of the force F_x , F_y , F_z is measured in N. How do we know that force is a vector? From experiments on static equilibrium which demonstrate that forces indeed add up following the standard vector addition rule of parallelogram (or, that they add up by components, which is the same thing).

Consider your Lab experiment



The force table (left) and its schematic representation (right)

$$\text{In equilibrium: } \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0\,, \;\Rightarrow\; \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

$$\text{In the example: } F_1 = 0.9\,N\,,\ \theta_1 = 180^o\ ;\ F_2 = 0.75\,N\,,\ \theta_2 = 315^o = -45^o \Rightarrow$$

$$F_{1x} = 0.9\cos(180^o) = -0.9\,,\ F_{1y} = 0.9\sin(180^o) = 0$$

$$F_{2x} = 0.75\cos(-45^o) = 0.53\,,\ F_{2y} = 0.75\sin(-45^o) = -0.53$$

$$F_{3x} = -(F_{1x} + F_{2x}) = 0.37\,,\ F_{3y} = -(F_{1y} + F_{2y}) = 0.53$$

$$\theta_3 = \tan^{-1}\frac{F_{3y}}{F_{3x}} = \tan^{-1}\frac{0.53}{0.37} \simeq 55^o$$

(check with picture that this is about correct; the angle $55^o + 180^o = 235^o$ has the same tan). Similarly, the magnitude

$$F_3 = \sqrt{.37^2 + .53^2} = 0.646\ (N)$$

III. 1-DIMENSIONAL MOTION

Position $x(t)$

Displacement:

$$\Delta x = x(t_2) - x(t_1)$$

Distance:

$$D \geq |\Delta x| \geq 0$$

Velocity:

$$v = \frac{\Delta x}{\Delta t}\,,\ \Delta t = t_2 - t_1$$

with a small Δt (later, we distinguish between *average* velocity with a finite Δt and *instantaneous* with $\Delta t \to 0$).

Speed:

$$s = \frac{D}{\Delta t} \geq |v| \geq 0$$

A. $v = const$

See fig. 4

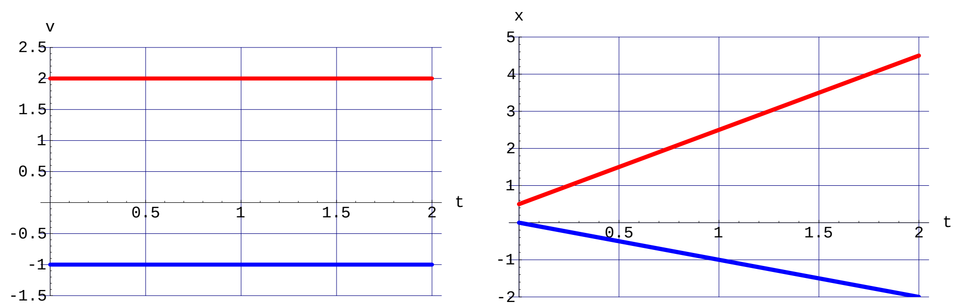


FIG. 4: Velocity (left) and position (right) plots for motion with constant velocity: Positive (red) or negative (blue). Note that area under the velocity line (positive or negative) corresponds to the change in position: E.g. (red) $2 \times 2 = 4.5 - 0.5$, or (blue) $2 \times (-1) = -2 - 0$.

Displacement

$$\Delta x = v\,\Delta t \tag{5}$$

Distance

$$D = |\Delta x| \, , \text{ for v=const only} \tag{6}$$

Speed

$$s = D/\Delta t = |v| \, , \text{ for v=const only} \tag{7}$$

Example. A motorcycle with $V_M = 60\, m/s$ is catching up with a car with $V_C = 40\, m/s$, originally $D = 200\, m$ ahead. When and where will they meet? Give the graphic solution (in class).

$$x_M = V_M t \ , \ x_C = V_C t + D \ , \text{ and } x_M = x_C \text{ - they meet}$$

$$V_M t = V_C t + D \, , \; thus, \; t = \frac{D}{V_M - V_C} = \frac{200}{60 - 40} = 10\, s$$

Above is the meeting time (note "relativity" - in a reference frame moving with V_M the motorcycle is stationary, while the car approaches it with speed of $V_M - V_C$). The meeting point - the distance from the original position of the motorcycle - is

$$x = V_M t = D \frac{V_M}{V_M - V_C} = 600\, m$$

Graphically, the solution is intersection in t, x plane.

B. Variable velocity

Example (trap!): A hiker goes from A to B with $S_1 = 2\,km/h$ and returns with $S_2 = 4\,km/h$. Find S_{av} .

$$S_1 = 2\,km/h; ,\ S_2 = 4\,km/h\ .\ \ S_{av} - ?$$

$$D = 2AB\,,\ t_1 = AB/S_1\,,\ t_2 = AB/S_2$$

$$S_{av} = \frac{D}{t_1 + t_2} = \frac{2AB}{AB/S_1 + AB/S_2} = \frac{2S_1S_2}{S_1 + S_2} \neq 3\,km/h$$

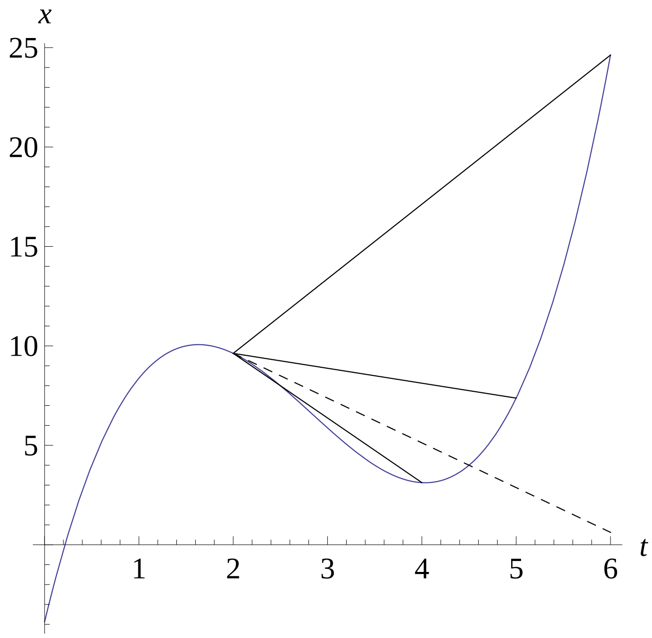


FIG. 5: A sample position vs. time plot (blue curve), and determination of the average velocities - slopes (positive or negative) of straight solid lines. Slope of dashed line (which is tangent to $x(t)$ curve) is the instantaneouus velocity at $t = 2$.

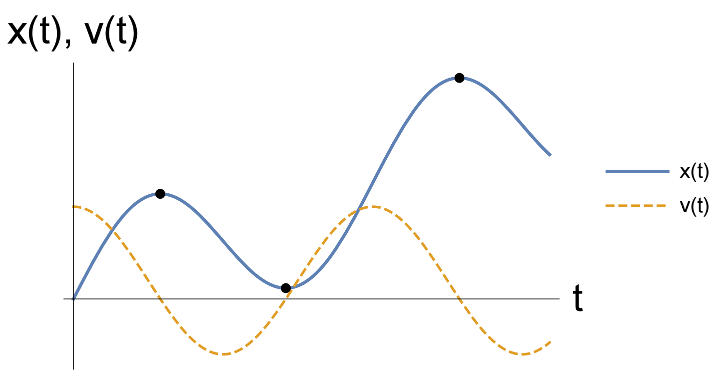


FIG. 6: Example. Instantaneous velocity as the slope of $x(t)$. Points indicate instances with $v(t) = 0$.

$$\text{Average velocity: } v_{av} = \frac{\Delta x}{\Delta t}\,, \quad \text{Instantaneous: } \quad v = \lim_{\Delta t \to 0} v_{av} = \frac{dx}{dt} \tag{8}$$

$$\text{Distance: } \quad D \geq |\Delta x|\,, \quad \text{Average speed: } \quad s_{av} = \frac{D}{\Delta t} \geq |v_{av}| \tag{9}$$

$$\text{Instantaneous speed: } s = \frac{dD}{dt} = |v| \tag{10}$$

2. Displacement from velocity

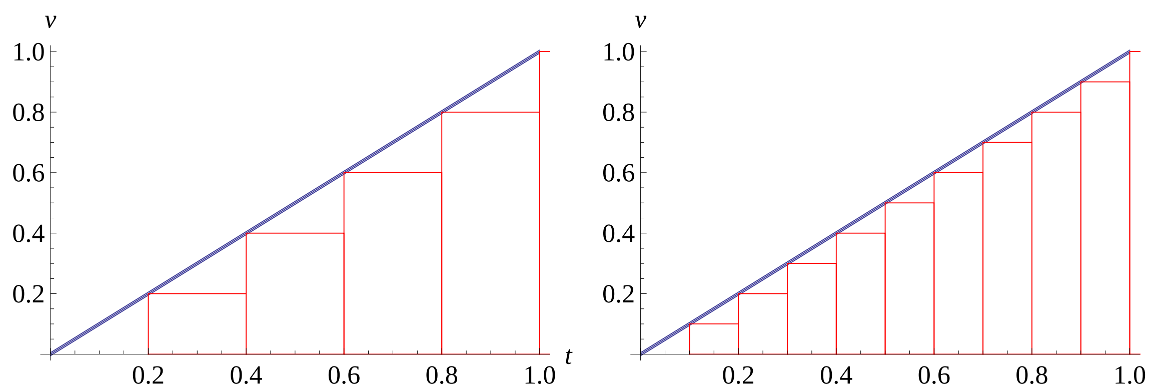


FIG. 7: Determination of displacement for a variable $v(t)$. During the ith small interval of duration Δt the velocity is replaced by a constant v_i shown by a horizontal red segment. Corresponding displacement is $\Delta x_i \approx v_i \cdot \Delta t$ (the red rectangular box). The total displacement $\Delta x = \sum \Delta x_i$ is then approximated by the area under the $v(t)$ curve.

$$\text{Displacement: } \quad \Delta x(t) = \text{"area" under the } v(t) \text{ curve } = \int_{t_1}^{t} v\,(t')\; dt' \tag{11}$$

$$\text{average:} \quad a_{av} = \frac{\Delta v}{\Delta t} \tag{12}$$

$$\text{instantaneous:} \ a = \lim_{\Delta t \to 0} a_{av} = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{13}$$

Geometric meaning: a - slope of $v(t)$. If $a(t)$ is known, Δv is the "area" under the $a(t)$ curve.

Example: Given (t in seconds, x in meters)

$$x(t) = -\frac{t^4}{4} + \frac{t^3}{3} + t^2 - t + 1$$

Find: a) v_{av} between $t_1 = 1\,s$ and $t_2 = 3\,s$; b) $v(t)$, $a(t)$

$$v_{av} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{-17/4 - 13/12}{3 - 1} = -\frac{8}{3}$$

b) from

$$\frac{d}{dt}t^n = nt^{n-1} \ : \ v(t) = -t^3 + t^2 + 2t - 1\,, \ a(t) = -3t^2 + 2t + 2$$

Example. A particle is moving according to $x = 20t^2$ (with x in meters, t in seconds. When $t = 2\,s$ (a) find a and (b) find v

$$v(t) = \frac{dx}{dt} = 40t\,, \ a = \frac{dv}{dt} = 40\,\frac{m}{s^2}$$

$$v(t=2) = 80\,\frac{m}{s}\,, \ a = 40\,\frac{m}{s^2}$$

4. $a = const$

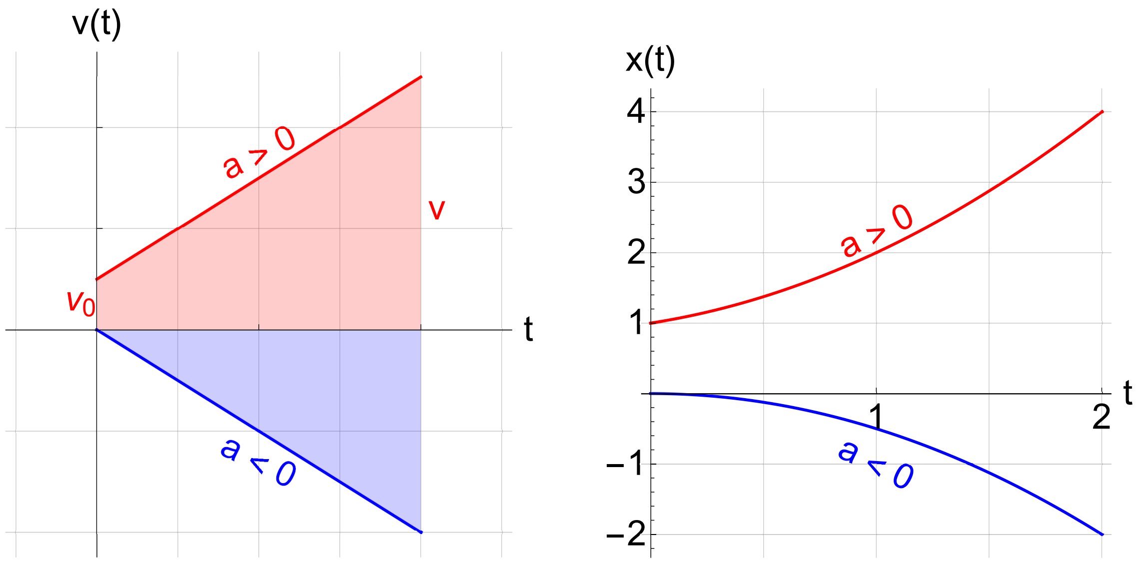


FIG. 8: Velocity (left) and position (right) plots for motion with constant acceleration: Positive (red) or negative (blue). Again, area under the velocity line (positive or negative) corresponds to the change in position. E.g. (red) $(2.5 + 0.5) \times 2/2 = 4 - 1$ or (blue) $(-2) \times 2/2 = -2 - 0$.

Notations: Start from $t = 0$, thus $\Delta t = t$; $v(0) \equiv v_0$.

$$\Delta v = at\,,\;\; v = v_0 + at \tag{14}$$

Displacement - area of the trapezoid in fig. 8 (can be negative!):

$$\Delta x = \frac{v_0 + v}{2} t = v_0 t + at^2/2 \tag{15}$$

A useful alternative: use $t = (v - v_0)/a$:

$$\Delta x = \frac{v_0 + v}{2}\frac{v - v_0}{a} = \frac{v^2 - v_0^2}{2a} \tag{16}$$

(A more elegant derivation follows from conservation of energy,... later)

SUMMARY: if

$$a = const\,, \tag{17}$$

$$v = v_0 + at \tag{18}$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2 \tag{19}$$

$$x - x_0 = \frac{v_0 + v}{2} t = \frac{v^2 - v_0^2}{2a} \tag{20}$$

Example. A motorcycle accelerates from rest for $t = 2.0\,s$ traveling $x = 20\,m$ during that time. Find acceleration a. *Solution:*. Select an equation which has no $v(t)$ (and $v_0 = 0$ since 'from rest' and $x_0 = 0$ for convenience)

$$x = \frac{1}{2} a t^2 \;\Rightarrow\; a = \frac{2x}{t^2} = \frac{2 \cdot 20}{2.0^2} = 10\,m/s^2$$

Example. A car accelerates from $v_0 = 5\,m/s$ to $v = 35\,m/s$ with $a = 3\,m/s^2$. (a) How far will it go? (b) How long will it take? *Solution:*. (a) time is not given, thus select the only formula which has no t:

$$x = \frac{v^2 - v_0^2}{2a} = \frac{35^2 - 5^2}{2 \times 3} = \ldots$$

(b) select a formula with no x

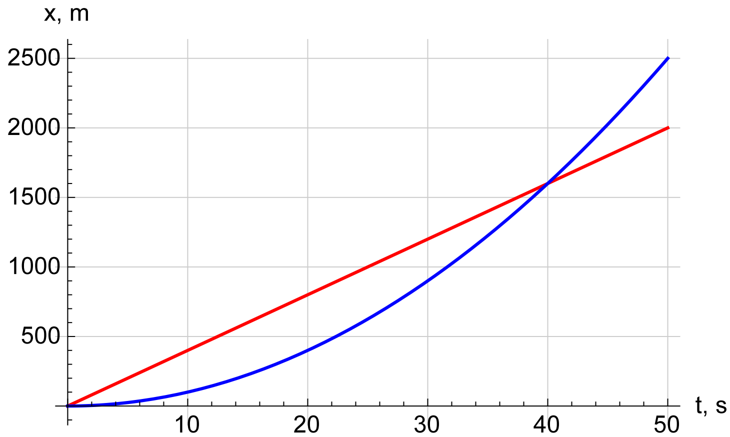
$$t = (v - v_0)/a = \ldots$$

Example. After the driver hits the brakes, the car skids for $10\,s$ a distance $100\,m$ before it stops. (a) Find the initial speed v_0; (b) find the deceleration. *Solution:*. (a) acceleration not given or asked for, thus use

$$x = \frac{v_0 + v}{2} t \text{ with } v = 0 \;\Rightarrow$$

$$v_0 = \frac{2x}{t} = \frac{2 \times 100}{10} = 20\,m/s\,, \text{ and } a = \frac{v - v_0}{t} = -\frac{v_0}{t} = \ldots \quad \text{(b)}$$

Example: meeting problems (car $V_C = 40\,m/s$, $a_C = 0$ and motorcycle $V_M = 0$, $a_M = 2\,m/s^2$)



$$X_C = V_C t\,,\ X_M = \frac{1}{2} a_M t^2$$

$$V_C t = \frac{1}{2} a_M t^2\,,\ t = 2V_C/a_M = 40\,s\,,\ X_{meet} = t \cdot V_C = 1600\,m = \frac{1}{2} a_M t^2$$

C. Free fall

(a)

Reminder:

$$\text{if } a = const \text{ then} \tag{21}$$

$$v = v_0 + at \tag{22}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{23}$$

$$x - x_0 = \frac{v_0 + v}{2}t = \frac{v^2 - v_0^2}{2a} \tag{24}$$

Free fall:

$$a \ \rightarrow \ -g\,,\ x \ \rightarrow \ y\,,\ x_0 \ \rightarrow \ y_0\ (or,\ H)$$

$$a = -g = -9.8\,m/s^2 \tag{25}$$

$$v = v_0 - gt \tag{26}$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{27}$$

$$y - y_0 = \frac{v_0^2 - v^2}{2g} \tag{28}$$

Example: max height:

$$v = 0\,,\ y_{\max} - y_0 = v_0^2/2g \tag{29}$$

Example: the Tower of Piza ($v_0 = 0\,,\ y_0 = H \simeq 55\,m$). Find t, v upon impact.

$$0 = H + 0t - gt^2/2\,,\ t = \sqrt{\frac{2H}{g}}$$

$$0 - H = -v^2/2g\,,\ v = \sqrt{2gH}$$

What if $v_0 = 10\,m/s$? (use $g \approx 10\,m/s^2$)

$$0 = H + v_0 t - \frac{1}{2} g t^2\,,\ 0 = 55 + 10t - 5t^2$$

$$t^2 - 2t - 11 = 0,\ t = 1 \pm \sqrt{1^2 + 11} = 1 + \sqrt{12} = \dots$$

(only positive root!)

$$0 - H = \frac{v_0^2 - v^2}{2g}\,,\ v^2 = v_0^2 + 2gH$$

$$v = \sqrt{v_0^2 + 2gH} \approx \sqrt{10^2 + 2 * 10 * 55} = \dots$$

If $v_0 = -10\,m/s$

$$0 = 55 - 10t - 5t^2$$

$$t^2 + 2t - 11 = 0,\ t = -1 \pm \sqrt{1^2 + 11} = -1 + \sqrt{12} = \dots$$

$v = \sqrt{v_0^2 + 2gH}$ (same: sign of v_0 does not matter!)

Other examples - ignore air friction

A shell is fired vertically up with $v_0 = 200\, m/s$. Find $h_{\max}$.

$$y - y_0 = \frac{v_0^2 - v^2}{2g}\,,\; y - y_0 = h_{\max}\,,\; v = 0 \text{ @ max } \Rightarrow$$

$$h_{\max} = \frac{v_0^2}{2g} = \frac{200^2}{2 * 9.8} \approx 2 \cdot 10^3\, m$$

A package is dropped from a helicopter moving upward at $v_0 = 20\, m/s$. If it takes $t = 15\, s$ before the package strikes the ground, (A) how high above the ground was the package when it was released if air resistance is negligible? (B) How long is the path?

$$(A)\; 0 = y_0 + v_0 t - \frac{1}{2} g t^2\,,\; y_0 = \frac{1}{2} g t^2 - v_0 t = \frac{1}{2} * 9.8 * 15^2 - 20 * 15 = \ldots$$

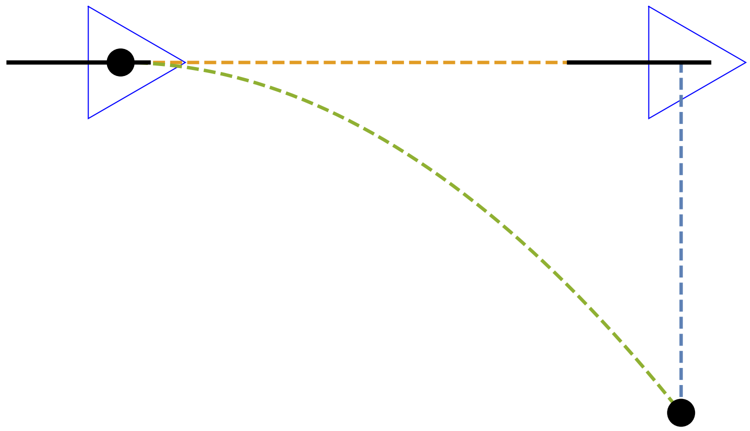
$$(B)\; D = y_0 + 2 \times h_{\max} = y_0 + 2\frac{v_0^2}{2g} = \ldots$$

Link to "hammer-vs-feather" :

`https://www.youtube.com/watch?v=KDp1tiUsZw8`

IV. PROJECTILE MOTION

A. Introduction: Object dropped from a plane



Given: $y_0 = H = 100\,m$ and $\vec{v}_0 = V\hat{i}$, with $V = 200\,m/s$ (horizontal). Find: (a) horizontal distance L to hit the ground, (b) the speed v upon impact and the angle and (c) position of the object relative to the plain. Solution:
Vertical motion (horizontal velocity does not matter!):

$$y(0) = H\,,\; v(0) = 0 \;\Rightarrow\; y(t) = H - \frac{1}{2}gt^2$$

Time to fall, $y(t) = 0$

$$t = \sqrt{2H/g}$$

(a) From x-direction

$$L = Vt = V\sqrt{2H/g} = 200\sqrt{2 \cdot 100/9.8} = \ldots$$

(b) Speed upon impact. From y-direction

$$y - y_0 = \frac{v_{0y}^2 - v_y^2}{2g} \;\Rightarrow\; v_y^2 = 2gH$$

(could use $v_y = -gt$ with calculated t). From x-direction

$$v_x = const = V \;\Rightarrow\; v^2 = v_x^2 + v_y^2 = V^2 + 2gH = 200^2 + 2 * 9.8 * 100 = \ldots$$

(the last formula is also valid for non-horizontal launch with V replaced by v_0, the full initial speed).

Angle of impact with horizontal:

$$\tan\theta = v_y/v_x = -\sqrt{2gH}/V$$

(c) since $v_x = const = V$ the object is right under the plane (!)

Another example: vertical toy cannon on a moving cart.

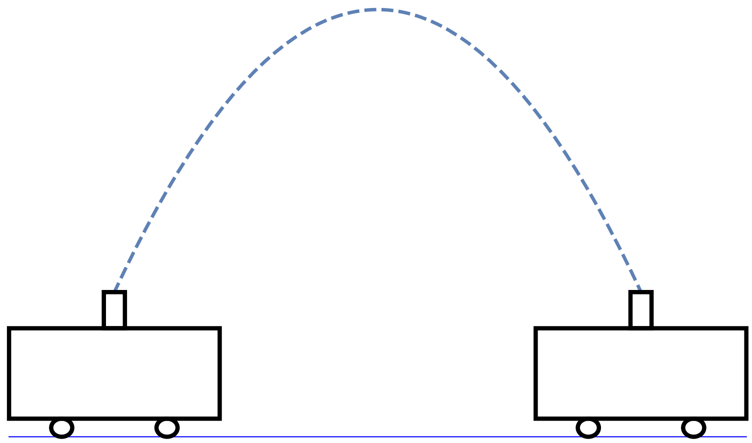


FIG. 9: The ball lands back into the cannon for any constant velocity of the cart (Galileo's relativity!). The maximum hight and the time of flight depend only on vertical velocity but not on the horizontal motion.

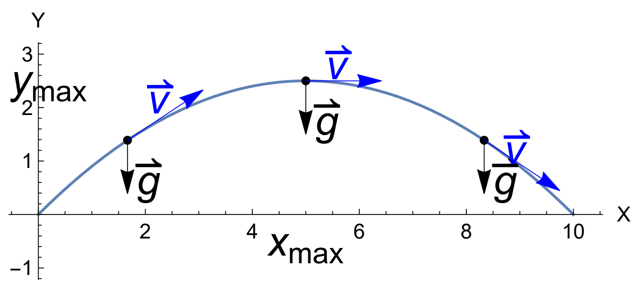
B. General

$$x\text{-axis horizontal, } y\text{-axis vertical (up): } \boxed{a_x = 0\ ,\quad a_y = -g} \tag{30}$$

$$\boxed{v_x = v_{0,x} = const\ ,\ v_y = v_{0,y} - gt} \text{ with } \boxed{v_{0,x} = v_0 \cos\theta\ ,\ v_{0,y} = v_0 \sin\theta} \tag{31}$$

$$\text{Displacement: } \boxed{x = x_0 + v_{0,x} t}\ ,\ \boxed{y = y_0 + v_{0,y} t - \frac{1}{2} g t^2} \tag{32}$$

$$\text{Also, } \boxed{y - y_0 = \frac{v_{0y}^2 - v_y^2}{2g}}$$



$$\text{Max elevation: } \boxed{y_{\max} - y_0 = \frac{v_{0,y}^2}{2g}}\ ,\ x_{\max} = \frac{v_{0,x} v_{0,y}}{g} \tag{33}$$

Range:

$$R = 2x_{\max} = 2\frac{v_0 \cos\theta\, v_0 \sin\theta}{g}$$

$$\boxed{R = \frac{v_0^2}{g} \sin(2\theta)} \tag{34}$$

Note maximum for $\theta = 45^o$.

Trajectory: (use $x_0 = y_0 = 0$). Exclude time, $t = x/v_{0,x}$. Then

$$y = x\frac{v_{0,y}}{v_{0,x}} - \frac{1}{2}g\frac{x^2}{v_{0,x}^2} = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2 \tag{35}$$

This is a parabola - see Fig. 10.

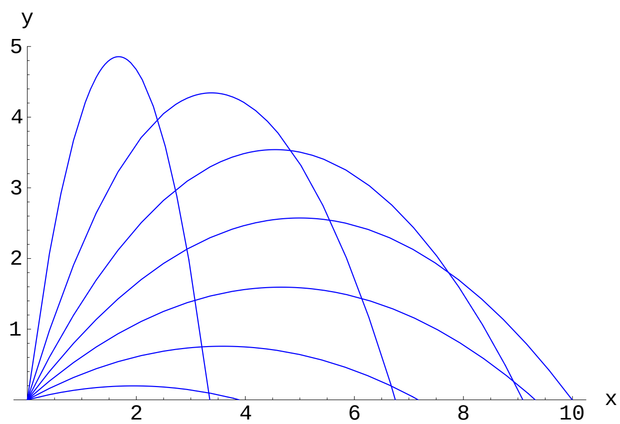


FIG. 10: Projectile motion for different values of the initial angle θ with a fixed value of initial speed v_0 (close to $10\,m/s$). Maximum range is achieved for $\theta = 45^o$.

C. Examples

Problem. A daredevil on a motorcycle wants to jump across an L=15m - wide river starting from a horizontal cliff which is H=10m high. What should be his initial speed V?

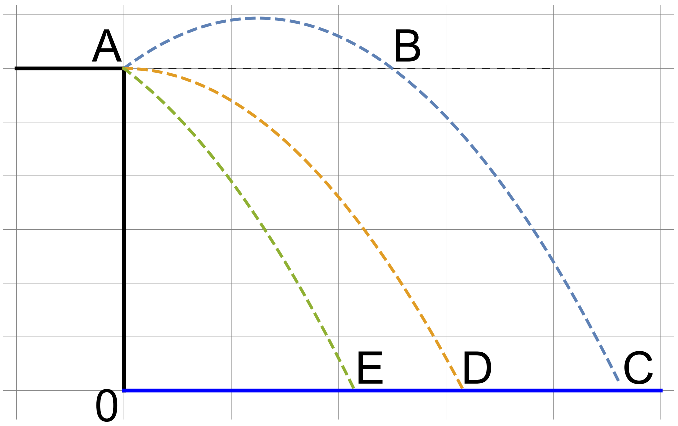
$$y = H + 0t - \frac{1}{2}gt^2 \;\Rightarrow\; t = \sqrt{\frac{2H}{g}} = \ldots$$

$$L = Vt \;\Rightarrow\; V = \frac{L}{t} = \ldots$$

Problem. A coastguard cannon is placed on a cliff $y_0 = 60\,m$ above the sea level. Three shells are fired at angles $\theta = 0$ and $\theta = \pm 30^o$ with horizontal, each with an initial speed $v_0 = 80\,m/s$. Find the following:

1. the horizontal distance x from the cliff to the point where each projectile hits the water

2. the speed upon impact



Solution: $\underline{\theta = 0}$ exactly as for the plane, $V = v_0$: First find t, time to hit the water (from vertical motion only!):

$$y = y_0 + 0t - \frac{1}{2}gt^2\,,\; y = 0 \;\Rightarrow\; t = \sqrt{\frac{2y_0}{g}} = \ldots\,, \quad \text{horizontal motion: } x = v_0 t = \ldots$$

(b) - horizontal component of the velocity $v_x = V$, vertical component $v_y = -gt = \ldots$

$$v = \sqrt{v_x^2 + v_y^2} = \ldots \approx 87.2\frac{m}{s}$$

$$\underline{\theta = +30^o}:\quad v_x = v_0\cos\theta \approx 69.3\frac{m}{s}\,,\; v_{0y} = v_0\sin\theta = 40\,\frac{m}{s}$$

Note $y = 0$ at the end. Find time from vertical motion only (use here $g \approx 10\,m/s^2$)

$0 = y_0 + v_{0y}t - \frac{1}{2}gt^2 \;\Rightarrow\; 0 = 60 + 40t - 5t^2$ or $t^2 - 8t - 12 = 0$ with $t \approx 9.3\,s$ (the positive root)

Horizontal distance: $x = v_x t = \ldots$

$$\text{Speed upon impact: } v = \sqrt{v_x^2 + (v_{0,y} - gt)^2} \approx 87.2\frac{m}{s}$$

Angle θ does not affect the final speed.

$\underline{\theta = -30^o}$ - same, but $v_{0,y} = -40\frac{m}{s}$ and $0 = 60 - 40t - 5t^2$ with $t \approx 1.3\,s$.

V. 2D MOTION

A. Introduction: Derivatives of a vector

Reminder from 1D motion:

$$\boxed{v = \frac{dx}{dt}}\,,\ \boxed{a = \frac{dv}{dt}}$$

$$\Delta v = \int_{t_1}^{t_2} a(t)\,dt\ ,\ \Delta x = \int_{t_1}^{t_2} v(t)\,dt$$

For 2D:

$$\vec{r}(t) = (x(t)\ ,\ y(t)) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = \left(\frac{dx}{dt}\ ,\ \frac{dy}{dt}\right) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

$$\vec{a}(t) = \frac{d^2}{dt^2}\vec{r} = \left(\frac{d^2x}{dt^2}\ ,\ \frac{d^2y}{dt^2}\right) = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j}$$

and similarly for integrals.

B. General

$$\text{Position: } \vec{r} = \vec{r}(t)$$

$$\text{Average velocity: } \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

(see Fig. 11).

$$\text{Instantaneous velocity: } \vec{v} = \lim_{\Delta t \to 0} \vec{v}_{av} = \frac{d\vec{r}}{dt}$$

$$\text{Average acceleration: } \vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\text{Instantaneous acceleration: } \vec{a} = \lim_{\Delta t \to 0} \vec{a}_{av} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

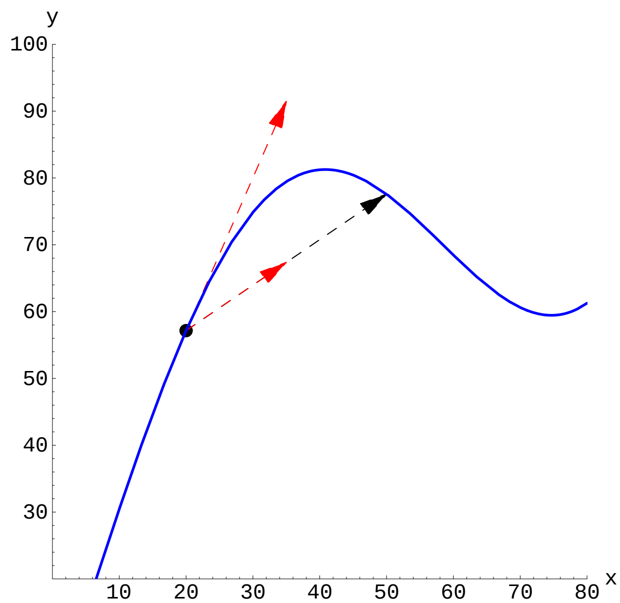


FIG. 11: Position of a particle $\vec{r}(t)$ (blue line), finite displacement $\Delta\vec{r}$ (black dashed line) and the average velocity $\vec{v} = \Delta\vec{r}/\Delta t$ (red dashed in the same direction). The instantaneous velocity at a given point is tangent to the trajectory.

C. $\vec{a} = const$

$$\Delta\vec{v} = \vec{a}\cdot\Delta t$$

or with $t_0 = 0$

$$\vec{v} = \vec{v}_0 + \vec{a}\cdot t \tag{36}$$

Displacement:

$$\vec{r} = \vec{r}_0 + \vec{v}_0\cdot t + \frac{1}{2}\vec{a}\cdot t^2 \tag{37}$$

(The above can be proven either by integration or by writing eq. (36) in components and using known 1D results).

Example. A particle moves according to

$$\vec{r}(t) = 2t^2\hat{\imath} + (3t+4)\hat{\jmath}$$

Find $\vec{v}(t)$ and $\vec{a}(t)$ at $t = 2\,s$
Solution. Consider $x(t) = 2t^2$ and $y(t) = 3t + 4$ separately

$$v_x = \frac{dx}{dt} = 4t\,,\ a_x = \frac{dv_x}{dt} = 4$$

$$v_y = \frac{dy}{dt} = 3\,,\ a_y = \frac{dv_y}{dt} = 0$$

$$\text{at t=2}\ \ v_x = 8\,,\ a_x = 4\,,\ v_y = 3\,,\ a_y = 0$$

$$\text{or}\ \ \vec{v}(t=2) = 8\hat{\imath} + 3\hat{\jmath}\,,\ \vec{a} = 4\hat{\imath}$$

Example. Describe the x and y motion for $\vec{a} = -\hat{\imath} + 2\hat{\jmath}$ (in m/s^2) and $\vec{v}_0 = 10\hat{\imath}$ (in m/s). At $t = 0$ the particle is at the origin. Solution

$$x(t) = 10t - \frac{1}{2}t^2\,,\ y(t) = \frac{1}{2}2t^2$$

Can eliminate t via (the simpler) $y(t)$ from which we get $t = \sqrt{y}$ and

$$x = 10\sqrt{y} - y/2$$

D. $\vec{a} = \vec{g} = 0\hat{i} - 9.8\hat{j}$ **(projectile motion)**

see previous section

HW examples

A car comes to a bridge during a storm and finds the bridge washed out. The driver must get to the other side, so he decides to try leaping it with his car. The side the car is on is H=21.1 m above the river, whereas the opposite side is a mere h=2.4 m above the river. The river itself is a raging torrent L=61.0 m wide.

$$t = \sqrt{\frac{2(H-h)}{g}}\,,\ v_x = L/t = \ldots$$

$$v_y = 0 - gt\,,\ v = \sqrt{v_x^2 + v_y^2}$$

$$\text{or } h - H = \frac{0 - v_y^2}{2g}\,,\ v_y = -\sqrt{2g(H-h)}\,,\ v = \ldots$$

E. Uniform circular motion

1. Preliminaries

Radian measure of an angle:

$$\boxed{l = r\theta}$$

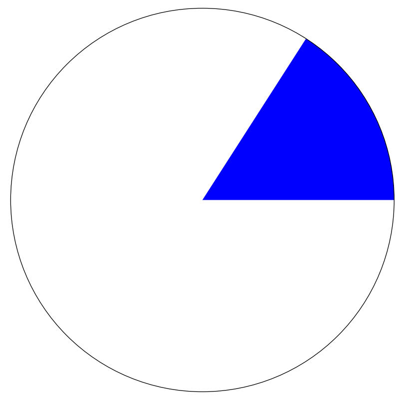


FIG. 12: Angle of $1\,rad \approx 57.3^o$. For this angle the length of the circular arc exactly equals the radius. The full angle, 360^o , is 2π radians.

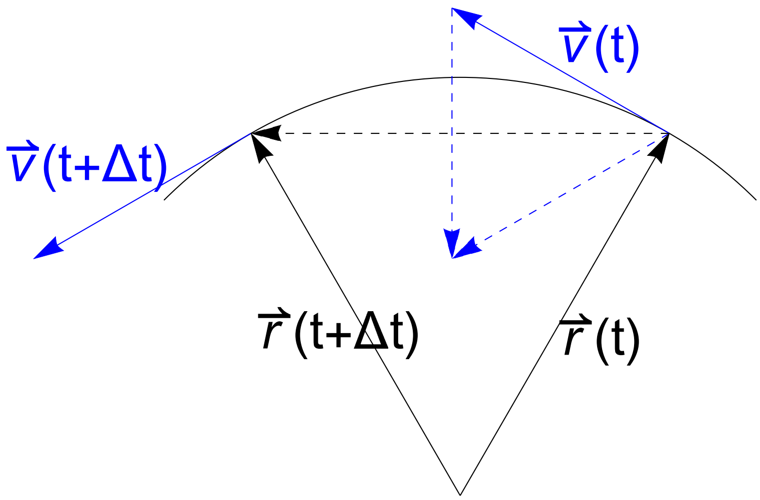
Consider motion around a circle with a constant speed v. The velocity $\vec{v}$, however, changes directions so that there is acceleration.

Period of revolution:

$$T = 2\pi r/v \tag{38}$$

with $1/T$ - "frequency of revolution". Angular velocity (in rad/s):

$$\boxed{\omega = \frac{2\pi}{T} = \frac{v}{r}} \tag{39}$$



Consider counterclockwise rotation with representative position vectors $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ (black) which are symmetric with respect to the vertical. Note that $\vec{v}$ (blue) is *always* perpendicular to $\vec{r}$. Thus, from geometry vectors $\vec{v}\,(t+\Delta t)\,, \vec{v}(t)$ and $\Delta\vec{v}$ (dashed blue) form a triangle which is similar to the one formed by $\vec{r}\,(t+\Delta t)\,, \vec{r}(t)$ and $\Delta\vec{r}$ (dashed black). Or,

$$\frac{|\Delta\vec{v}|}{v} = \frac{|\Delta\vec{r}|}{r}$$

$$a = \lim_{\Delta t\to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \frac{v}{r}\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}$$

Or

$$\boxed{a_c = \frac{v^2}{r} = \omega^2 r} \tag{40}$$

We can use derivatives with the major relation

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t)\ ,\quad \frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)\ , \tag{41}$$

One has

$$\vec{r}(t) = (x,\, y) = (r\cos\omega t,\, r\sin\omega t)$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-r\omega\sin\omega t,\, r\omega\cos\omega t)$$

$$\text{Note: } \vec{r}(t)\cdot\vec{v}(t) = -r^2\omega\cos(\omega t)\sin(\omega t) + r^2\omega\sin(\omega t)\cos(\omega t) = 0$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-r\omega^2\cos\omega t,\, -r\omega^2\sin\omega t\right) = -\omega^2\vec{r} \tag{42}$$

which gives not only magnitude but also the direction of acceleration opposite to $\vec{r}$, i.e. towards the center.

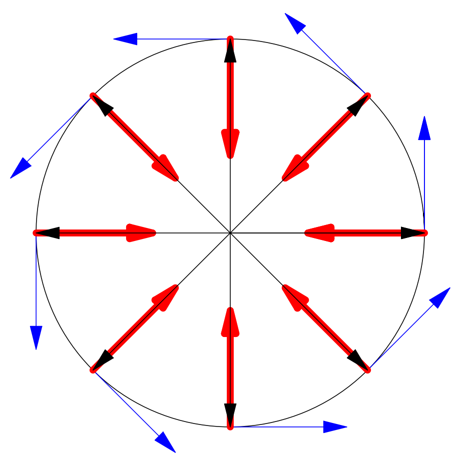


FIG. 13: Position (black), velocities (blue) and acceleration (red) vectors for a uniform circular motion in counter-clockwise direction.

VI. NEWTON'S LAWS

A. Force

1. Units

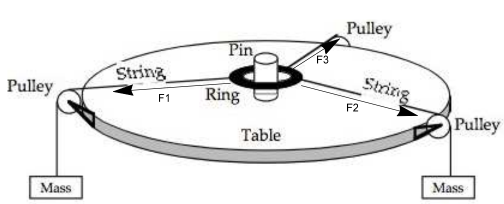
"Newton of force"

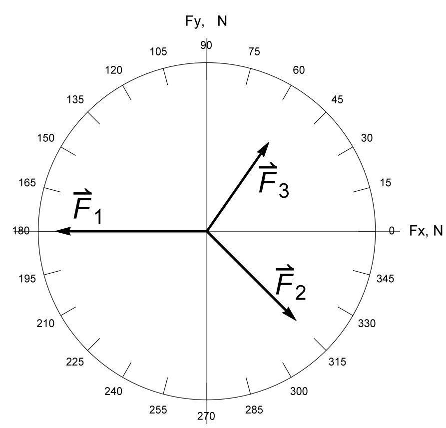
$$N = kg\frac{m}{s^2} \tag{43}$$

2. Vector nature

From experiment, action of two independent forces $\vec{F}_1$ and $\vec{F}_2$ is equivalent to action of the resultant

$$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{44}$$

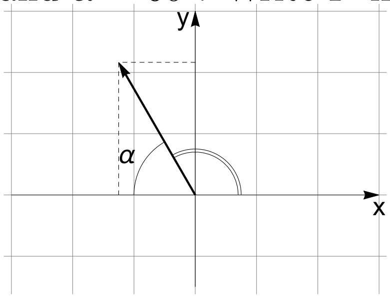




The force table (left) and its schematic representation (right)

In equilibrium: $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0\,,\ \Rightarrow\ \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$

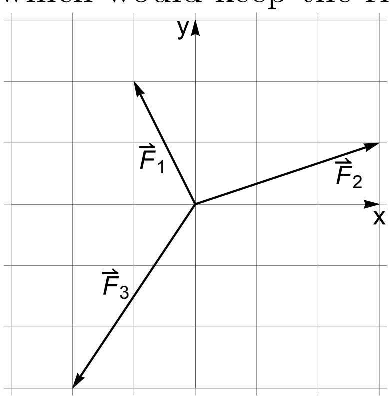
Example. In the picture below the magnitude of force $F \equiv \left|\vec{F}\right| = 2.5\,N$ and $\alpha = 60^o$. Write $\vec{F}$ in unit vector notations.



$F_x = -F\cos\alpha = F\cos(180^o - \alpha) = -1.25\,,\ F_y = F\sin\alpha = F\sin(180^o - \alpha) = 2.165 \ \Rightarrow$

$$\vec{F} = -1.25\hat{\imath} + 2.165\hat{\jmath}$$

Example. Two forces $\vec{F}_1 = -\hat{\imath} + 2\hat{\jmath}$ and $\vec{F}_2 = 3\hat{\imath} + \hat{\jmath}$ (in Newtons) are applied to the plastic ring in the middle of the force table. Find the 3rd force which would keep the ring in equilibrium.



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \ \Rightarrow\ \vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right) = -\left(2\hat{\imath} + 3\hat{\jmath}\right) = -2\hat{\imath} - 3\hat{\jmath}$$

Find the angle α between $\vec{F}_2$ and the x-axis:

$$\cos\alpha = F_{2,x}/F_2 = 3/\sqrt{3^2+1^2} = 3/\sqrt{10}\,,\ \alpha = \ldots$$

Find the angle β between $\vec{F}_1$ and $\vec{F}_2$:

$$\cos\beta = \vec{F}_1 \cdot \vec{F}_2/(F_1 * F_2) = (-1*3+2*1)/(\sqrt{1^2+2^2} * \sqrt{3^2+1^2}) = \ldots,\ \beta = \ldots$$

$$\text{Force of gravity: } \boxed{\vec{F}_g = m\vec{g}} \tag{45}$$

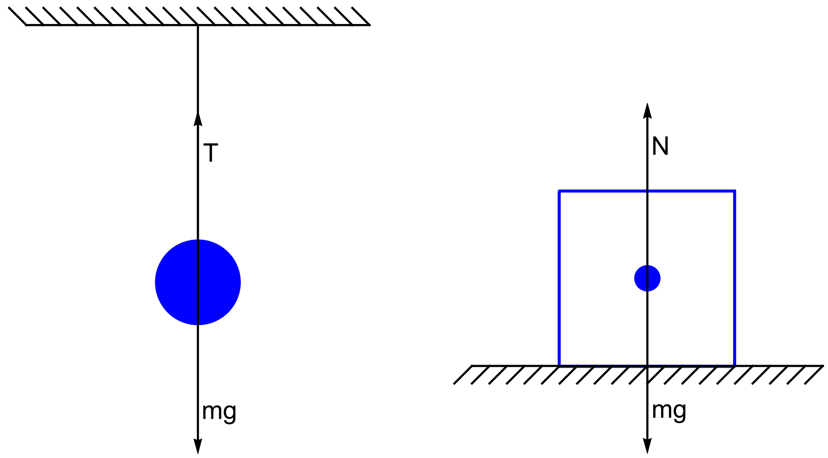
(sometimes called "weight" which is not always correct). Force $m\vec{g}$ is applied to the center of the body (center-of-mass, as we learn later.)

Tension $\vec{T}$, with magnitude T constant along a string (even if there is a massless pulley which changes the direction of $\vec{T}$).

Normal force $\vec{N}$ is perpendicular to the surface and acts on the body. Common to represent $\vec{N}$ as applied to the center of a body, (though in reality it is applied to the surface of contact).

$$\text{In equilibrium: } \vec{F}_{net} \equiv \sum_i \vec{F}_i = 0 \tag{46}$$

(all forces are applied to the same body! Note only "+" in the sum, *regardless* of the actual direction of vectors).

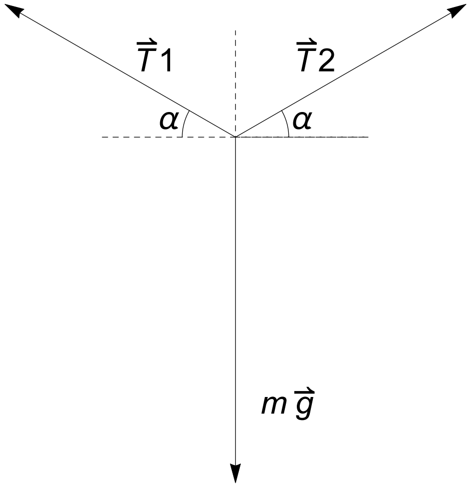


Equilibrium of a body under the action of gravity and tension (left, $m\vec{g}+\vec{T}=0$) or gravity and normal force (right, $m\vec{g}+\vec{N}=0$). In projections on a vertical axis with the positive direction - up)

$$\text{left: } T - mg = 0 \;\Rightarrow\; T = mg \;\; \text{(Note: as a rule, all symbols are positive)}$$

$$\text{right: } N - mg = 0 \;\Rightarrow\; N = mg \text{ (for } \vec{a} = 0 \text{ only!)}$$

Example: "bird in the middle of a cord"



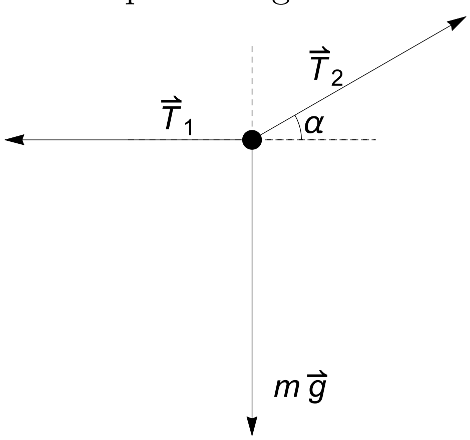
$$\text{From symmetry: } |\vec{T}_1| = |\vec{T}_2| \equiv T \ , \ x \text{ - automatic}$$

$$\text{from y : } T\sin\alpha + T\sin\alpha - mg = 0\,, \ \Rightarrow \ T = mg/(2\sin\alpha)$$

$$\text{limits: } \alpha = 90^o\,, \ T = mg/2$$

$$\alpha \to 0\,, \ T \to \infty$$

Example: "weight between two cords"



$$\vec{T}_1 + \vec{T}_2 + m\vec{g} = 0$$

$$x: \quad -T_1 + T_2\cos\alpha + 0 = 0$$

$$y: \quad 0 + T_2\sin\alpha - mg = 0$$

$$T_2 = mg/\sin\alpha\,, \ T_1 = T_2\cos\alpha = mg\cot\alpha$$

B. The 3 Laws of motion (Newton)

1. If $\vec{F} = 0$ (no net force) then $\vec{v} = const$

2.

$$\boxed{\vec{F} = m\vec{a}} \tag{47}$$

3.

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}} \tag{48}$$

Notes: the 1st Law is *not* a trivial consequence of the 2nd one for $\vec{F} = 0$, but rather it identifies inertial reference frames where a free body moves with a constant velocity.

In the 3rd Law forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are applied to *different* bodies; both forces, however, are of the same physical nature (e.g. gravitational attraction).

Example (1D): Velocity, in m/s, of an $m = 2\,kg$ particle is a polynomial function of time t (in seconds):

$$v(t) = t^4 - t^3 - t^2 + t + 4\,.\ \text{Find}\ F(0)$$

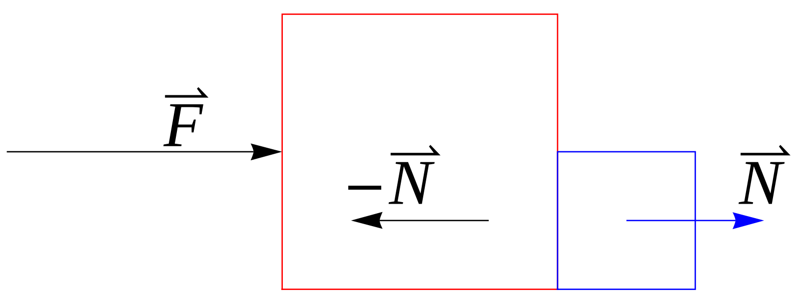
$$\text{From}\ \frac{d}{dt}t^n = nt^{n-1}\ \text{find}\ a(t) = \frac{d\,v(t)}{dt} = 4t^3 - 3t^2 - 2t + 1$$

$$F(t) = ma(t)\ \Rightarrow\ F(0) = ma(0) = 2\,kg\ \times 1\,\frac{m}{s^2} = 2\,N$$

The following examples will be discusses in class:

- finding force between two accelerating blocks, finding force between cars in accelerating train
- hanging block pulling a block on a frictionless surface
- apparent weight in an elevator
- (*) Atwood machine
- block on inclined plane (no friction)

Example. Two blocks: a horizontal force $\vec{F}$ is applied to a block with mass M (red) which in turn pushes a block with mass m (blue); ignore friction. Find $\vec{N}$ and $\vec{a}$. (note that the 3rd Law is already used in the diagram).

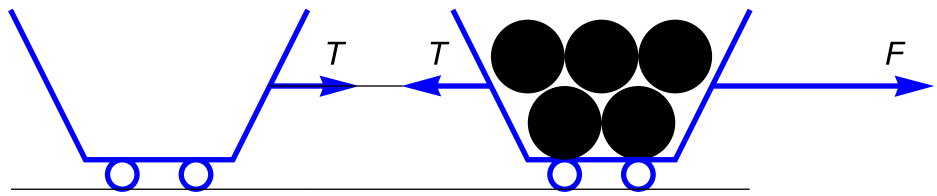


Solution: first treat the 2 blocks as a single solid body (at this stage ignore N which is an internal force). From 2nd Law for this "body" (x axis is horizontal, in direction of acceleration)

$$a = F/(M + m)$$

Now note that the blue block m is accelerated *only* due to force N. Thus, from 2nd Law for mass m alone:

$$N = ma$$



Example. A train has 2 cars: loaded with M=18,000 kg (first) and empty with m=2,000 kg (last). Find the tension T in the coupler connecting the cars if the applied force is F=40,000 N.

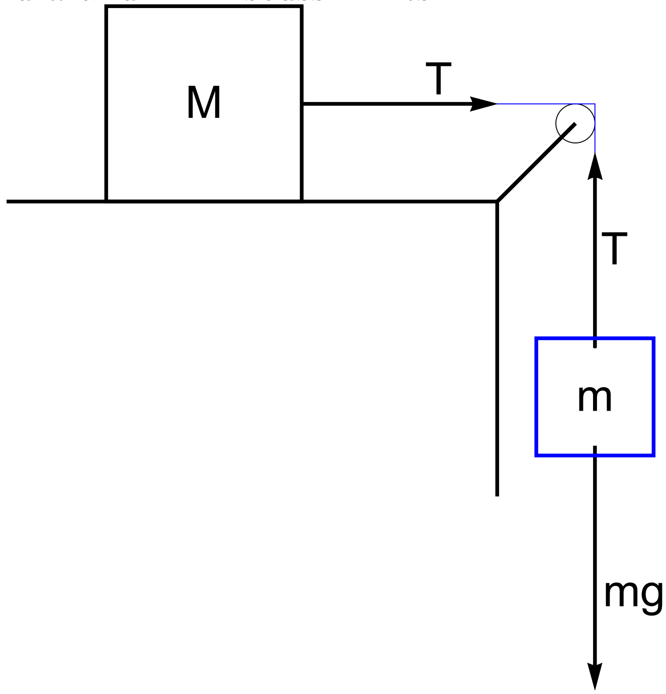
Solution. First, find a treating the train as a single body:

$$a = F/(M + m) = 4 \cdot 10^4/(2 \cdot 10^3 + 18 \cdot 10^3) = 2\,m/s^2$$

The last car alone is accelerated *only* by tension T. Thus,

$$T = ma = 2 \cdot 10^3 \times 2 = 4 \cdot 10^3\,N$$

Example: hanging block m pulling a block M on a frictionless surface. Find a and T. Discuss limits.



$$\text{mass } M\text{, axis horizontal:} \quad T = Ma$$

$$\text{mass } m\text{, axis down:} \quad mg - T = ma$$

Note: two equations for two unknowns (a and T).

$$mg = Ma + ma \;\Rightarrow\; a = g\frac{m}{M+m}$$

$$T = Ma = g\frac{mM}{M+m}$$

Limits:

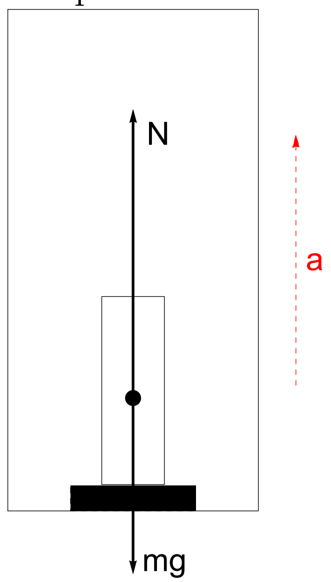
$M \to \infty$ (does not move). $a \to 0$, $T \approx mg$, as expected.

$M \to 0$ (no resistance to free fall). $a \to g$, $T \to 0$.

Quick solution. Use only *external* force mg and consider acceleration of combined mass $M + m$. (This is non-rigorous, but can be useful for verification). This immediately gives

$$mg = (M+m)a \;\Rightarrow\; a = g\frac{m}{M+m} \text{ as before}$$

Weight in an elevator: Find the "apparent weight" -reading of floor scale- of a person of mass m if the elevator accelerates up with acceleration $\vec{a}$.

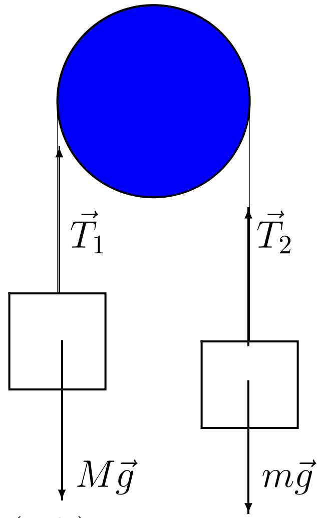


Solution. Forces on the person: $m\vec{g}$ (down, applied to center-of-mass) and $\vec{N}$ - up (reaction of the floor, applied to feet, but show as applied to center for simplicity). [A force $-\vec{N}$ -not shown- acts on the floor and determines "apparent weight"]. The 2nd Law for the person

$$-mg + N = ma \Rightarrow N = m(g + a)$$

Note: $N > mg$ for $\vec{a}$ up and $N < mg$ for $\vec{a}$ down; $N = 0$ for $\vec{a} = \vec{g}$.

Advanced. Atwood machine:



Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has negligible rotational inertia, so that both strings have the same tension $\left|\vec{T}_1\right| = \left|\vec{T}_2\right| = T$

Quick solution. Use external forces only and combined mass $M + m$.

$$Mg - mg = (M+m)a \;\Rightarrow\; a = g\frac{M-m}{M+m}$$

If need tension, write individual 2nd Laws:

$$Mg - T_1 = Ma \;\Rightarrow\; T_1 = M(g-a) = Mg\left(1 - \frac{M-m}{M+m}\right) = \frac{2mMg}{M+m}$$

$$T_2 - mg = ma\,,\; T_2 = m(g+a) = \ldots \text{ (same)}$$

Limits:

$$M = m\,,\; a = 0\,,\, T_1 = T_2 = Mg$$

$$m = 0\,,\; a = g\frac{M-0}{M+0} = g\,,\, T_1 = T_2 = 0$$

Academic solutions. Start with 2nd Law(s) for each body separately; use $T_1 = T_2 = T$:

$$Mg - T = Ma\,,\; T - mg = ma$$

Add together: T gets canceled and $(M-m)g = (M+m)a$ - same as before.

Frictionless incline - Fig. 14.

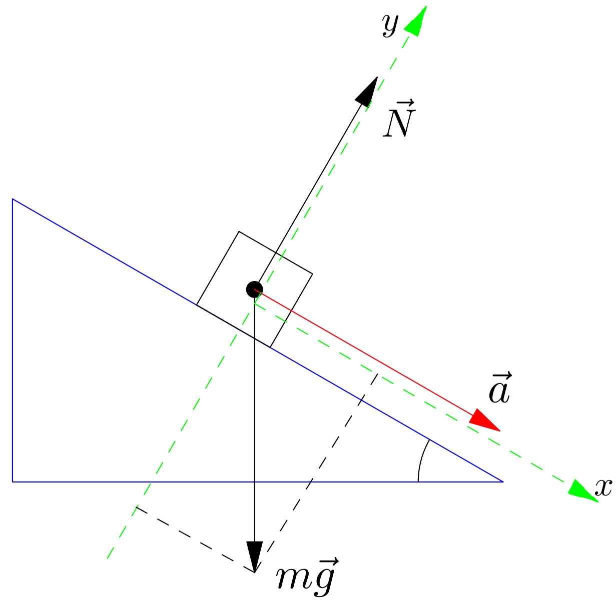


FIG. 14: A block on a frictionless inclined plane which makes an angle θ with horizontal.

- identify forces, $m\vec{g}$ and $\vec{N}$ in our case and the assumed acceleration $\vec{a}$ (magnitude still to be found).
- write the 2nd Law (vector form!) $\vec{N} + m\vec{g} = m\vec{a}$
- select a "clever" system of coordinates x, y.
- write down projections of the vector equation on the x, y axes, respectively (note, components of the force of gravity in such coordinates are always $\boxed{mg\sin\theta}$ downhill parallel to the incline and $-mg\cos\theta$ in perpendicular direction):

$$x: \quad mg\sin\theta = ma$$

$$y: \quad N - mg\cos\theta = 0$$

the x-equation will give acceleration

$$\boxed{a = g \sin\theta}$$

(which is already the solution); the y equation determines N.

- before plugging in numbers, a good idea is to check the limits. Indeed, for $\theta = 0$ (horizontal plane) $a = 0$ no acceleration, while for $\theta = \pi/2$ one has $a = g$, as should be for a free fall.

A few practical remarks to succeed in such problems.

- The original diagram should be BIG and clear. If so, you will use it as a FBD, otherwise you will have to re-draw it separately with an extra possibility of mistake.

- In the picture be realistic when dealing with "magic" angles of 30, 45, 60 and 90 degrees. Otherwise, a clear picture is more important than a true-to-life angle.

- Vectors of forces should be more distinct than anything else in the picture; *do not* draw arrows for projection of forces - they can be confused with real forces if there are many of them.

- the force of gravity in the picture should be immediately identified as $m\vec{g}$ (using an extra tautological definiton, such as $\vec{F}_g = m\vec{g}$ adds an equation and confuses the picture).

- If only one body is of interest, do not draw any forces which act on other bodies (in our case that would be, e.g. a force $-\vec{N}$ which acts on the inclined plane).

- select axes only *after* the diagram is completed and the 2'd Law is written in vector form. As a rule, in dynamic problems one axes is selected in the direction of acceleration (if this direction can be guessed).

Example. A block of mass M slides down an incline which is $L = 4\,m$ long and makes $\theta = 30^o$ with horizontal. No friction. Find the speed v at the bottom of the incline.

$$\text{from dynamics: } a = g\sin\theta \text{ (and } M \text{ does not matter)}$$

$$\text{from kinematics: } L = \frac{v^2 - v_0^2}{2a}\,,\ v_0 = 0\ \Rightarrow$$

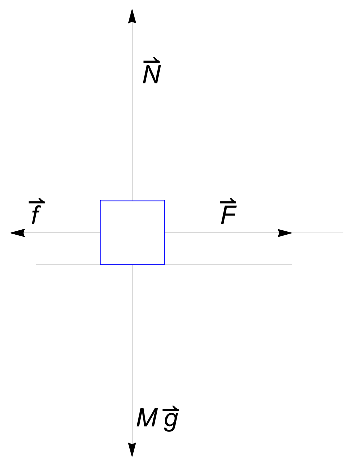
$$v^2 = 2aL = 2gL\sin\theta = 2gh \text{ with } h = L\sin\theta\,, \text{ vertical displacement}$$

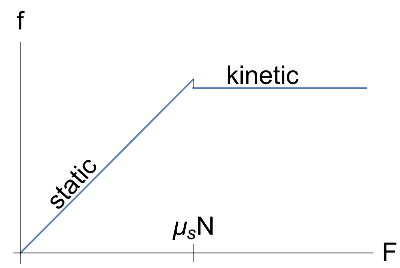
$$v = \sqrt{2gh} = \sqrt{2\cdot 9.8\cdot 4\cdot \sin 30^o} = \ldots$$

(the same will be later re-derived from conservation of energy)

VII. NEWTON'S LAWS: APPLICATIONS TO FRICTION AND TO CIRCULAR MOTION

A. Force of friction





Left: $\vec{F}$ - external force, $\vec{f}$ - friction force, kinetic if the block is moving ($F \geq f$), static if not moving ($f = F$).
Right: The f vs. F dependence. Note: the *maximum* static friction is usually slightly bigger than kinetic.

Force of friction on a moving body:

$$f = \mu N \tag{49}$$

Direction - against velocity; μ (or μ_k) - kinetic friction coefficient.

Static friction:

$$f_s \leq \mu_s N \tag{50}$$

with μ_s - static friction coefficient.

1. *Example: block on inclined plane*

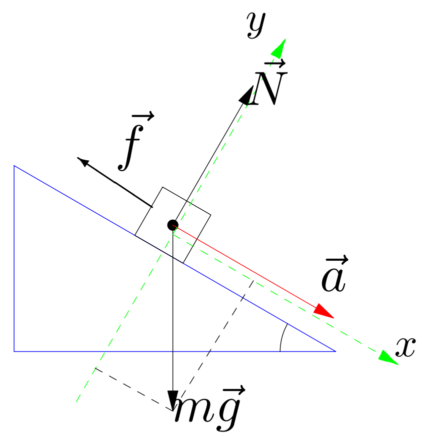


FIG. 15: A block sliding down an inclined plane with friction. The force of friction $\vec{f}$ is opposite to the direction of motion and equals $\mu_k N$.

- identify forces, $\vec{f}$, $m\vec{g}$ and $\vec{N}$ and the acceleration $\vec{a}$ (magnitude still to be found).
- write the 2nd Law (vector form!): $\vec{f} + \vec{N} + m\vec{g} = m\vec{a}$
- select a "clever" system of coordinates and write down projections of the vector equation on the x, y axes:

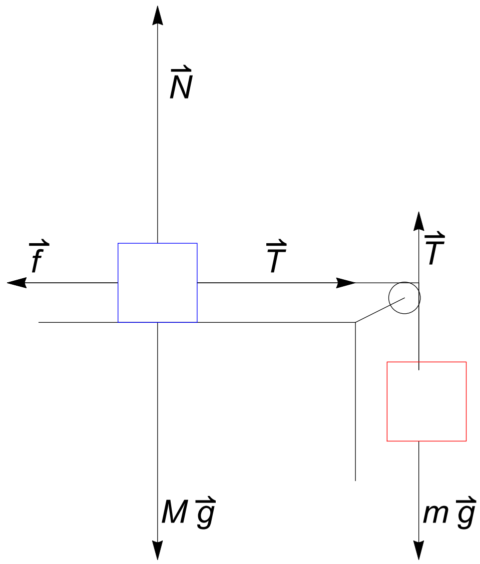
$$x : \quad -f + mg\sin\theta = ma$$

$$y : \quad N - mg\cos\theta = 0 \;\Rightarrow\; N = mg\cos\theta$$

- relate friction to normal force: $f = \mu_k N = \mu_k mg\cos\theta$. This goes into the above equation for the x-axis: $-\mu_k mg\cos\theta + mg\sin\theta = ma \;\Rightarrow$

$$a = g\,(\sin\theta - \mu_k\cos\theta) \;>\; 0$$

- keeps moving: $\boxed{\mu_k \leq \tan\theta}$, starts moving: $\boxed{\mu_s < \tan\theta}$



Quick solution: First, recall without friction. External force mg, thus

$$mg = (M+m)a \;\Rightarrow\; a = g\frac{m}{m+M}$$

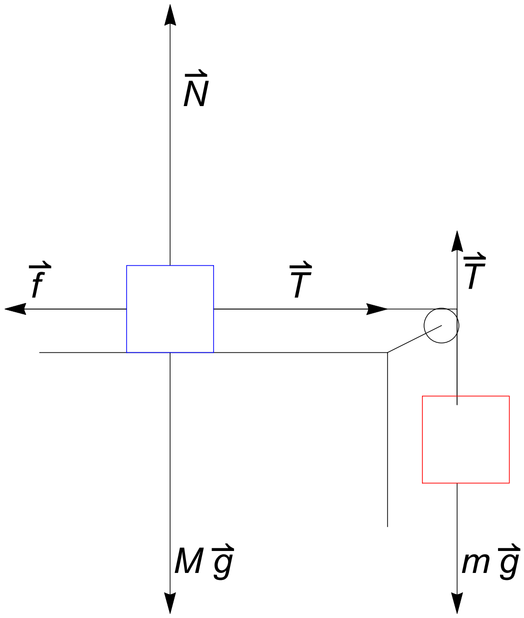
$$\text{for mass } M \text{ on the table: } T = Ma \;\Rightarrow\; T = g\frac{Mm}{m+M}$$

With friction: Assume motion. Use $\boxed{f = \mu_k mg\cos\theta}$ for a general incline and $\boxed{f = \mu_k mg}$ for a horizontal surface, $\theta = 0$.

$$\text{external forces } mg \text{ and } f\text{, net } mg - f = mg - \mu_k Mg \;\Rightarrow$$

$$mg - \mu_k Mg = (M+m)a \;\Rightarrow\; a = \frac{mg - \mu_k Mg}{M+m} = g\frac{m - \mu_k M}{M+m} > 0$$

(if $\mu_k M > m$ motion is impossible - friction too strong, see below).



If not moving:

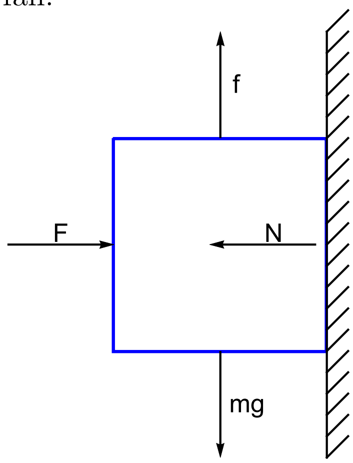
$$\text{Hanging mass: } mg - T = 0 \Rightarrow T = mg$$

$$\text{Mass on the table: } T - f = 0 \Rightarrow f = T = mg$$

$$\text{Restriction: } f \leq \mu_s M g \Rightarrow \mu_s M \geq m$$

Recall that friction force f is related to normal force N by $f = \mu_k N$ (kinetic) or $f \leq \mu_s N$ (static). Usually, N is determined by Mg (horizontal surface) or by $Mg\cos\theta$ (inclined). But not always...

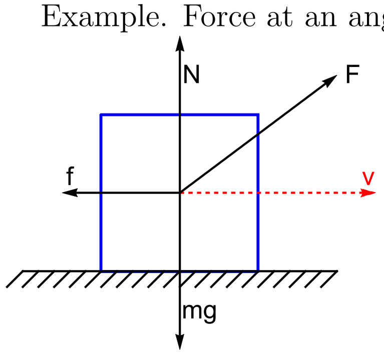
Example. Block pressed to a rough vertical wall. Find the minimal force F for it not to fall.



$$\text{from horizontal: } N = F\,, \quad \text{from vertical: } f = mg$$

$$f \leq \mu_s N \;\Rightarrow\; mg \leq \mu_s F\,, \text{ or } F \geq mg/\mu_s$$

Example. Force at an angle - reduced friction.



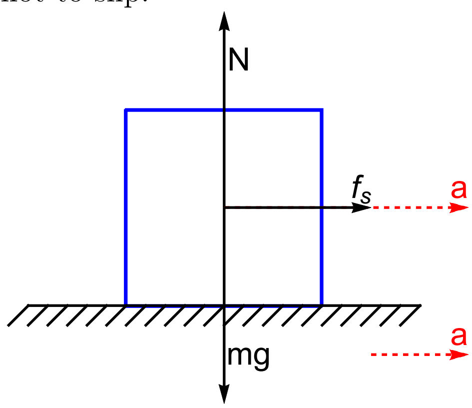
$$x: \quad -f + F\cos\theta = 0\,, \quad y: \quad N - mg + F\sin\theta = 0\,, \quad f = \mu N$$

$$N = mg - F\sin\theta \;\Rightarrow\; -\mu(mg - F\sin\theta) + F\cos\theta = 0\,, \; F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

$$F = \min \text{ if } \cos\theta + \mu\sin\theta = \max$$

$$0 = \frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = -\sin\theta + \mu\cos\theta \;\Rightarrow\; \theta = \tan^{-1}\mu$$

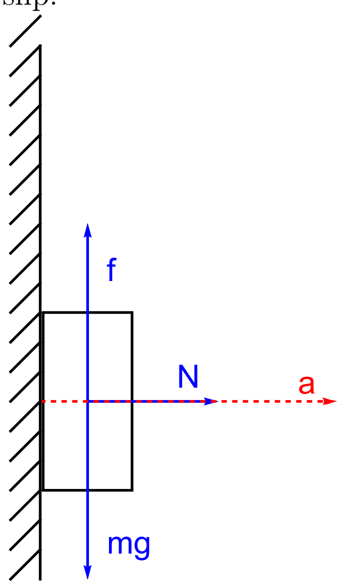
Example. Coin on accelerating rough horizontal surface. Find maximum a for the coin not to slip.



$$\text{from horizontal: } f_s = ma\,, \quad \text{from vertical: } N = mg$$

$$f \leq \mu_s N \;\Rightarrow\; ma \leq \mu_s mg\,, \text{ or } a \leq \mu_s g$$

Example. The same as above for a vertical wall. Find the *minimal* a for the coin not to slip.



$$\text{from horizontal: } N = ma\,, \quad \text{from vertical: } f = mg$$

$$f \leq \mu_s N \;\Rightarrow\; mg \leq \mu_s ma\,, \text{ or } a \geq g/\mu_s$$

Note: if μ_s is small, a needs to be VERY large.

B. Centripetal force

Newton's laws are applicable to *any* motion, centripetal including. Thus, for centripetal force of any physical origin

$$F_c = ma_c \equiv m\frac{v^2}{r} = m\omega^2 r \tag{51}$$

Centripetal force is always directed towards center, perpendicular to the velocity - see Fig. 16. Examples: tension of a string, gravitational force, friction.

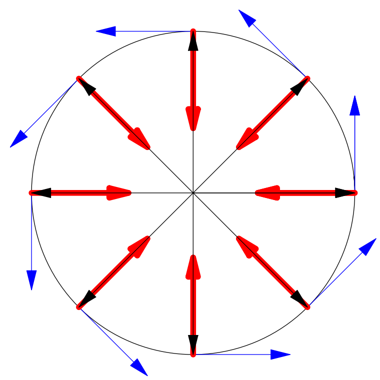


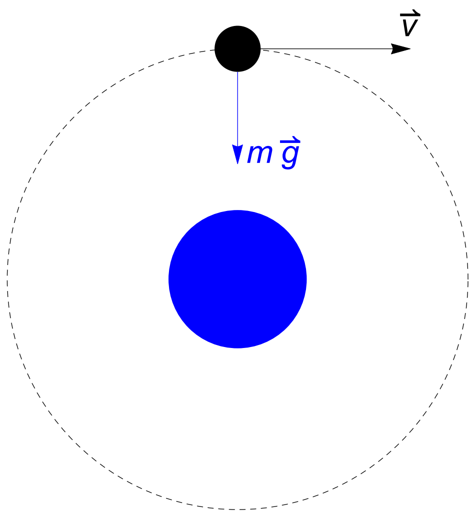
FIG. 16: Position (black), velocities (blue) and centripetal force (red) vectors for a uniform circular motion in counter-clockwise direction. The direction of force coincides with centripetal acceleration (towards the center). The value of centripetal force at each point is determined by the vector sum of actual physical forces, e.g. normal force and gravity in case of a Ferris Wheel, or tension plus gravity in case of a conic pendulum.

Simple example: The centripetal force is due to tension.
A particle with mass $m = 4.0\,kg$ is attached to a string with length $l = 1.0\,m$ and is moving at a constant speed around a horizontal circle. It takes $t = 3.0\,s$ to complete one revolution. Find the magnitude of the tension force F in the string. (Ignore gravity)
Solution: Since $F = ma_c$, need acceleration. Use

$$a_c = \omega^2 r \text{ with } r = l = 1.0\,m \text{ and } \omega = 2\pi/t = \frac{2\pi}{3.0} = \ldots$$

$$F = ma_c = m\omega^2 r = \ldots$$

1. Example of Satellite. "Centripetal force" is force of gravity

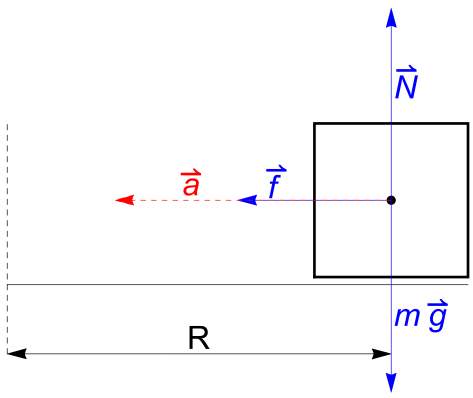


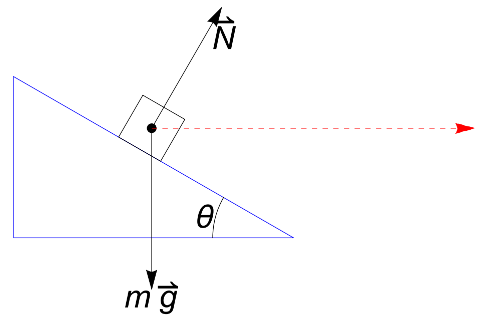
$$\vec{F}_g = m\vec{a}_c \text{ and } \vec{F}_g = m\vec{g} \;\Rightarrow\; \vec{a}_c = \vec{g}$$

$$a_c = \frac{v^2}{r} \;\Rightarrow\; \frac{v^2}{r} = g \;\Rightarrow\; \boxed{v = \sqrt{gr}}$$

$$\text{for Earth, low orbit } r \simeq 6400\,km\text{: } v = \left(9.8\frac{m}{s^2} \times 6.4 \times 10^3 \times 10^3 m\right)^{1/2} \simeq 8 \times 10^3 \frac{m}{s} = 8\frac{km}{s}$$

$$\text{Period of revolution: } \frac{2\pi r}{v} \approx \frac{40,000\,km}{8\,km/s} = 5000\,s \text{ (for Earth)}$$





Left: car making a left turn of radius R (view from back); find the max speed for a given friction coefficient μ_s. Right: car making a turn on an inclined road (no friction force); find the speed v.

$$\text{horizontal road with friction, 2nd Law}: \ \vec{N} + \vec{f} + m\vec{g} = m\vec{a}$$

$$x\text{-axis - towards the center (dashed): } f = ma = mv^2/R$$

$$y\text{-axis, up: } N - mg = 0$$

and

$$f \leq \mu_s N = \mu_s mg \quad \Rightarrow$$

$$v^2/R \leq \mu_s g\,, \ v_{\max} = \sqrt{\mu_s g R}$$

$$\text{incline}: \ \vec{N} + m\vec{g} = m\vec{a}$$

$$x: \quad N \sin\theta = mv^2/R$$

$$y: \quad N\cos\theta - mg = 0$$

$$mg\tan\theta = mv^2/R$$

$$\tan\theta = \frac{v^2}{gR}$$

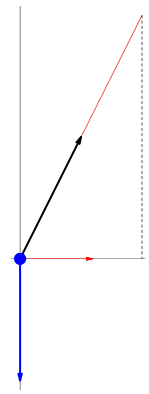


FIG. 17: A conic pendulum rotating around a vertical axis (dashed line). Two forces gravity (blue) and tension (black) create a centripetal acceleration (red) directed towards the center of rotation. The angle with vertical is θ, length of the s tring is L and radius of revolution is $r = L\sin\theta$.

3. Advanced: Conic pendulum

See Fig. 17, forces will be labeled in class.

One has the 2nd Law:

$$\vec{T} + m\vec{g} = m\vec{a}_c$$

In projections:

$$x: \quad T\sin\theta = ma_c = m\omega^2 r$$

$$y: \quad T\cos\theta - mg = 0$$

Thus,

$$\omega = \sqrt{\frac{g}{L\cos\theta}} \simeq \sqrt{\frac{g}{L}}$$

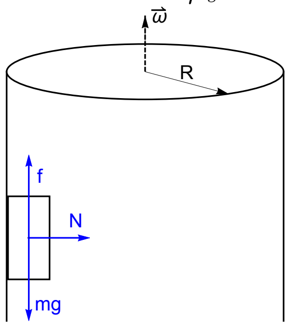
(the approximation is valid for $\theta \ll 1$). The period of revolution

$$\frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{L}{g}}$$

Note that mass and angle (if small) do not matter.

4. "Barrel-of-fun"

In the "barrel-of-fun" attraction a person stays in a spinning room with his back against the wall. Suddenly, the floor falls out but the person does not (!) Find the minimal ω if $R = 5\,m$ and $\mu_s = 0.3$.



$$m\vec{g} + \vec{N} + \vec{f} = m\vec{a}_c$$

$\vec{a}$ in the diagram - to the right (towards center). Thus select x-axis -right, y-axis - up

$$x: \quad N = ma_c\,, \quad y: \quad f - mg = 0$$

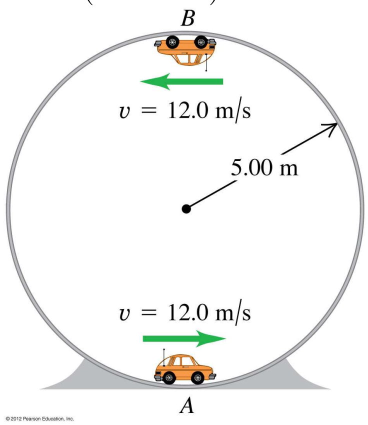
and $f \leq \mu_s * N$. Thus,

$$mg \leq \mu_s m a_c\,,\; g \leq \mu_s a_c = \mu_s \omega^2 R$$

$$\omega^2 \geq \frac{g}{\mu_s R}\,,\; \omega_{\min} = \sqrt{\frac{g}{\mu_s R}} = \sqrt{\frac{9.8}{0.3 * 5}} \approx 2.6\,\frac{rad}{s}$$

5. "Loop-of-death"

Find N (""weight") of the toy car ($m = 0.3\,kg$) at the top and the bottom parts of the circle ($R = 5\,m$) if $v = 12\,m/s$. Find $v_{\min}$ for the car not to fall.



$$\vec{N} + m\vec{g} = m\vec{a}_c$$

BOT.: $N = N_1$, up; a_c - up

TOP: $N = N_2$, down; a_c - down

In each case x-axis - along $\vec{a}_c$:

$$\text{BOT.: } N_1 - mg = ma_c\,,\; N_1 = mg + ma_c = mg + mv^2/R = ...$$

$$\text{TOP: } N_2 + mg = ma_c\,,\; N_2 = ma_c - mg = mv^2/R - mg = ...\; > 0$$

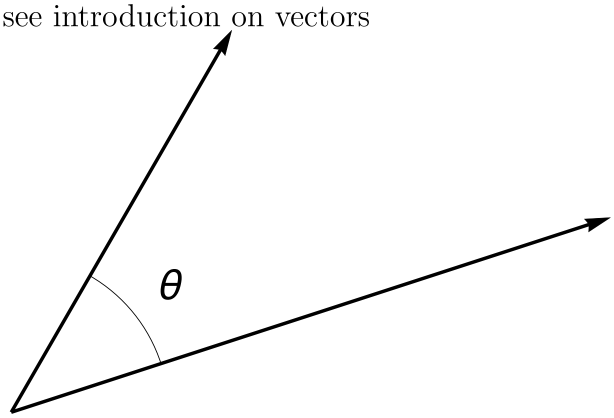
Smallest speed: $N_2 = 0$

$$mv_{\min}^2/R - mg = 0\,,\; v_{\min}^2/R = g\,,\; v_{\min} = \sqrt{Rg} = \sqrt{5 * 9.8} \simeq 7\,\frac{m}{s}$$

VIII. WORK

A. Scalar (dot) product in 3D

see introduction on vectors



$$\vec{a} \cdot \vec{b} = ab \cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}\,,\ b = \sqrt{b_x^2 + b_y^2 + b_z^2}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab}$$

Example: $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} = (1, 2, 3)$, $\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k} = (2, -4, 1)$. Find $\vec{a} \cdot \vec{b}$ and find θ.

$\vec{a} \cdot \vec{b} = 1 \times 2 + 2 \times (-4) + 3 \times 1 = -3\,,\ a = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}\,,\ b = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{-3}{\left(\sqrt{14}\right) \times \left(\sqrt{21}\right)}\,,\ \theta \approx 100^o$$

Note: if $\cos\theta < 0$, then $\theta > 90^o$.

B. Units

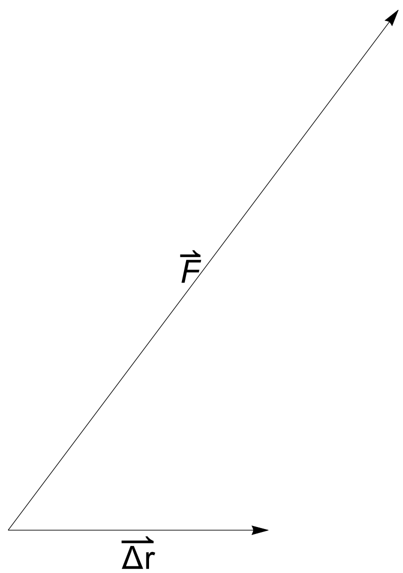
Joule (J):

$$1\,J = 1\,N \cdot m = kg\frac{m^2}{s^2} \tag{52}$$

C. Definitions

Constant force:

$$W = \vec{F} \cdot \Delta\vec{r} \equiv F\Delta r\cos\theta \equiv F_x\Delta x + F_y\Delta y + F_z\Delta z \tag{53}$$



Example: work by force of gravity ; x-"east", y-"North", z-up

$$F_x = 0\,,\ F_y = 0\,,\ F_z = -mg$$

$$\boxed{W_g = -mg\Delta z} \tag{54}$$

(and Δx , Δy do not matter!)

Let $m = 3\,kg$, and the particle is moved from $\vec{r}_1 = 2\hat{\imath} - 4\hat{\jmath} + \hat{k}$ to $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ (in *meters*). Find W_g

$$W_g = -3\,kg \cdot 9.8\frac{m}{s^2} \cdot (3-1)\,m \simeq -60\,J$$

Example:

$$\vec{F} = \vec{i} + 2\vec{j}\,,\ \vec{r}_1 = 3\vec{i} + 4\vec{j}\,,\ \vec{r}_2 = 6\vec{i} - 4\vec{j}$$

$$\Delta\vec{r} = 3\vec{i} - 8\vec{j}\,,\ W = 3 \cdot 1 + (-8) \cdot 2 = -13\,(J)$$

Example:

$$\vec{F} = 2\vec{j}\,,\ \vec{r}_1 = 3\vec{i} + 4\vec{j} + 5\vec{k}\,,\ \vec{r}_2 = 6\vec{i} - 4\vec{j} + \vec{k}$$

$$W = (-4 - 4) \cdot 2 = -16\,(J)$$

(x and z components of displacement do not matter!)

Example: An $M = 80\,kg$ skydiver falls $200\,m$ with a constant speed of $100\,m/s$. Find the work done by viscous air friction.

Since $v = const$

$$F_y - Mg = 0\,,\ \text{thus } F_y = Mg$$

(otherwise speed does not matter!)

$$\Delta y = -200\,m\,,\ W = Mg\,\Delta y = 80 \cdot 9.8 \cdot (-200) = -\ldots$$

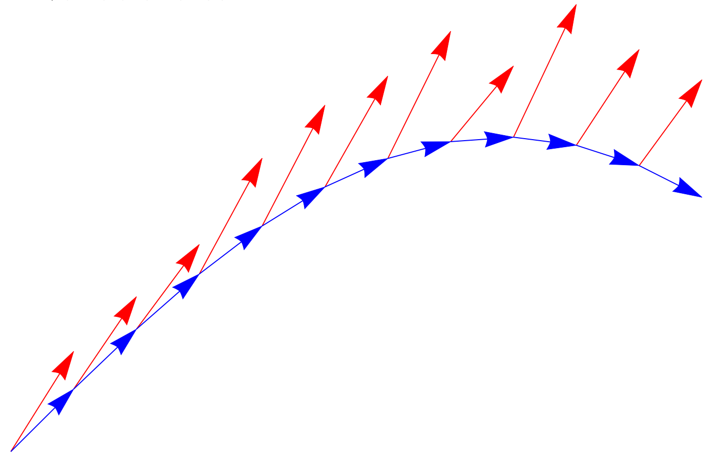
[Alternatively: the work done by gravity $W_g = -Mg\Delta y = 80 \cdot 9.8 \cdot 200 > 0$ (since goes down), and $W = -W_g < 0$]

Example: An $10\,kg$ projectile is displaced $200\,m$ horizontally and $50\,m$ vertically with respect to its initial position. Find W_g .

Horizontal motion does not matter!

$$W_g = -Mgy = -10 \cdot 9.8 \cdot 50 \simeq -5\,kJ$$

Variable force:



Let us break the path from $\vec{r}_1$ to $\vec{r}_2$ in small segments $\Delta\vec{r}_i$ (blue), each with a force $\vec{F}_i \simeq const$ (red). Then

$$W_i \simeq \vec{F}_i \cdot \Delta\vec{r}_i$$

and total work

$$W = \sum_i W_i \;\to\; \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{55}$$

Note: forces which are perpendicular to displacement do not work, e.g. the centripetal force, normal force.

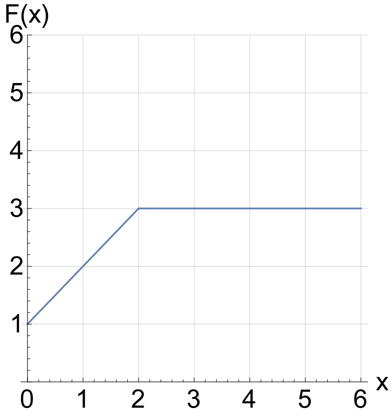
D. 1D motion and examples

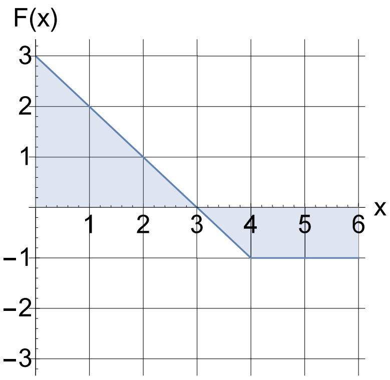
For motion in x-direction only

$$W = \int_{x_1}^{x_2} F_x \, dx \tag{56}$$

If F_x is given by a graph, work is the area under the curve (can be negative!).

Example: find work from graphs for $0 \leq x \leq 6$



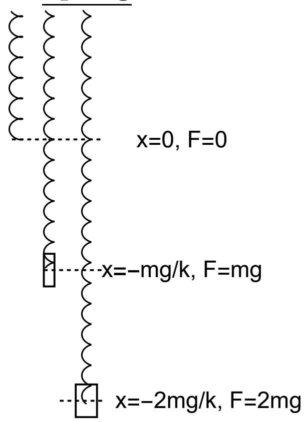


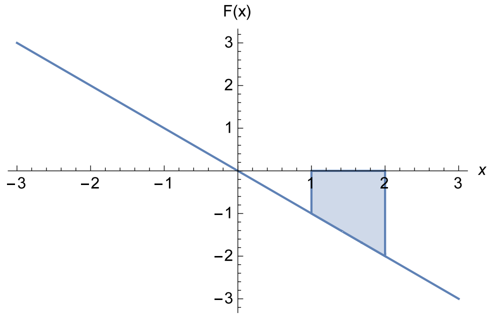
$$\text{left}: \; W = 16 \, J \, ,$$

$$\text{right}: \; W = \frac{1}{2} 3 \cdot 3 + \frac{1}{2} [(6-3) + (6-4)](-1) = 2 \, J$$

(note: part of area can be negative)

Spring:





Left: the "experiment". Elongation of the spring is proportional to the stretching force. Right: the Hook's law. The shaded area is the (negative) work done by the spring when stretching from x_1 to $x_2 > x_1$.

$$F = -kx \tag{57}$$

k - "spring constant" (also known as "Hook's law"). Work done by the spring

$$W_{sp} = \frac{1}{2}(F_2 + F_1)(x_2 - x_1) = \frac{1}{2}(-kx_2 - kx_1)(x_2 - x_1) = -\frac{k}{2}(x_2 + x_1)(x_2 - x_1) =$$

$$= -\frac{1}{2}kx_2^2 + \frac{1}{2}kx_1^2$$

$$W_{sp} = +\frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \tag{58}$$

(If the spring is stretched from rest, the term with x_1^2 will be absent and the work done by the spring will be negative for any $x_2 \neq 0$.)

Example. When a $3.6\,kg$ mass hangs from a spring it is extended by $x_1 = 12\,cm$. An extra force is applied to extend the spring by additional $\Delta x = 8\,cm$. What is the work done by the spring between x_1 and x_2?

$$\text{first, find } k\text{: } kx_1 = mg \;\Rightarrow\; k = \frac{mg}{x_1} = \frac{3.6 \cdot 9.8}{0.12} = \ldots \simeq 300\,\frac{N}{m}$$

$$x_2 = x_1 + \Delta x = 20\,cm \;\Rightarrow\; W = \frac{k}{2}(x_1^2 - x_2^2) = \frac{300}{2}(0.12^2 - 0.2^2) = -3.8\,J$$

IX. KINETIC ENERGY

A. Definition and units

$$\boxed{K = \frac{1}{2}mv^2} \tag{59}$$

or if many particles, the sum of individual energies.

Units: J (same as work).

Note: $v^2 = v_x^2 + v_y^2 + v_z^2$, thus $K = \frac{1}{2}m\left(v_x^2 + v_y^2 + \ldots\right)$.

B. Relation to work

1. *Constant force*

$$W = F_x\Delta x + F_y\Delta y = m\left[a_x\Delta x + a_y\Delta y\right]$$

According to kinematics

$$\Delta x = \frac{v_x^2 - v_{0x}^2}{2a_x}\ , \quad \Delta y = \frac{v_y^2 - v_{0y}^2}{2a_y}$$

and

$$a_x\Delta x = \frac{v_x^2 - v_{0x}^2}{2}\ , \quad a_y\Delta y = \frac{v_y^2 - v_{0y}^2}{2}$$

Thus,

$$W = \Delta K \tag{60}$$

which is the "work-energy" theorem.

Examples: gravity and friction (in class).

(gravity). Max hight for vertical initial v:

$$-mgh = 0 - \frac{1}{2}mv^2\ ,\ h = \frac{v^2}{2g}$$

(friction). The "policeman problem". Given: L, μ, find v_o

Friction: $f = \mu mg$.

$$W = -fL = -\mu mgL \ < 0 \ (!) \tag{61}$$

$$\Delta K = 0 - \frac{1}{2}mv_0^2 \tag{62}$$

$$-mv_0^2/2 = -\mu mgL \tag{63}$$

$$v = \sqrt{2\mu gL} \tag{64}$$

(friction+another force). A heavy crate is pushed from A to B (with $AB = 3\,m$) by a force $P = 200\,N$ at 60^o to horizontal. Find the work W_f done by friction if $K_A = 300\,J$ and $K_B = 100\,J$.

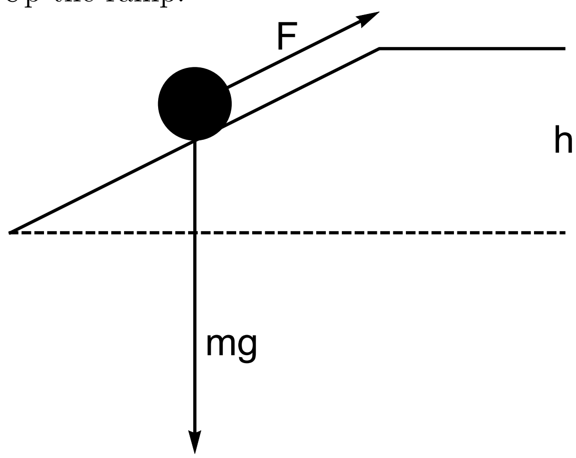
$$K_B - K_A = W_f + W_P\,,\ W_P = P \cdot AB \cdot \cos 60^o \tag{65}$$

$$W_f = K_B - K_A - W_P = 100 - 300 - 3 \cdot 200 \cdot \frac{1}{2} = -500\,J$$

(Note: negative!)

Ramp (no friction)

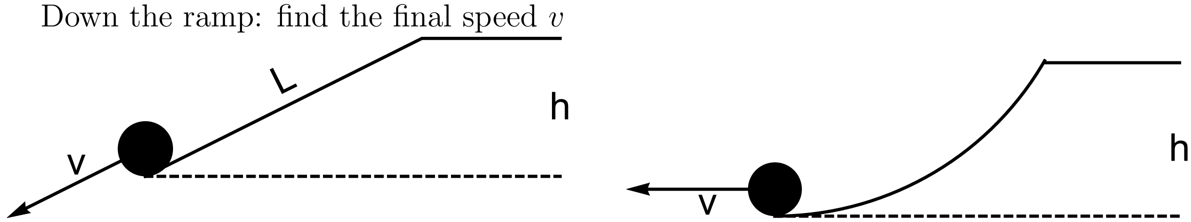
Up the ramp:



$$F = mg\sin\theta = mg\frac{h}{L} < mg\,(!)$$

$$\text{work } W = FL = mgh \ - \ \text{same}$$

Down the ramp: find the final speed v



$$\text{Kinetic energy: } K - 0 = W_g = mgh \Rightarrow K = mgh$$

$$\text{speed: from } K = \frac{1}{2}mv^2\,,\ v = \sqrt{2gh}$$

Note: shape of the slope does not matter; kinetic energy is proportional to mass, final v is mass-independent

Again, break the displacement path into a large number N of small segments Δr_i . For each

$$W_i = \Delta K_i \equiv K_i - K_{i-1}$$

Thus,

$$W = \sum_i^N W_i = (K_1 - K_0) + (K_2 - K_1) + \ldots + (K_{i-1} - K_{i-2}) + (K_i - K_{i-1}) + \ldots + (K_N - K_0) \equiv \Delta K$$

or

$$\boxed{W = \Delta K} \tag{66}$$

which is the "work-energy" theorem in a general form (also, can be applied to a system of particles).

Example: spring. A mass $M = 3\,kg$ is attached to a spring with $k = 10\,N/m$. The spring is at equilibrium (neither stretched nor compressed). The mass is given an initial speed of $v_0 = 2\,m/s$. Find the maximum absolute deviations from equilibrium.

$$\Delta K = \frac{1}{2} M \left(v^2 - v_0^2 \right) = -\frac{1}{2} M v_0^2 \text{ since } v = 0 \text{ at max deviation}$$

$$W_{sp} = -\frac{k}{2} \left(x_{\max}^2 - x_0^2 \right) = -\frac{k}{2} x_{\max}^2 \text{ since } x_0 = 0$$

$$\Delta K = W_{sp} \Rightarrow -\frac{1}{2} M v_0^2 = -\frac{k}{2} x_{\max}^2 \Rightarrow$$

$$x_{\max}^2 = v_0^2 \frac{M}{k} \,,\; x_{\max} = \pm v_0 \sqrt{\frac{M}{k}} = \pm 2 \sqrt{\frac{3}{10}} \simeq \pm 1.1\,m$$

C. Power

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$$

Units: *Watts*. $1\,\mathcal{W} = 1\,J/s$

Re-derivation of work-energy theorem:

$$\frac{d}{dt}K = \frac{d}{dt}\frac{m}{2}v^2 = \frac{m}{2}\frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right) = m\frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v} = \frac{dW}{dt}$$

Example: An $M = 500\,kg$ horse is running up an $\alpha = 30^o$ slope with $v = 4\,m/s$. Find P:

$$P = mg\sin\alpha \cdot v \simeq 500 \cdot 9.8\frac{1}{2}4 \approx 10^4\,\mathcal{W}$$

$1\,hp \simeq 746\,\mathcal{W}$.

Example. For a fast bike the air resistance is $\sim v^2$. How much more power is needed to double v?

$$F(v) = \alpha \times v^2 \ \ \text{(unknown } \alpha\text{)}$$

$$P(v) = F(v) \times v = \alpha \times v^3 \ \Rightarrow \ P(2v) = 8 \times P(v)$$

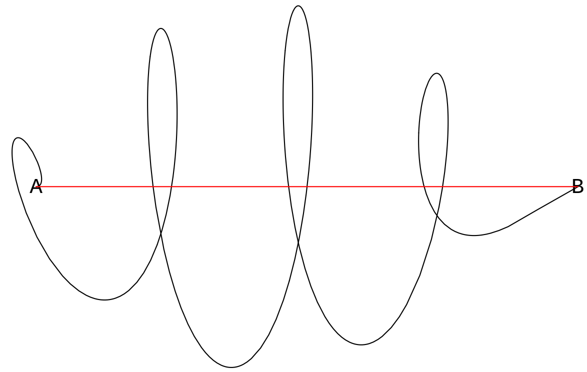


FIG. 18: Generally, the work of a force between points A and B depends on the actual path. However, for some "magic" (conservative) forces the work is path-independent. For such forces one can introduce potential energy U and determine work along *any* path as $W = U_A - U_B = -\Delta U$.

Dr. Vitaly A. Shneidman, Phys 111, 7th Lecture

X. POTENTIAL ENERGY

A. Some remarkable forces with path-independent work

See Fig. 18 and caption.

Examples:

- Constant force

$$W = \sum \vec{F} \cdot \Delta \vec{r}_i = \vec{F} \cdot \sum \Delta \vec{r}_i = \vec{F} \cdot \Delta r = \vec{F} \cdot \vec{r}_B - \vec{F} \cdot \vec{r}_A \tag{67}$$

with potential energy

$$U(\vec{r}) = -\vec{F} \cdot \vec{r} \tag{68}$$

Example force of gravity with $F_x = 0\,, F_y = -mg$ and

$$\boxed{U_g = mgh} \tag{69}$$

- Elastic (spring) force

$$W \simeq -\sum k \frac{x_i + x_{i+1}}{2} (x_{i+1} - x_i) = -\frac{k}{2} \sum x_{i+1}^2 - x_i^2 = -\frac{k}{2} \left(x^2 - x_0^2\right) \tag{70}$$

with potential energy

$$\boxed{U_{sp}(x) = \frac{1}{2}kx^2} \tag{71}$$

- Other: full force of gravity $F = -mg$ with r-dependent g, and any force which does not depend on velocity but depends only on distance from a center.

- Non-conservative: kinetic friction (depends on velocity, since points against $\vec{v}$)

B. Relation to force

$$U(x) = -\int F(x)\,dx \tag{72}$$

$$F = -dU/dx \tag{73}$$

XI. CONSERVATION OF ENERGY

Start with

$$\Delta K = W$$

If only conservative forces

$$W = -\Delta U \tag{74}$$

thus

$$\boxed{K + U = const \equiv E} \tag{75}$$

Examples:

- Maximum hight.

$$E = mgh + \frac{1}{2}mv^2$$

$$0 + \frac{1}{2}mv_0^2 = mgh_{\max} + 0$$

$$h_{\max} = \frac{v_0^2}{2g}$$

- Galileo's tower. Find speed upon impact

$$mgh + 0 = 0 + \frac{1}{2}mv^2$$

$$v^2 = 2gh$$

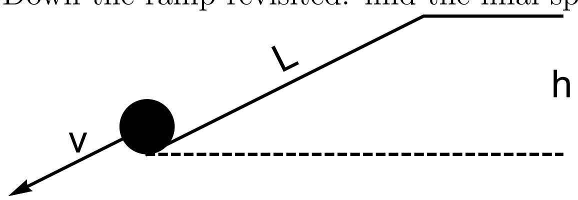
- Coastguard cannon (from recitation on projectiles). Find speed upon impact

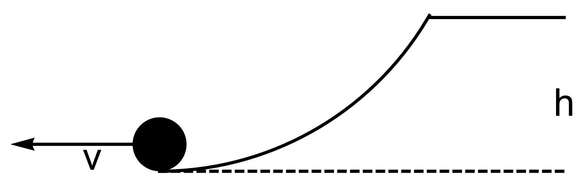
$$mgH + \frac{1}{2}mv_0^2 = 0 + \frac{1}{2}mv^2$$

$$v^2 = v_0^2 + 2gH$$

(note that the angle does not matter!).

- Down the ramp revisited: find the final speed v



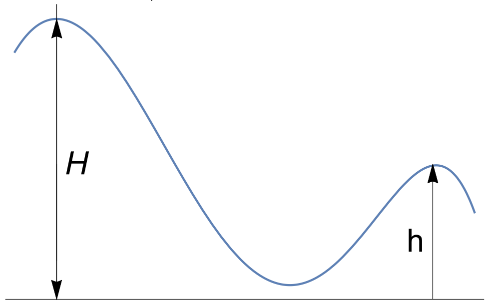


$$\text{initial mech. energy: } 0 + mgh\,, \text{ final mech.energy: } \frac{1}{2}mv^2 + 0$$

$$\Rightarrow\ \frac{1}{2}mv^2 = mgh\,,\ v = \sqrt{2gh}$$

Note: shape of the slope or angle or mass do not matter

- A skier with no initial velocity slides down from a hill which is $H = 18\,m$ high and then, without losing speed, up a smaller hill which is $h = 10\,m$ high. What is his speed in m/s at the top of the smaller hill? Ignore friction.

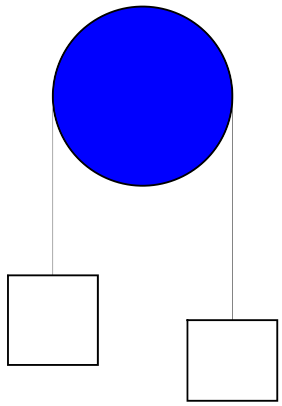


$$\text{(full initial energy) } MgH + 0 \ = \ Mgh + \frac{1}{2}Mv^2 \text{ (full final)}$$

$$v^2 = 2g(H - h) > 0\,, \ v = \ldots$$

- In the Atwood machine the heavier body on left has mass $M = 1.01\,kg$, while the lighter body on right has mass $m = 1\,kg$. The system is initially at rest, there is no friction and the mass of the pulley is negligible.

  Find the speed after M lowers by $h = 50\,cm$; use *only* energy considerations.



$$\text{(M on top)}: \; E = Mgh + 0 + 0 \;=\; mgh + \frac{1}{2}(M+m)v^2 \; \text{(m on top)}$$

$$v^2 = 2gh\frac{M-m}{M+m}\,, \; v = \ldots$$

What if we need a? For smaller mass, for example

$$h = \frac{v^2 - v_0^2}{2a} = \frac{v^2}{2a}, \Rightarrow a = \frac{v^2}{2h} = g\frac{M-m}{M+m}$$

- A spring with given k, m is stretched by X meters and released. Find $v_{\max}$.

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$\frac{1}{2}kX^2 + 0 = 0 + \frac{1}{2}mv_{\max}^2\,, \; v_{\max} = X\sqrt{k/m}$$

- A block approaches the spring with speed v_0 . Find the maximum compression

$$\text{(full final energy)}: \; \frac{1}{2}kx_{\max}^2 + 0 \;=\; 0 + \frac{1}{2}mv_0^2 \; \text{(full initial energy)}$$

$$|x_{\max}| = v_0\sqrt{m/k}$$

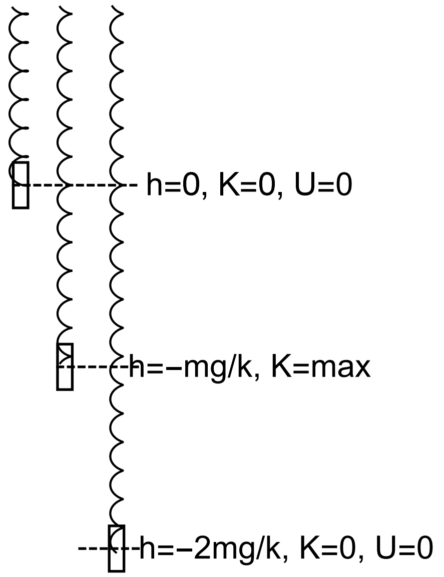
- find speed v at some given location x

  first find kinetic energy at x

$$K(x) + U(x) = E = 0 + \frac{1}{2}mv_0^2$$

$$K(x) = \frac{1}{2}mv_0^2 - U(x) = \frac{1}{2}mv_0^2 - \frac{1}{2}kx^2 \; > \; 0$$
$$v(x) = \sqrt{2K(x)/m}\,,\; |x| \le |x_{max}|$$

Example. A block of $m = 10\,kg$ is placed on a vertical spring with $k = 40\,N/m$, originally unstretched . Find the max compression distance and $v_{\max}$.



From work-energy theorem: $\Delta K = 0 - 0 = 0 \;\Rightarrow, W = W_g + W_s = 0$

$$W_g = -mgh\,,\; W_s = -\frac{1}{2}kh^2 \;\Rightarrow h = -2mg/k = \ldots$$

From energy conservation: $U(h) = U_g(h) + U_s(h) = mgh + \frac{1}{2}kh^2$

$$E = K + U = K(0) + U(0) = 0,\; K_{fin} = 0 \;\Rightarrow\; U_{fin} = 0 \;\Rightarrow\; U_{fin} = 0$$
$$mgh + \frac{1}{2}kh^2 = 0\,,\; h = -\frac{2mg}{k}$$

Maximum speed is achieved when the block passes through equilibrium at $h = -mg/k$ (middle picture).

$$E = K + U = 0 \;\Rightarrow\; K_{\max} = -U_g\left(-\frac{mg}{k}\right) - U_s\left(-\frac{mg}{k}\right) = mg * \frac{mg}{k} - \frac{1}{2}k\left(-\frac{mg}{k}\right)^2 = \frac{1}{2}\frac{(mg)^2}{k}$$
$$\frac{1}{2}mv_{\max}^2 = K_{\max} \;\Rightarrow\; v_{\max}^2 = (mg)^2/(mk)$$

Alternnatively, could consider deviation from new equilibrium with *only* spring energy:

$$h_{eq} = -\frac{mg}{k}\,,\; x = h - h_{eq}\,,\; U = \frac{1}{2}kx^2\,,\; x_{\max} = -h_{eq}\,,\; v_{\max} = \pm\sqrt{\frac{k}{m}}x_{\max}$$

A. Conservative plus non-conservative forces

For conservative forces introduce potential energy U, and then define the total *mechanical* energy

$$E_{mech} = K + U$$

One has

$$W = -\Delta U + W_{non-cons} \tag{76}$$

Then, from work-energy theorem

$$\boxed{\Delta E_{mech} = \Delta\left(K + U\right) = W_{non-cons}} \tag{77}$$

Examples:

- Galileo's Tower revisited (with friction/air resistance). Given: $H = 55\,m$, $M = 1\,kg$ and $W_f = -100\,J$ is lost to friction (i.e. the mechanical energy is lost, the thermal energy of the mass M and the air is increased by $+100\,J$). Find the speed upon impact.

$$\left(0 + \frac{1}{2}Mv^2\right) - (mgH + 0) = W_f < 0$$

$$\frac{1}{2}Mv^2 = Mgh + W_f\,,\ v^2 = 2gH - \frac{2|W_f|}{M} = 2 \times g \times 55 - 2 \times \frac{100}{1}\,,\ v \sim 30\frac{m}{s}$$

- Sliding crate (down). Given L, m, θ, μ find the speed v at the bottom.

$$h = L\sin\theta\,,\ f = \mu mg\cos\theta\,,\ W_f = -fL$$

$$(0 + \frac{1}{2}mv^2) - (mgh + 0) = W_f \ \Rightarrow\ v^2 = 2gh - \frac{2}{m}|W_f|$$

$$v^2 = 2gL\sin\theta - \frac{2}{m}\mu Lmg\cos\theta = 2gL(\sin\theta - \mu\cos\theta) \geq 0 \ \Rightarrow\ \mu < \tan\theta \text{ (or will not slide)}$$

- Sliding crate (up). Given v find L (a) no friction; (b) with friction

$$\text{(a):}\ \frac{1}{2}mv^2 + 0 = mgh\,,\ h = v^2/(2g)\,,\ L = h/\sin\theta$$

$$\text{(b):}\ W_f = -fL \ \Rightarrow\ (mgh + 0) - (0 + \frac{1}{2}mv^2) = -fL = -\mu mg\cos\theta \times \frac{h}{\sin\theta}$$

$$mgh(1 + \mu\cot\theta) = \frac{1}{2}mv^2\,,\ h = \frac{v^2}{2g(1 + \mu\cot\theta)} \ \Rightarrow\ L = \frac{h}{\sin\theta} = \frac{v^2}{2g(\sin\theta + \mu\cos\theta)}$$

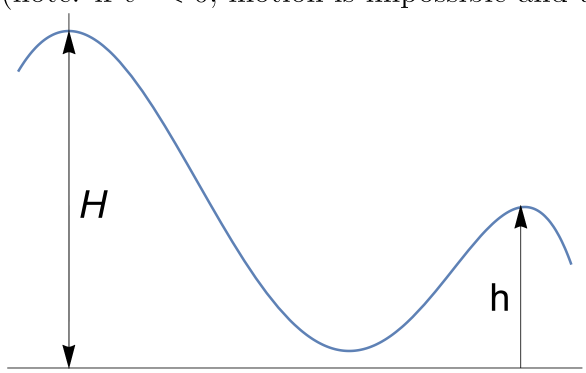
- The same skier as before, but assume a small average friction force of $40\,N$ and the combined length of the slopes $L = 200\,m$. The mass of the skier is $M = 80\,kg$; find the speed at the top of the smaller hill.

$$W_f = -fL < 0$$

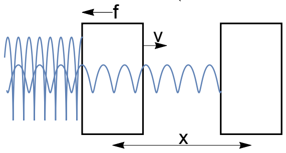
(full final energy - full initial) $(Mgh + \frac{1}{2}Mv^2) - (MgH + 0) \;=\; W_f$(work by non-cons. force)

$$v^2 = 2g(H-h) + \frac{2W_f}{M} \geq 0\,,\; v = \ldots$$

(note: if $v^2 < 0$, motion is impossible and the skier will stop before reaching the 2nd top.)



Example (*Advanced*) .



A block $m = 2\,kg$ is attached to an unstretched spring with spring constant $k = 20\,N/m$ and is given an initial speed $v = 10\,m/s$ along a rough horizontal surface with $\mu = 0.5$. At which distance x will the block stop during the first swing to the right?

$$E_i = \frac{1}{2}mv^2 + 0\,,\; E_f = 0 + \frac{1}{2}kx^2$$

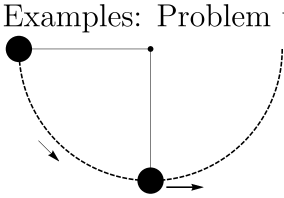
$$E_f - E_i = W_{n.-c.} = -fx \text{ (work done by friction)}$$

$$f = \mu mg \;\Rightarrow\; \frac{1}{2}kx^2 - \frac{1}{2}mv^2 = -\mu mgx \text{ (a quadratic equation for x)}$$

$$x^2 + 2\mu g\frac{m}{k} - \frac{m}{k}v^2 = 0$$

$$x = -\mu g\frac{m}{k} + \sqrt{(\mu g\frac{m}{k})^2 + \frac{m}{k}v^2}$$

Examples: Problem which are too hard without energy



A pendulum of length L swings starting from a horizontal position (i.e. $\theta_0 = \pi/2$). Find the speed V at the lowest point. Generalize for arbitrary θ_0.

$$E = mgh + \frac{1}{2}mv^2$$

$$\text{init. } \theta_0 = \frac{\pi}{2}\text{: } h = L\,,\; v = 0\,;\; \text{fin. } \theta = 0\text{: } h = 0\,,\; v = V$$

$$mgL = \frac{1}{2}mV^2, V^2 = 2gL$$

Arbitrary θ_0:

$$\text{init. } \theta = \theta_0\text{: } h = L(1 - \cos\theta)\,,\; v = 0\,;\; \text{fin. } \theta = 0\text{: } h = 0\,,\; v = V$$

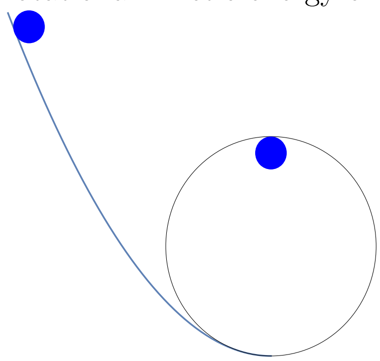
$$mgL(1 - \cos\theta) = \frac{1}{2}mV^2\,,\; V^2 = 2gL(1 - \cos(\theta))$$

What if need T? Lowest point

$$T - mg = mV^2/r\,,\; r = L \;\Rightarrow\; T = mg + m\frac{V^2}{L}$$

.

In the "loop-the-loop" demo a small ball is sent along a looping track of radius R. Find the minimal initial hight H so that the ball makes the loop. (Ignore at this stage the rotational kinetic energy of the spinning ball).



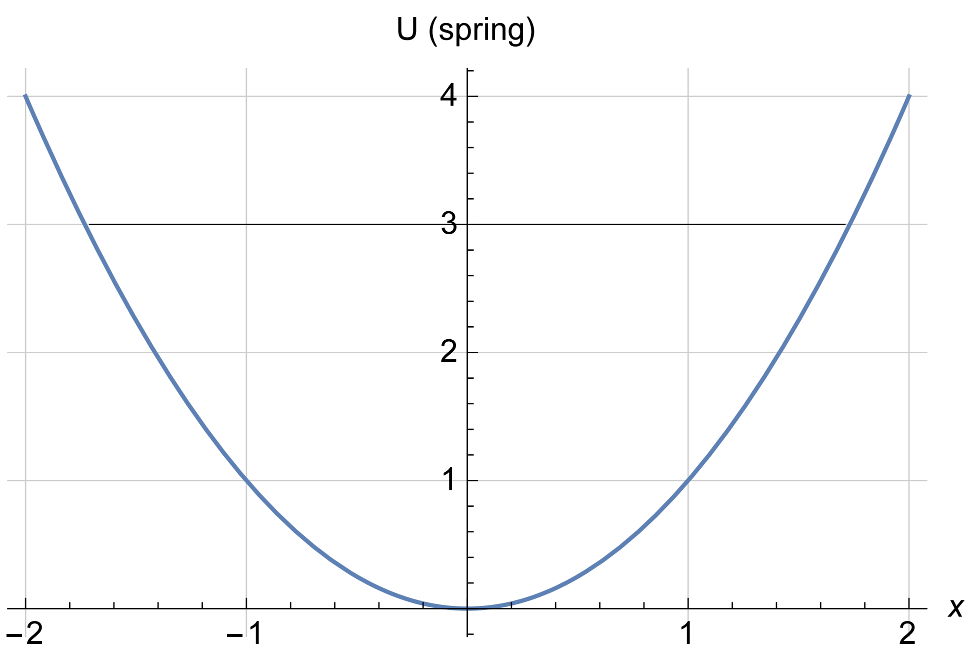
$$E = mhg + \frac{1}{2}mv^2$$

$$\text{init.: } h = H,\ v = 0\,;\ \text{top of the loop: } h = 2R, v = V$$

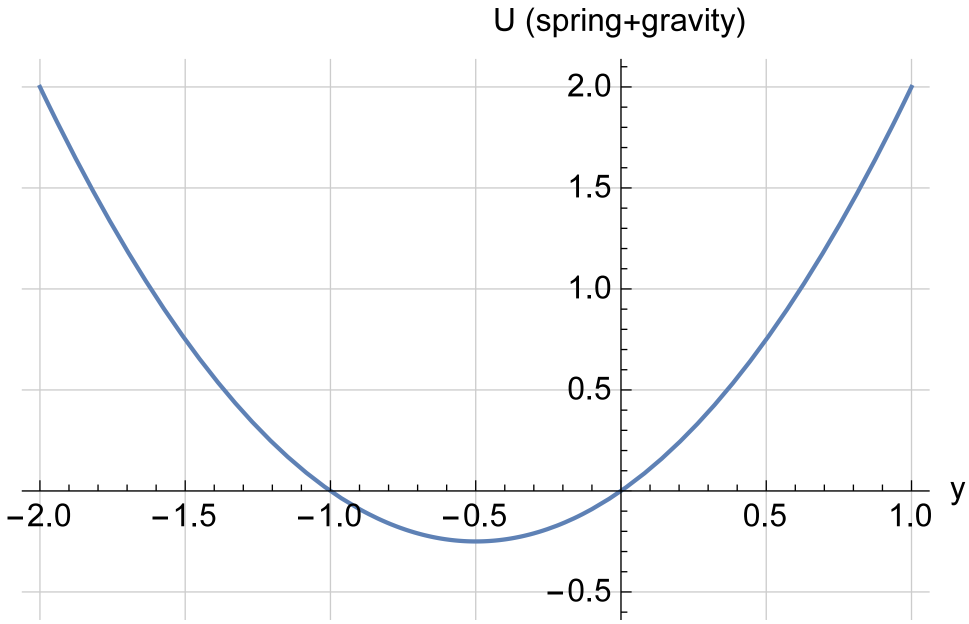
$$mgH = 2mgR + \frac{1}{2}mV^2 \text{ and } \frac{V^2}{R} \geq g \text{ not to fall}$$

$$gH_{\min} = 2gR + \frac{1}{2}gR\,,\ H_{\min} = 2.5R$$

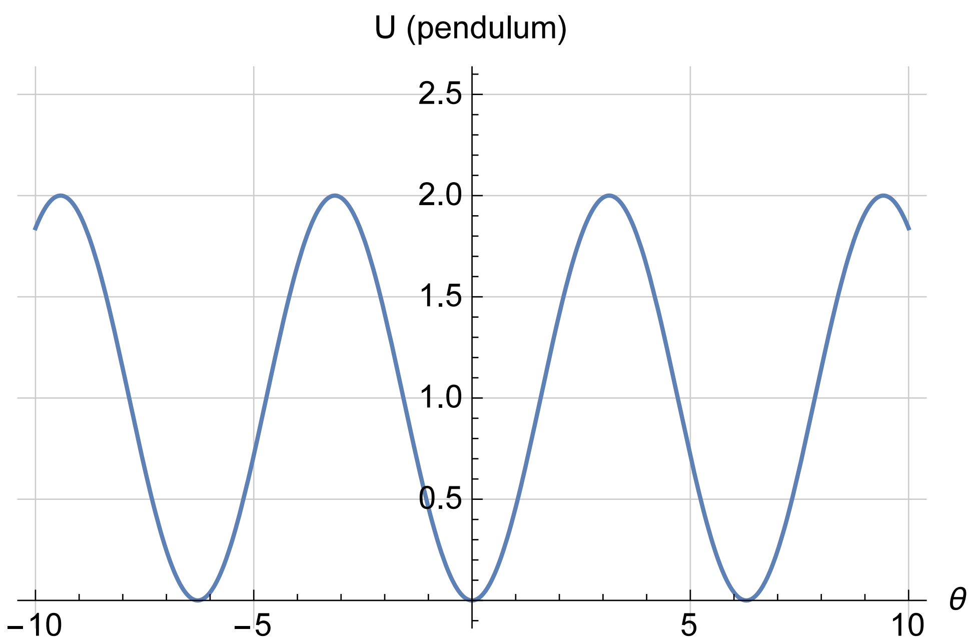
B. Advanced: Typical potential energy curves



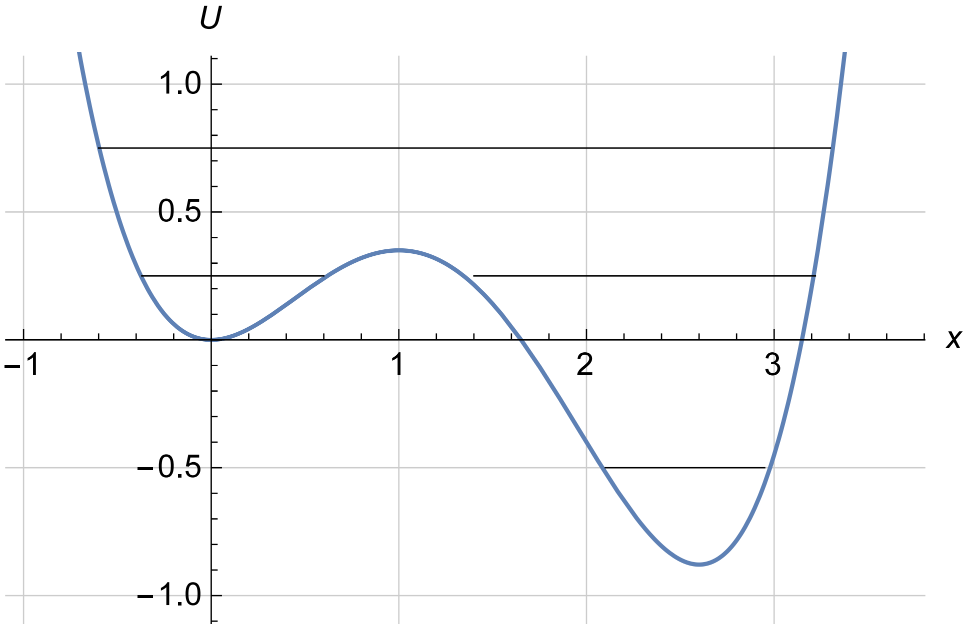
$$U = \frac{1}{2}kx^2 \tag{78}$$



$$U = \frac{1}{2}ky^2 + mgy \tag{79}$$



$$U = mgL(1 - \cos\theta) \tag{80}$$



$$K = E - U \geq 0\,,\;\; v = \pm\sqrt{2K/m} \tag{81}$$

C. (Advanced) "Forces of inertia"

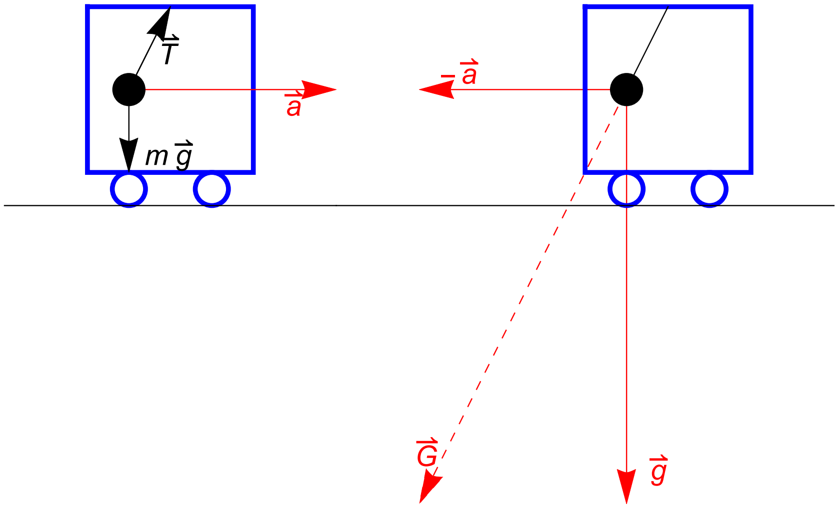
$$\vec{F} = m\vec{a}$$

Define

$$\vec{F}_i = -m\vec{a}$$

$$\vec{F} + \vec{F}_i = 0$$

"statics".



Deflection of a pendulum in an accelerating car. Left: from the point of inertial reference frame. Right: from the point an accelerating reference frame associated with the car; note a new "field of gravity" $-\vec{a}$ which combines with regular $\vec{g}$.

Regular solution (ϕ - angle with vertical):

$$\vec{T} + m\vec{g} = m\vec{a}$$

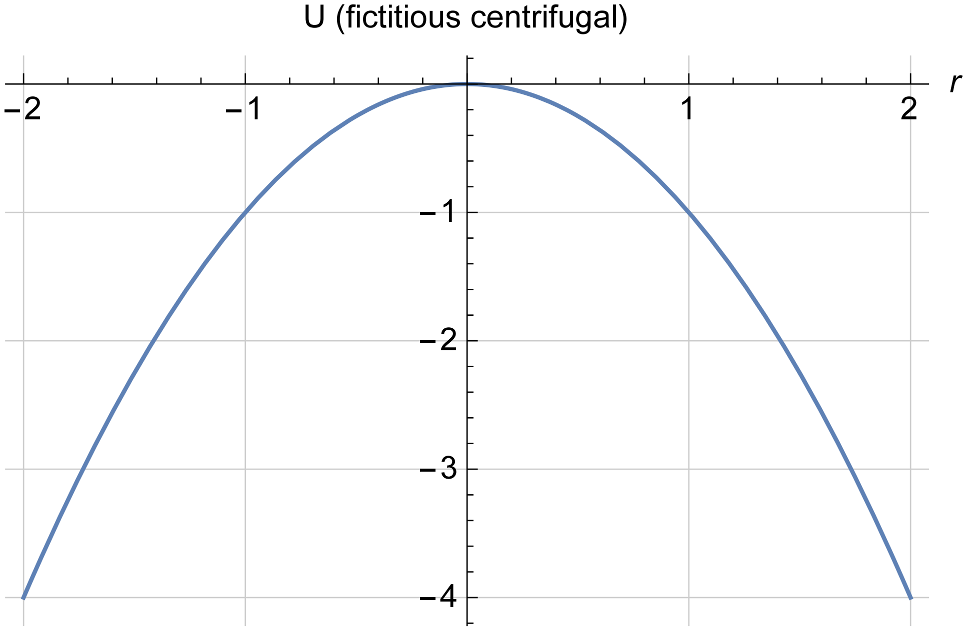
$$x: \; T \sin\phi = ma\,, \quad y: \; T\cos\phi - mg = 0$$

Thus,

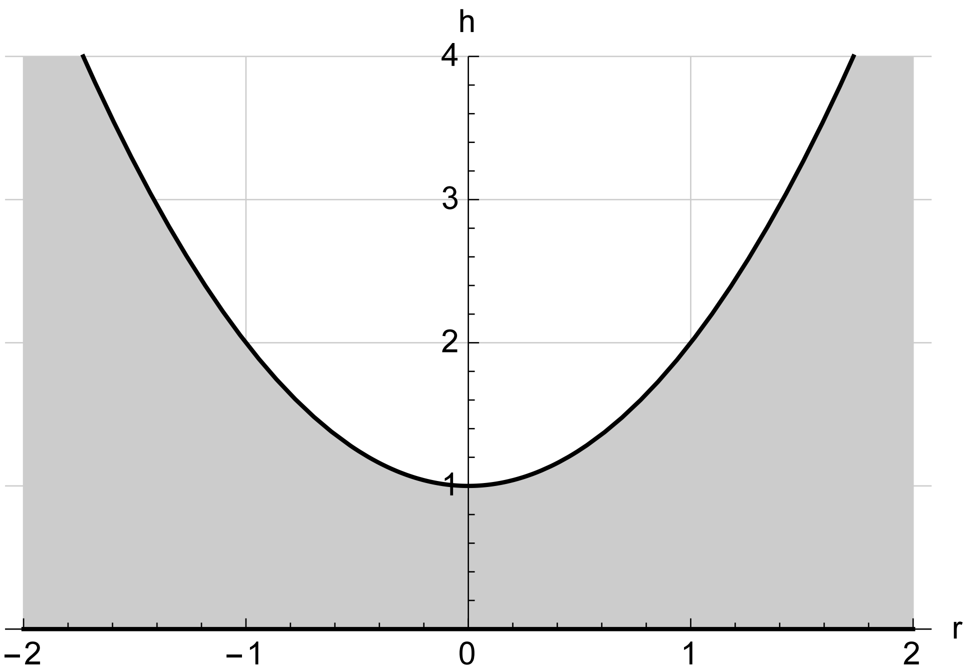
$$\tan\phi = \frac{a}{g}$$

Solution in non-inertial reference frame: no acceleration; pendulum hangs in the direction of new "full gravity" $\vec{G} = \vec{g} + (-\vec{a})$. Thus, $\tan\phi = a/g$, the same.

D. Advanced: Fictitious "centrifugal energy"



$$U = -\frac{1}{2} m\omega^2 r^2 \tag{82}$$



$$U = U_g + U_{cent} = mgh - \frac{1}{2} m\omega^2 r^2 = const \tag{83}$$

$$h(r) = h(0) + \frac{\omega^2}{g} r^2 \tag{84}$$

E. Advanced: Mathematical meaning of energy conservation

Start from 2nd Law with $F = F(x)$ (not v or t !)

$$m\ddot{x} = F(x) \mid \times \dot{x} \tag{85}$$

$$\dot{x}\ddot{x} = \frac{d}{dt}(\dot{x})^2/2$$

$$\dot{x}F(x) = \frac{d}{dt}\int F(x)\,dx = -\frac{d}{dt}U(x)$$

Thus

$$\frac{d}{dt}(K+U) = 0 \tag{86}$$

$$K + U = const = E \tag{87}$$

XII. MOMENTUM

A. Definition

$$\text{One particle:} \quad \boxed{\vec{p} = m\vec{v}}\ (\text{vector})\,,\ K = \frac{1}{2}mv^2 = \frac{p^2}{2m}\ (\text{scalar}) \tag{88}$$

$$\text{System:} \quad \vec{\mathcal{P}} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \tag{89}$$

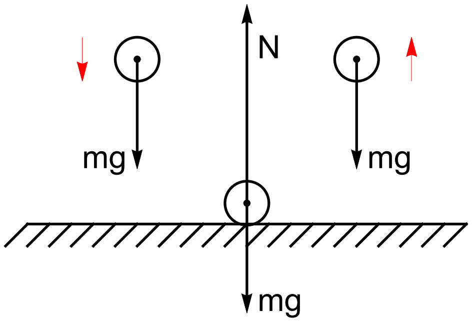
Units: $kg \cdot m/s$ (no special name).

B. 2nd Law in terms of momentum (Single particle)

$$m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \Rightarrow \boxed{\frac{d\vec{p}}{dt} = \vec{F}} \tag{90}$$

$$\Delta\vec{p} = \int_{t_1}^{t_2} dt\,\vec{F}(t) = \vec{F}_{av}\,\Delta t \tag{91}$$

$\int_{t_1}^{t_2} dt\,\vec{F}(t)$ **-impulse**. "It takes an impulse to change momentum".

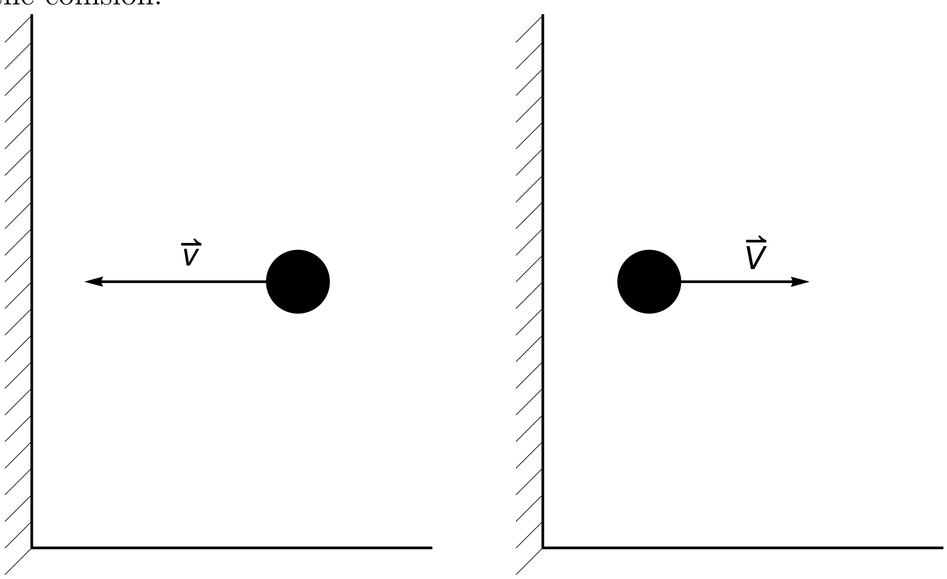


Question. How does N compare to mg?

$$\text{A) } N = mg\,, \text{ B) } N = 2mg\,, \text{ or C) } N > 10,000mg$$

$$\text{collision time: } t \sim \frac{2d}{v_s} \sim 10^{-4}\, s \;\Rightarrow\; N \sim \frac{\Delta p}{t} \sim 2mv10^4/s$$

Example. A fast ball of mass m=0.2 kg moving with velocity $\vec{v} = -50\hat{i}$ (in m/s) strikes a vertical wall as shown. It rebounds with $\vec{V} = 30\hat{i}\, m/s$ after a brief collision with the wall lasting about $\Delta t = 0.04\, s$. Estimate the average force (in N) which acts on the ball during the collision.



$$\vec{p}_2 = m\vec{V}\,,\ \vec{p}_1 = m\vec{v}\,, \text{ thus } \Delta\vec{p} = m\left(\vec{V} - \vec{v}\right) = 0.2(30+50)\hat{i} = 16\hat{i}\, kg\frac{m}{s}$$

$$F_{av} = \frac{\Delta p}{\Delta t} = 400\, N$$

Example. A steel ball with mass $m = 100\,g$ falls down on a horizontal plate with $v = 3\,m/s$ and rebounds with $V = 2\,m/s$ up. (a) find the impulse from the plate; (b) estimate F_{av} if the collision time is $1\,ms$.

Select the positive direction up. Then, $p_1 = -3 \times 0.1 = -0.3\,kg\,m/s$ and after the collision $p_2 = mV = 0.2\,kg\,m/s$. (a) Impulse $p_2 - p_1 = 0.5\,kg \cdot m/s$. (note: $|p_1|$ and $|p_2|$ add up!). (b) $F_{av} = 0.5/0.001 = 500\,N$ (large!).

Example. An $M = 70\,kg$ Olympic diver from an $H = 10\,m$ platform stops $\Delta t = 0.1\,s$ after hitting the surface. Find the average force on the diver during this time.

First, find velocity v when hitting the water (axis - up). From energy conservation while in air:

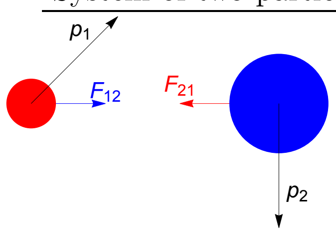
$$\frac{1}{2}Mv^2 = MgH\,,\ v = -\sqrt{2gH} \simeq -14\,\frac{m}{s}$$

Then, find the impulse and average force while entering the water

$$\Delta p = M \cdot 0 - Mv = 0 - 70(-14) = 980\,kg\frac{m}{s}\,,\ F_{av} = 980/0.1 \approx 10^4\,N$$

C. Conservation of momentum in a closed system

System of two particles with NO external forces:



$$d\vec{p}_1/dt = \vec{F}_{12}\,,\ d\vec{p}_2/dt = \vec{F}_{21}\,,\ \vec{F}_{12} + \vec{F}_{21} = 0 \tag{92}$$

$$\vec{\mathcal{P}} = \vec{p}_1 + \vec{p}_2 \ \Rightarrow\ \frac{d\vec{\mathcal{P}}}{dt} = \vec{F}_{12} + \vec{F}_{21} = 0 \tag{93}$$

$$\boxed{\vec{\mathcal{P}} = const} \tag{94}$$

and the same for any number of particles.

XIII. COLLISIONS

Very large forces F acting over very short times Δt, with a finite impulse $F_a\,\Delta t$. Impulse of "regular" forces (e.g. gravity) over such tiny time intervals is negligible, so the colliding bodies are almost an insolated system with

$$\vec{\mathcal{P}} \simeq const$$

before and after the collision. With energies it is not so simple, and several options are possible.

A. Inelastic

$$\vec{\mathcal{P}} = const \;, \quad K \neq const \tag{95}$$

1. *Perfectly inelastic*

$$\text{After collision: } \vec{V}_1 = \vec{V}_2 \tag{96}$$

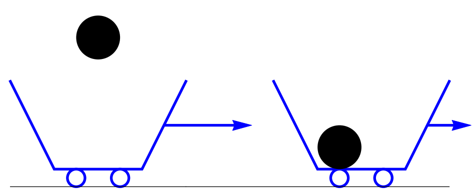


FIG. 19: Example (1D). A chunk of black coal of mass m falls vertically into a wagon with mass M, originally moving with velocity V. (Only) the horizontal component of momentum is conserved: $m \cdot 0 + M \cdot V = (M+m)v_{final}$

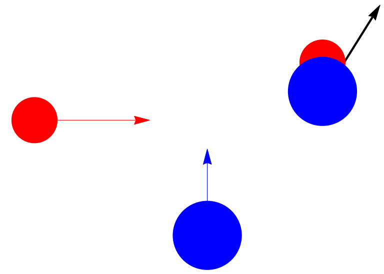


FIG. 20:

Example (2D). A car m (red) moving East with velocity v collides and hooks up with a truck M moving North with velocity V. Find the magnitude and direction of the resulting velocity V_1. Solution:
Direction:

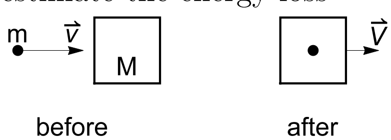
$$\tan\theta = \frac{V_{1y}}{V_{1x}} = \frac{P_y}{P_x} = \frac{MV}{mv}$$

Magnitude:

$$(M+m)\vec{V}_1 = m\vec{v} + M\vec{V}$$

$$V_{1x} = mv/(M+m)\,,\ V_{1y} = MV/(M+m)\,,\ V_1 = \sqrt{V_{1x}^2 + V_{1y}^2}$$

Example. A bullet of mass $m = 9\,gram$ is fired horizontally with unknown speed v at a wooden block of mass $M = 4.5\,kg$ resting on a frictionless table. The bullet hits the block and becomes completely embedded within it. After that, the block, with the bullet in it, is traveling at speed $V = 2\,m/s$. (a) Calculate the initial speed of the bullet in m/s;(b) estimate the energy loss



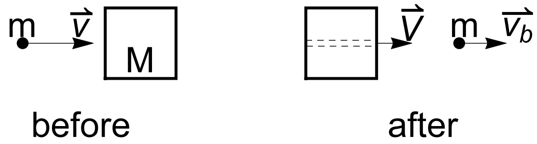
$$mv + M\cdot 0 = (m+M)V\ ,\ v = V\frac{M+m}{m} \simeq 1000\,\frac{m}{s}$$

$$\Delta K = \frac{1}{2}(M+m)V^2 - \frac{1}{2}mv^2 = 0.5\cdot 4.5\cdot 2^2 - 0.5\cdot 0.009\cdot 1000^2 \approx -8\,kJ < 0$$

Note: in an inelastic collision energy is always lost:

$$\frac{1}{2}(M+m)V^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2\right) < 0$$

Example. The same block and bullet as above, but now the initial speed of the bullet is known to be $700\,m/s$ and it passes *through* the block, emerging on the other side with $v_b = 300\,m/s$. Find the speed of the block V in m/s after the collision.

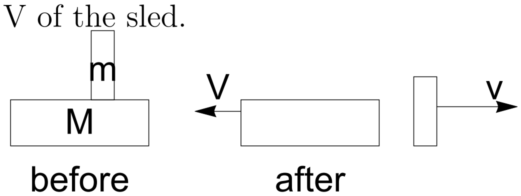


$$mv + M \cdot 0 = mv_b + MV\;,\; V = (v - v_b)\frac{m}{M} = \ldots$$

Example. The same, but initially the block is also moving with $v_B = 5\,m/s$ in the same direction as the bullet. The bullet gets stuck. Find the final speed V of block+bullet.

$$mv + Mv_B = (m + M)V\;,\; V = \frac{mv + Mv_B}{m + M} = \ldots$$

Example. An m=50 kg boy stands still on an M=25 kg sled on an icy lake. He then jumps forward off the sled with horizontal speed v=4 m/s relative to ice. Find the velocity V of the sled.



$$\text{momentum before } 0 = MV + mv \text{ momentum after}$$

$$V = -v(m/M) = -4(50/25) = \ldots$$

2. Explosion

A rocket with $M, \vec{v}$ brakes into m_1, $\vec{V}_1$ and m_2, $\vec{V}_2$ with $m_1 + m_2 = M$.

$$M\vec{v} = m_1\vec{V}_1 + m_2\vec{V}_2$$

Energy is increased.

B. Elastic

$$\vec{\mathcal{P}} = const \ , \quad K = const \tag{97}$$

Example: 1D elastic collision of a body $m\,,v$ with an identical stationary body $M = m$

$$\text{momentum: } mv + 0 = mV_1 + mV_2, \ , \ \text{energy: } \frac{1}{2}mv^2 = \frac{1}{2}mV_1^2 + \frac{1}{2}mV_2^2$$

$$v^2 = (V_1 + V_2)^2 \ \text{ and } \ v^2 = V_1^2 + V_2^2$$

$$\text{thus} \quad V_1 V_2 = 0 \quad \Rightarrow \quad \boxed{V_1 = 0\,,\, V_2 = v} \ \text{ or } \ V_1 = v,\, V_2 = 0$$

1. *Advanced: 2D elastic collision of two identical masses*

Collisions of 2 identical billiard balls (2nd originally not moving).

$$\vec{\mathcal{P}}\ (\text{before collision}) \;=\; \vec{\mathcal{P}}\ (\text{after})$$

$$K\ (\text{before collision}) \;=\; K\ (\text{after})$$

Momentum conservation gives

$$\vec{v} = \vec{V}_1 + \vec{V}_2$$

Energy conservation gives

$$v^2/2 = V_1^2/2 + V_2^2/2$$

or

$$\vec{V}_1 \cdot \vec{V}_2 = 0$$

which is a 90^o angle for any $V_2 \neq 0$.

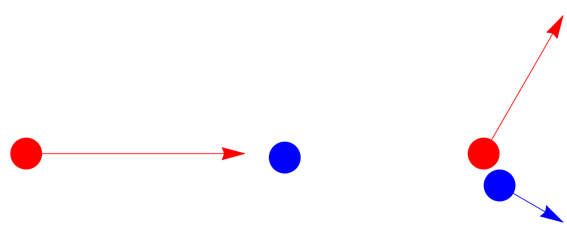


FIG. 21: Off-center elastic collision of 2 identical billiard balls (one stationary). Both energy and momentum are conserved. Note that the angle between final velocities is always 90^o.

2. *Advanced: 1D collision, $m \neq M$*

Collision of a body $m\,, v$ with a stationary body M.
From conservation of momentum:

$$m\,(v - V_1) = M V_2$$

From conservation of energy:

$$\frac{m}{2}\left(v^2 - V_1^2\right) = \frac{M}{2} V_2^2$$

Divide 2nd equation by the 1st one ($V_2 \neq 0$!) to get

$$v + V_1 = V_2$$

Use the above to replace V_2 in equation for momentum to get

$$V_1 = \frac{m - M}{m + M} v$$

and then

$$V_2 = \frac{2m}{M + m} v$$

Note limits and special cases:

- $m = M$: $V_1 = 0$, $V_2 = v$ ("total exchange")
- $M \to \infty$ (collision with a wall): $V_1 = -v$ (reflection)
- m ("tennis racket") $\gg$ M ("tennis ball"): $V_2 = 2v$

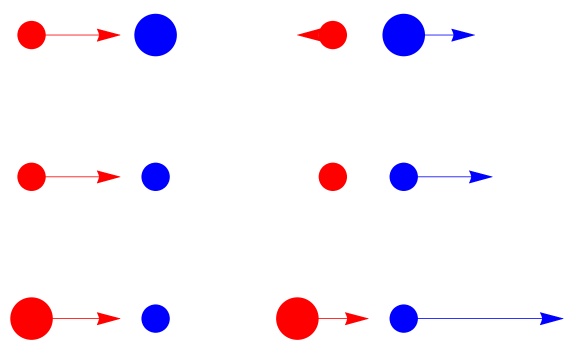


FIG. 22: Elastic collision of a moving missile particle m (red) with a stationary target M. (a) $m < M$; (b) $m = M$; (c) $m > M$. In each case both energy and momentum are conserved.

XIV. CENTER OF MASS (CM)

A. Definition

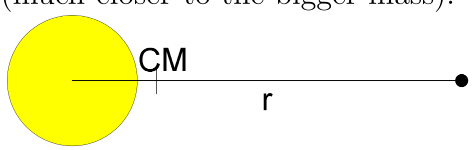
$$\vec{\mathcal{R}} = \frac{1}{M}\sum_i m_i \vec{r}_i \ , \quad M = \sum_i m_i \tag{98}$$

Example: CM of a two-particle system ("Sun-Earth", $r = 150 \cdot 10^6\, km$, from center of Sun on the x axis towards Earth; $M \simeq 2 \cdot 10^{30}\, kg$, $m \approx 6 \cdot 10^{24}\, kg$)

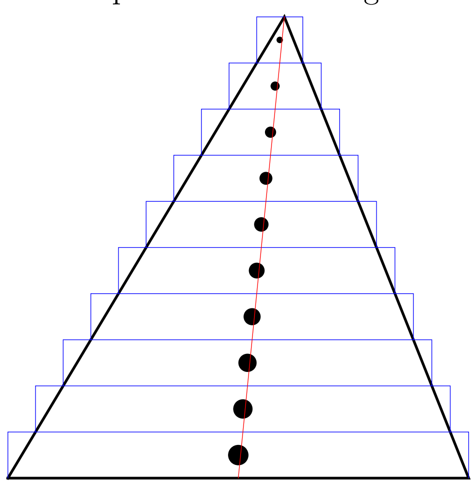
$$m_1 = M,\ \vec{r}_1 = (0,0)\ ;\ m_2 = m,\ \vec{r}_2 = (r,0)$$

$$\vec{\mathcal{R}} = \frac{1}{M+m}[(0,0) + m(r,0)] \approx \frac{m}{M}(r,0)$$

(much closer to the bigger mass).



Example: CM of a triangle.



For a uniform triangle the CM is at the intersection of medians.

B. Relation to total momentum

$$\vec{V}_{cm} = \frac{d}{dt}\vec{\mathcal{R}} = \frac{1}{M}\sum_i m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\vec{\mathcal{P}}$$

$$\boxed{\vec{\mathcal{P}} = M\vec{V}_{CM}} \tag{99}$$

C. 2nd Law for CM

Consider two particles:

$$d\vec{p}_1/dt = \vec{F}_{12} + \vec{F}_{1,ext}$$

$$d\vec{p}_2/dt = \vec{F}_{21} + \vec{F}_{2,ext}$$

$$\frac{d}{dt}\vec{\mathcal{P}} = \vec{F}_{1,ext} + \vec{F}_{2,ext} \equiv \vec{F}_{ext}$$

but

$$\frac{d}{dt}\vec{\mathcal{P}} = M\frac{d}{dt}\vec{V}_{cm} = M\vec{a}_{cm}$$

thus

$$M\vec{a}_{CM} = \vec{F}_{ext} \tag{100}$$

Example: acrobat in the air - CM moves in a simple parabola.

If $\vec{F}_{ext} = 0$

$$\vec{V}_{CM} = const \tag{101}$$

Examples: boat on a lake, astronaut.

D. Advanced: Energy and CM

$$\vec{v}_i = \vec{v}_i' + \vec{V}_{CM}$$

Note:

$$\sum_i m_i \vec{v}_i' = 0$$

$$E = \frac{1}{2}\sum_i m_i \left(\vec{v}_i' + \vec{V}_{CM}\right)^2$$

$$E = \frac{1}{2}\sum_i m_i \left(v_i'\right)^2 + \left(\sum_i m_i \vec{v}_i' \cdot \vec{V}_{CM}\right) + \frac{1}{2}MV_{CM}^2$$

$$E = \frac{1}{2}\sum_i m_i \left(v_i'\right)^2 + \frac{1}{2}MV_{CM}^2$$

2nd term - KE of the CM, 1st term - KE *relative* to CM. (will be very important for rotation).

Advanced Collisions of elementary particles

$$K = \frac{p^2}{2m} \tag{102}$$

only for $v \ll c$. General

$$E = \sqrt{m^2c^4 + c^2p^2}$$

Two limits:

$$m = 0\,,\ E = cp$$

("photon", $v = c$) and

$$m \neq 0\,,\ v \ll c : E \approx mc^2 + \frac{p^2}{2m}$$

See also:

- circular motion with $v = const$ (kinematics, centripetal acceleration) and
- circular motion with $v = const$ (dynamics, centripetal force)

XV. KINEMATICS OF ROTATION

A. Radian measure of an angle

see Fig. 23. Arc length

$$l = r\theta \tag{103}$$

if θ is measured in radians.

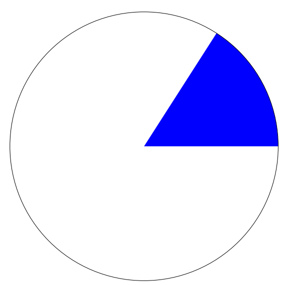


FIG. 23: Angle of $1\,rad \approx 57.3^o$. For this angle the length of the circular arc exactly equals the radius. The full angle, 360^o, is 2π radians.

B. Angular velocity

Notations: ω (omega). Units: rad/s

Definition: If constant,

$$\omega = \frac{2\pi}{T}\,,\ T \text{ - period of revolution}$$

$$\text{General: } \omega = \frac{d\theta}{dt} \approx \frac{\Delta\theta}{\Delta t} \tag{104}$$

Conversion from revolution frequency ($\omega = const$):

Example: find ω for $45\, rev/min$

$$45\,\frac{rev}{min} = 45\,\frac{2\pi\, rad}{60\, s} \approx 4.7\frac{rad}{s}$$

C. Connection with linear velocity and centripetal acceleration

$$v = \frac{dl}{dt} = \frac{d(r\theta)}{dt} = \omega r\,,\ \boxed{\omega = \frac{v}{r}} \tag{105}$$

$$\boxed{a_c = v^2/r = \omega^2 r} \tag{106}$$

D. Angular acceleration

Definition:

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}} \tag{107}$$

Units:

$$[\alpha] = rad/s^2$$

E. Connection with tangential acceleration

$$a_\tau = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{108}$$

(very important for rolling problems!)

F. Rotation with $\alpha = const$

Direct analogy with linear motion:

$$x \ \rightarrow \ \theta \ \ , \quad v \ \rightarrow \ \omega \ \ , \quad a \ \rightarrow \ \alpha$$

$$\omega = \omega_0 + \alpha t \, , \ \theta - \theta_0 = \frac{\omega + \omega_0}{2} t \tag{109}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \, , \ \theta - \theta_0 = \frac{\omega^2 - \omega_0^2}{2\alpha} \tag{110}$$

New: connections between θ (in rads) and N (in revs) and between ω in rad/s and frequency $f = 1/T$ in rev/s:

$$\boxed{\theta = 2\pi N} \, , \quad \boxed{\omega = 2\pi f} \tag{111}$$

Examples:

- A spinning wheel slows from $\omega_0 = 30\, rad/s$ to a stop in $t = 0.2\, s$. a) find angular acceleration α. b) What is the angular change θ in radians ?

$$\alpha = \frac{0 - \omega_0}{t} = -150\, rad/s^2\,,\ \theta = \frac{\omega_0^2 - 0}{2\alpha} = 3\, rad$$

- A disk starts from rest and accelerates at $\alpha = 3\, rad/s^2$ to an angular speed of $\omega = 6\, rad/s$. What is the angular change θ in radians? How long does it take?

$$\theta = \frac{\omega^2 - 0}{2\alpha} = 6\, rad\,,\ t = \frac{\omega - 0}{\alpha} = 2\, s$$

- Given $\omega(0) = 5\, rad/s\,,\ N = 100\, rev$ (till stop!). Find α.

$$2\pi N = \theta,\, \theta = \frac{0^2 - \omega(0)^2}{2\alpha} \;\Rightarrow\; \alpha = -\frac{\omega(0)^2}{2\theta} = \ldots$$

XVI. KINETIC ENERGY OF ROTATION AND ROTATIONAL INERTIA

A. The formula $K = 1/2\, I\omega^2$

For any point mass

$$K_i = \frac{1}{2} m_i v_i^2 \tag{112}$$

For a solid rotating about an axis

$$v_i = \omega r_i \tag{113}$$

with r_i being the distance from the axis and ω, the angular velocity being *the same* for every point. Thus, the full kinetic energy is

$$K = \sum_i K_i = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \equiv \frac{1}{2} I\omega^2 \tag{114}$$

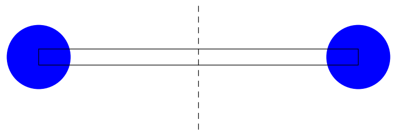
Here I, the rotational inertia, is the property of a body, independent of ω (but sensitive to selection of the rotational axis):

$$\boxed{I = \sum_i m_i r_i^2}\,, \quad \boxed{K = \frac{1}{2} I\omega^2} \tag{115}$$

For continuous distribution of masses the sum is replaced by an integral - will later consider several specific examples.

B. Rotational Inertia: Examples

1. Collection of point masses



Two identical masses m at $x = \pm a/2$. Rotation in the xy plane about the z-axis through the CM.

$$I = 2 \cdot m(a/2)^2 = m\frac{a^2}{2} \tag{116}$$

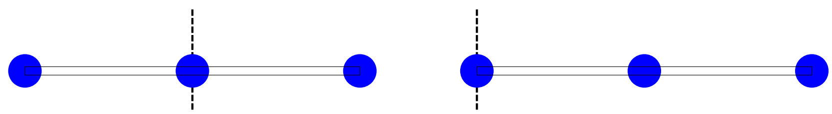
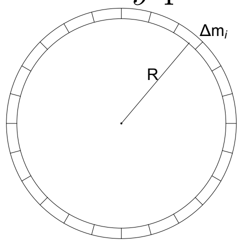


FIG. 24: 3 identical point masses m on a massless rod of length L. Left: axis through CM. $I_{CM} = m(L/2)^2 + m \cdot 0 + m(L/2)^2 = mL^2/2$. Right: axis through the end. $I_{end} = m \cdot 0 + m(L/2)^2 + mL^2 = (5/4)mL^2 > I_{CM}$ (!).

Hoop of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis.



Let us break up the hoop into small fragments with masses Δm_i, where i is the number of the fragment and masses Δm_i do not have to be identical. The distance of each fragment from the center is R and thus its contribution to I is $\Delta m_i R^2$. Thus

$$I = \sum_i \Delta m_i R^2 = R^2 \sum_i \Delta m_i = R^2 M$$

or

$$\boxed{I_{\text{hoop}} = MR^2}$$

(the same for hollow cylinder -which is a stack of hoops- rotating about the axis)

3. Rod

Consider a uniform rod of mass M of length L. Select the origin at the center of the rod, which thus spans between at $x = \pm L/2$. Rotation in the xy plane about the z-axis through the center of mass. We start with

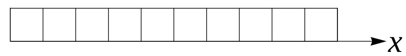


FIG. 25: Evaluating rotational inertia of a rod

breaking the rods into tiny segments of length dx. Each segment has a mass Mdx/L and is located at a distance x from the origin. Thus, the elementary contribution of a single segment to I is

$$dI = M\frac{dx}{L}x^2 \;\Rightarrow\; I = \int_{-L/2}^{L/2} Mx^2\frac{1}{L}dx =$$

$$= \frac{M}{L}2\int_0^{L/2} x^2 dx = \frac{M}{L}2\frac{x^3}{3}\bigg|_0^{L/2} = \frac{m}{L}\frac{2}{3}\left(\frac{L}{2}\right)^3 \quad \text{or}$$

$$\boxed{I_{\text{rod}} = \frac{ML^2}{12}} \tag{117}$$

4. Disk

Uniform disk of mass M, radius R in the xy plane, center at the origin. Rotation in the xy plane about the z-axis.

Planar density

$$\sigma = M/\left[\pi R^2\right]$$

Elementary area

$$dS = 2\pi r\, dr$$

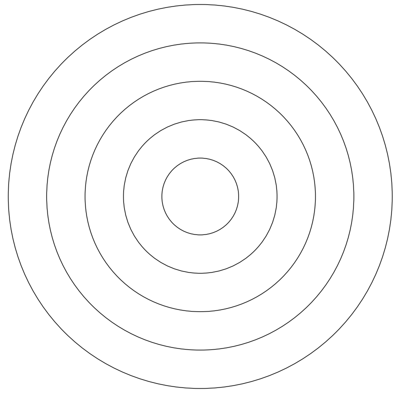


FIG. 26: Evaluating rotational inertia of a disk

$$I_{\rm disk} = \int_0^R dr\, 2\pi r \sigma r^2 = \frac{1}{2} M R^2 \tag{118}$$

(the same for solid cylinder about the axis).

5. *Advanced: Solid and hollow spheres*

Solid sphere: slice it into collection of thin disks of thickness dz and radius $r = \sqrt{R^2 - z^2}$

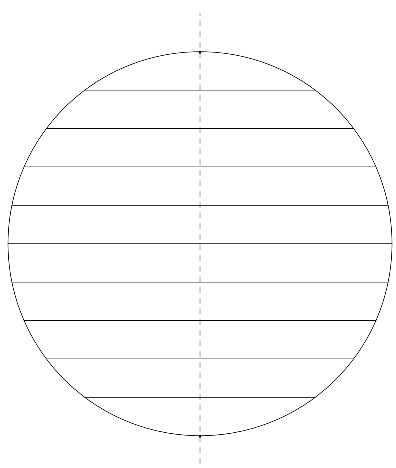


FIG. 27: Evaluating rotational inertia of a sphere. Solid sphere - collection of disks; hollow sphere (shell) - collection of hoops.

$$I_{\text{sph}} = \int_{-R}^{R} dz \, \frac{1}{2}\pi r^2 \rho r^2 = \frac{2}{5} M R^2 \tag{119}$$

Hollow:

$$I_{h.sph} = \frac{2}{3} M R^2$$

C. Parallel axis theorem

$$\boxed{I = I_{cm} + MD^2} \tag{120}$$

with I_{cm} being rotational inertia about a parallel axis passing through the center of mass and D - distance to that axis.

Proof:

$$\vec{R}_{cm} = \frac{1}{M} \sum_i \vec{r}_i m_i$$

Introduce

$$\vec{r}_i' = \vec{r}_i - \vec{R}_{cm}$$

with

$$\sum_i \vec{r}_i' m_i = 0$$

and

$$I_{cm} = \sum_i m_i \left(r_i'\right)^2$$

Now

$$I = \sum_i m_i \left(\vec{r}_i' + \vec{D}\right)^2 = I_{cm} + MD^2 + 2\vec{D} \cdot \sum_i \vec{r}_i' m_i$$

where the last sum is zero, which completes the proof.

Example: rod M, L about the end

$$I_{CM} = ML^2/12$$

with $D = L/2$:

$$I_{rod,end} = ML^2/12 + M(L/2)^2 = ML^2/3$$

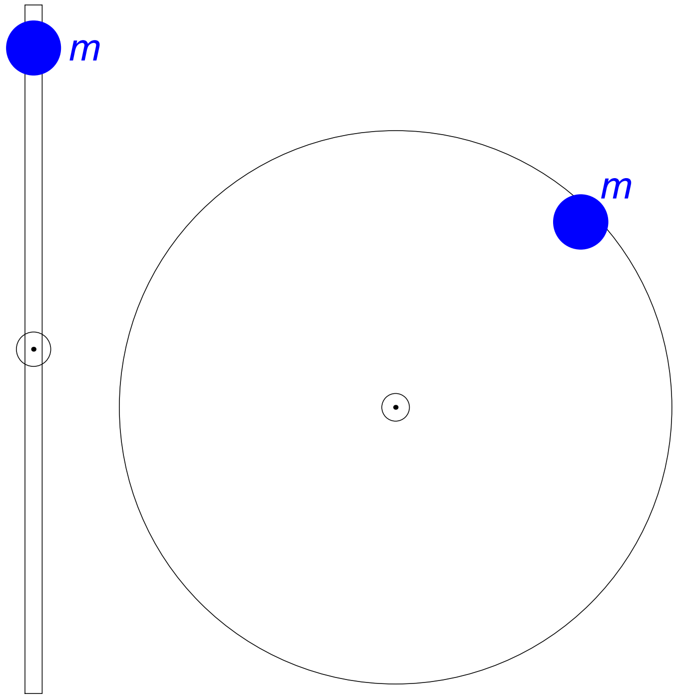


FIG. 28: Modification of I by a point mass m added at distance a from center. Left: rod mass M, length L. Right: disk of mass M and radius R.

$$rod:\ \ I = I_{rod} + ma^2 = ML^2/12 + ma^2 = L^2(M/3 + m)/4 \ \text{ if } a \simeq L/2$$

$$disk:\ \ I = I_{disk} + ma^2 = MR^2/2 + ma^2 = R^2(M/2 + m)\ \ if\ a \simeq R$$

D. Conservation of energy, including rotation

$$\boxed{K + U = const} \tag{121}$$

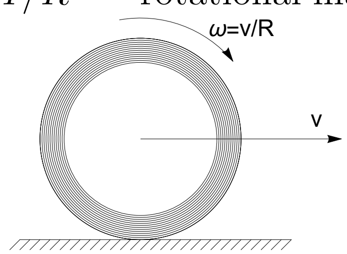
where K is the *total* kinetic energy (translational and rotational for all bodies) and U is total potential energy (typically, *not* changed by rotation).

$$K_{rot} = \frac{1}{2} I\omega^2 \,, \; K_{trans} = \frac{1}{2} MV^2$$

If ω and V are connected (as in rolling):

$$\omega = \frac{V}{R} \;\Rightarrow\; K = \frac{1}{2} MV^2 + \frac{1}{2} V^2 \frac{I}{R^2} = \frac{1}{2}\left(M + \frac{I}{R^2}\right) V^2$$

I/R^2 - "rotational mass"



Example. As a result of a car accident a tire of mass M got detached from a wheel and rolled away with linear speed of v_0. a) Find the full kinetic energy of the tire, treating it approximately as a "hoop". b) if the tire rolls uphill, at which hight h will it stop?

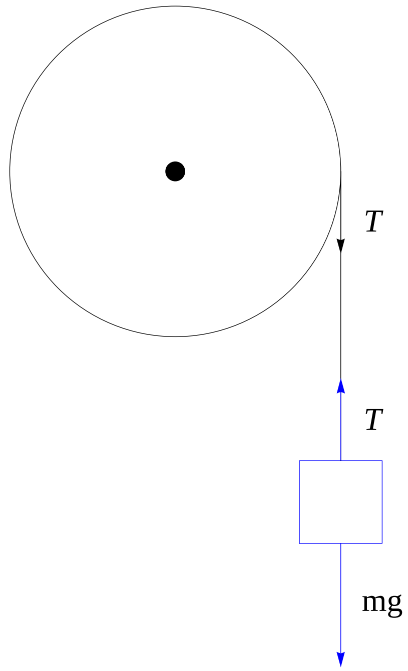
$$\text{(a) } K = K_{trans} + K_{rot} = \frac{1}{2} M v_0^2 + \frac{1}{2} I\omega_0^2 \,, \; \omega_0 = v/R$$

$$K = \frac{1}{2} M v_0^2 + \frac{1}{2} I \frac{v_0^2}{R^2} = \frac{1}{2} M v_0^2 \left(1 + \frac{I}{MR^2}\right) \,, \; \frac{I}{MR^2} = 1(\text{"hoop"})$$

$$K = \frac{1}{2} M v_0^2 (1+1) = ...$$

$$\text{(b) from energy conservation } Mgh = K \,, \; h = v_0^2/g$$

E. "Bucket falling into a well"



Example. Suppose mass m goes distance h down starting from rest. Find final velocity v. Pulley is a disk M, R.

Solution: ignore T (!), use energy only.

- energy conservation

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$

- constrains

$$v = \omega R$$

Thus

$$v^2 = 2gh\frac{m}{m + I/R^2}$$

If $I = MR^2/2$ (disk)

$$v^2 = 2gh\frac{1}{1 + M/(2m)}\ , \quad v = \dots$$

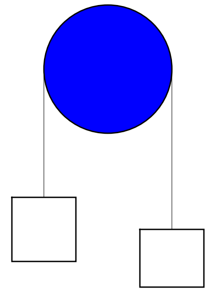


FIG. 29: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

1. *Advanced: Atwood machine*

Example. Suppose the larger mass goes distance h down starting from rest. Find final velocity v.

- energy conservation. Initial: potential energy Mgh only; final: potential mgh plus kinetic of both moving masses and the rotating pulley.

$$mgh + \frac{1}{2}(M+m)v^2 + \frac{1}{2}I\omega^2 = Mgh$$

- constrains

$$v = \omega R\,,\ \omega = \frac{v}{R}$$

Thus

$$\frac{1}{2}(M+m)v^2 + \frac{1}{2}I\frac{v^2}{R^2} = Mgh - mgh$$

$$\frac{1}{2}v^2\left[(M+m) + \frac{I}{R^2}\right] = gh(M-m)$$

$$v^2 = 2gh\frac{M-m}{M+m+I/R^2}$$

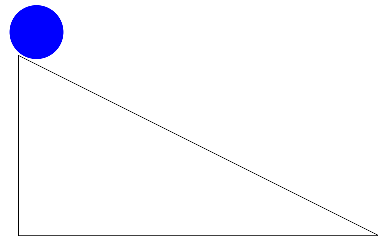


FIG. 30: Rolling down of a body with mass m, rotational inertia I and radius R. Potential energy at the top, mgh equals the full kinetic energy at the bottom, $mv^2/2 + I\omega^2/2$.

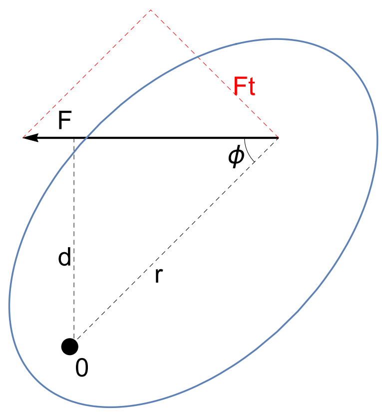
Suppose the body rolls vertical distance $h = L\sin\theta$ starting from rest. Find final velocity v.

- energy conservation
$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$$
- constrains
$$\omega = \frac{v}{R} \;\Rightarrow\; \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = mgh$$

Thus
$$v^2 = 2gh\frac{1}{1 + I/\left(mR^2\right)}$$
$$hoop: \; I = MR^2 \; , \;\; v^2 = 2gh\frac{1}{1+1}$$
$$disk: \; I = MR^2/2 \; , \;\; v^2 = 2gh\frac{1}{1+1/2}$$

The disk wins!

XVII. TORQUE



A. Definition

Consider a point mass m at a fixed distance r from the axis of rotation. Only motion in tangential direction is possible. Let F_t be the tangential component of force. The *torque* is defined as (τ is the lower case Greek "tau"):

$$\boxed{\tau = F_t r = F r \sin\phi = F d} \tag{122}$$

with ϕ being the angle between the force and the radial direction. $r\sin\phi = d$ is the "lever arm". Counterclockwise torque is positive, and for several forces torques add up.

Units:

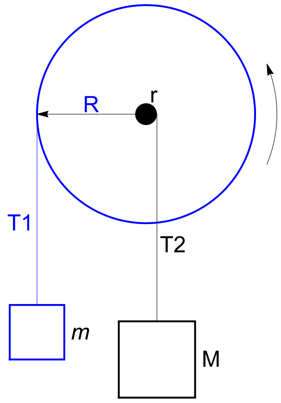
$$[\tau] = N \cdot m \text{ (same as } Joules\text{)}.$$

Why the same units as work? Consider the solid in above picture turning by a small angle $\Delta\theta$ with the point of the application of the force moving in a circular arc with length $\Delta l = r\Delta\theta$. The work W is then

$$W = F_t \Delta l = \tau\Delta\theta$$

Since $\Delta\theta$ is dimensionless (measured in *radians*), formally the units for τ look similar to those used for W; it is better, however to keep units for τ as $N \cdot m$.

2D convention:



(not to scale!)

positive direction is indicated

$$\tau_+ = T_1 R\,,\ \tau_- = T_2 r$$

$$\tau = \tau_+ - \tau_-$$

In equilibrium:

$$\tau_+ = \tau_-\ ,\ T_1 = mg\,, T_2 = Mg\ \Rightarrow\ R \times mg = r \times Mg\ \ (\text{"lever rule"})$$

B. 2nd Law for rotation

Start with a single point mass. Consider the tangential projection of the 2nd Law

$$F_t = ma_t$$

Now multiply both sides by r and use $a_t = \alpha r$ with α the angular acceleration.

$$\tau = mr^2\alpha$$

For a system of particles m_i each at a distance r_i and the same α ("solid"!)

$$\tau_i = m_i r_i^2 \alpha \;\Rightarrow\; \sum_i \tau_i = \left(\sum_i m_i r_i^2\right) \alpha \;\Rightarrow$$

$$\boxed{\sum \tau = I\alpha} \tag{123}$$

C. Application of $\tau = I\alpha$. Examples.

Example. In a relatively new Olympic sports of curling a flat round stone ("rock") is used with mass $M \simeq 20\,kg$ and radius $R \simeq 20\,cm$. The rock is given the initial angular velocity $\omega_0 = 100\,rad/s$ and it slows down to $\omega = 10\,rad/s$ after 50 revolutions. (a) find the torque due to fiction and (b) find the work done by friction to stop the rock

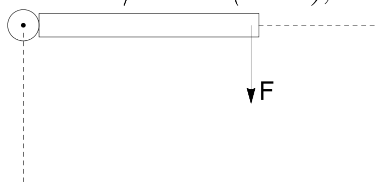
$$\text{(a):}\ \alpha = \frac{\omega^2 - \omega_0^2}{2\theta}\,,\ \theta = 2\pi * 50\,rad \;\Rightarrow\; \alpha = \frac{10^2 - 100^2}{2 * 2\pi * 50} \approx -16\,\frac{rad}{s^2}$$

$$I = \frac{1}{2}MR^2 = 0.4\,kg * m^2 \;\Rightarrow\; |\tau| = I|\alpha| = 6.4\,N * m$$

$$\text{(b):}\ W_f = \Delta K = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = -\frac{1}{2} * 0.4(100^2 - 10^2) = ...$$

1. Revolving door

How long will it take to open a heavy, freely revolving door by 90 degrees starting from rest, if a constant force F is applied at a distance r away from the hinges at an angle 90^0, as shown in the figure? Make some reasonable approximations about parameters of the door, F and r. For I you can use $1/3\,ML^2$ ("rod"), with L being the horizontal dimension.



Force applied to a door (top view).

Solution: 1) find torque; 2) use 2nd law for rotation to find α; 3) use kinematics to estimate t

1.

$$\tau = Fr\sin 90^0 = Fr$$

2.

$$\alpha = \tau/I = 3Fr/(ML^2)$$

if $r = L$

$$\alpha = 3F/(ML)$$

3. From

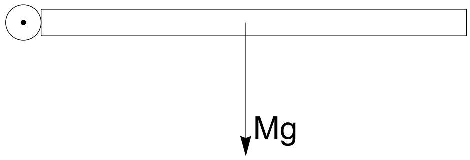
$$\theta = 1/2\,\alpha t^2$$

$$t = \sqrt{2\theta/\alpha} = \sqrt{\pi\frac{I}{FL}}$$

Using, e.g. $M = 30\,kg$, $F = 30\,N$, $L = 1\,m$ one gets $t \sim 1\,s$, which is reasonable.

2. Rotating rod

Consider a rod of length l and mass M falling freely under the force of gravity. Find the angular acceleration α and the linear acceleration a of the end.



$$\tau = Mgl \times \frac{1}{2}\,, \; I = \frac{1}{3}Ml^2 \;\Rightarrow$$

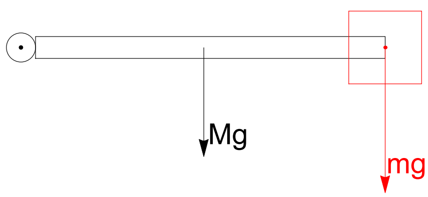
$$\alpha = \frac{\tau}{I} = (1/2\,Mgl) \Big/ \left(\frac{1}{3}Ml^2\right) = \frac{3}{2}\frac{g}{l}$$

Linear acceleration of the end:

$$a = \alpha l = \frac{3}{2}g \; > \; g \; (!)$$

3. Rod with a point mass m at the end.

A small mass m is added at the end of the rod. Will it go faster or slower compared to previous problem?



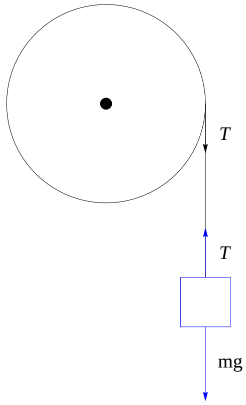
Solution: new I and τ:

$$I \;\rightarrow\; \frac{1}{3}Ml^2 + ml^2\,, \;\; \tau \;\rightarrow\; \frac{1}{2}\,Mgl + mgl$$

Thus,

$$\alpha = \frac{\tau}{I} = \frac{1/2\,Mgl + mgl}{1/3\,Ml^2 + ml^2} = \frac{1/2\,Mgl(1 + 2m/M)}{1/3\,Ml^2(1 + 3m/M)} = \frac{3}{2}\frac{g}{l} \times \frac{1 + 2m/M}{1 + 3m/M} < \frac{3}{2}\frac{g}{l}$$

4. "Bucket falling into a well" revisited.



$$\text{2nd Law for rotation: } \alpha = \frac{\tau}{I} = T\frac{R}{I} \Rightarrow T = \alpha\frac{I}{R}$$

$$\text{2nd Law for linear motion: } ma = mg - T \;\; = \;\; mg - \alpha\frac{I}{R}$$

$$\text{Constrain: } \alpha = \frac{a}{R} \;\Rightarrow\; ma = mg - a\frac{I}{R^2} \;\; \text{or} \;\; a\frac{I}{R^2} + ma = mg$$

Thus,

$$a = g\frac{1}{1 + I/\left(mR^2\right)}$$

Advanced Recall that the same result can be obtained from from energy:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\,,\ \omega = v/R$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\frac{I}{R^2}v^2\,,\ v^2 = 2gh\frac{1}{1 + I/mR^2} = \frac{2gh}{1 + M/(2m)}$$

$$a = \frac{v^2}{2h} = g\frac{1}{1 + I/\left(mR^2\right)}$$

5. Advanced: Atwood machine revisited.

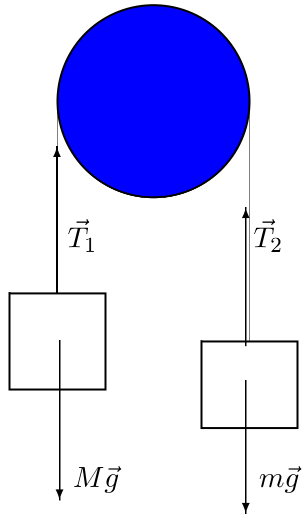


FIG. 31: Atwood machine. Mass M (left) is almost balanced by a slightly smaller mass m. Pulley has rotational inertia I and radius R.

Let T_1 and T_2 be tensions in left string (connected to larger mass M) and in the right string, respectively.

- 2nd Law(s) for each body (and for the pulley with $\tau = (T_1 - T_2)\, R$)
- constrains $a = \alpha R$

From 2nd Law(s):

$$Mg - T_1 = Ma\,,\; T_2 - mg = ma\,,\; (T_1 - T_2) = I\alpha/R$$

Add all 3 together to get (with constrains)

$$(M - m)g = (M + m)a + I\alpha/R = (M + m + I/R^2)a$$

which gives

$$a = g\frac{M - m}{M + m + I/R^2}$$

and $\alpha = a/R$. What if need tension?

$$T_1 = Mg - ma \,<\, Mg\;,\;\; T_2 = mg + ma \,>\, mg$$

6. Rolling down incline revisited.

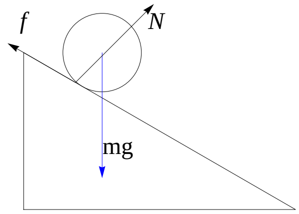


FIG. 32: Rolling down of a body with mass m, rotational inertia I and radius R. Three forces act on the body: $\vec{f}$ - static friction at the point of contact, up the plane; $\vec{N}$ - normal reaction, perpendicular to the plain at the point of contact and $m\vec{g}$ is applied to CM. Note that only friction has a non-zero torque with respect to CM with value $\tau = -fR$.

$$\text{2nd Law (linear): } \vec{f} + \vec{N} + m\vec{g} = m\vec{a} \;\Rightarrow\; -f + mg\sin\theta = ma$$

with x-axis down the incline.

$$\text{Constrains: } \alpha = \frac{a}{R}$$

$$\text{2nd Law (rotation): } fR = I\alpha \;\Rightarrow\; f = I\frac{\alpha}{R} = I\frac{a}{R^2}$$

Using this to replace f in $-f + mg\sin\theta = ma$ one obtains

$$-a\frac{I}{R^2} + mg\sin\theta = ma \;\Rightarrow\; mg\sin\theta = ma + a\frac{I}{R^2} \quad \text{or}$$

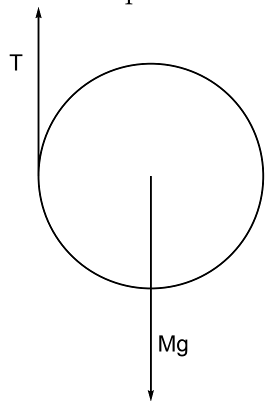
$$a = g\sin\theta\,\frac{1}{1 + I/(mR^2)}$$

Advanced. The same can be obtained from energy conservation:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\,,\; \omega = \frac{v}{R}$$

$$v^2 = \frac{2gh}{1 + I/(mR^2)}\,,\; a = \frac{v^2}{2L}, L = \frac{h}{\sin\theta} \;\Rightarrow\; a = g\sin\theta\,\frac{1}{1 + I/(mR^2)}$$

Example: Primitive yo-yo.



$$\alpha = \tau/I\,,\; \tau = Tr \;\Rightarrow\; T = \frac{I\alpha}{r}$$

$$\alpha = \frac{a}{r} \;\Rightarrow\; T = \frac{Ia}{r^2}$$

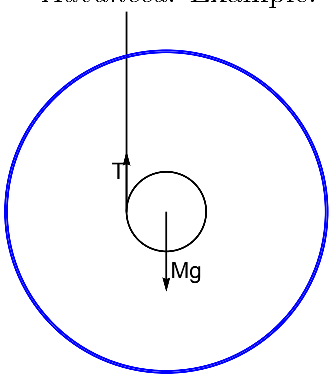
$$Mg - T = Ma \;\Rightarrow\; Mg - \frac{Ia}{r^2} = Ma$$

$$a = g\frac{1}{1 + I/Mr^2} = \frac{2}{3}g \text{ for solid disk with } I = \frac{1}{2}Mr^2$$

$$a = \frac{2}{3}g\,,\; \alpha = \frac{2}{3}\frac{g}{r}$$

Note that always $a = (2/3)g$, regardless of parameters of the disk.

Advanced. Example: Real yo-yo.



Everything similar, but now $I \approx \frac{1}{2}MR^2$ where $R \gg r$ is the radius of the large (blue) disk. Thus

$$a = g\frac{1}{1 + I/Mr^2} \approx g\frac{1}{1 + R^2/2r^2} \approx g\frac{2r^2}{R^2} \ll g$$

D. Torque as a vector

Cross product - see Introduction on vectors

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{124}$$

Example. Find $\vec{\tau}$ if $\vec{F} = \hat{i} - 2\hat{k}$ (in Newtons) and $\vec{r} = 3\hat{i} + 4\hat{k}$ meters. (Note: $\vec{r}$ goes first!)

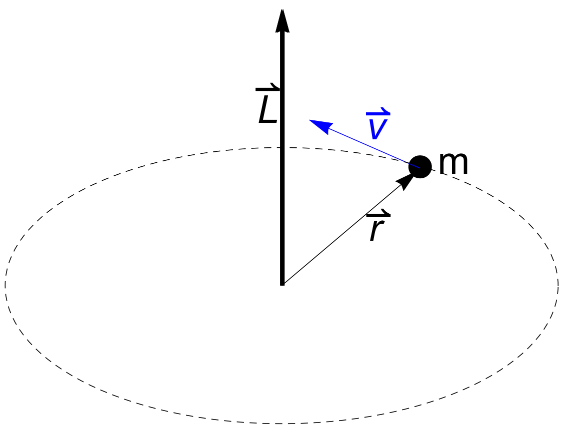
$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 4 \\ 1 & 0 & -2 \end{vmatrix} = \hat{i} \begin{vmatrix} 0 & 4 \\ 0 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = -\hat{j}\,\{3(-2) - 4 \cdot 1\} = 10\hat{j}\, N \cdot m$$

XVIII. ANGULAR MOMENTUM $\mathcal{L}$

A. Single point mass

$$\boxed{\vec{\mathcal{L}} = \vec{r} \times \vec{p}} \tag{125}$$

with $\vec{p} = m\vec{v}$, the momentum.



Example. Find $\vec{\mathcal{L}}$ for circular motion.

Solution: Direction - along the axis of rotation (as $\vec{\omega}$!). Magnitude:

$$\mathcal{L} = mvr \sin 90^o = mr^2\omega$$

or

$$\vec{\mathcal{L}} = mr^2\vec{\omega} \tag{126}$$

B. System of particles

$$\vec{\mathcal{L}} = \sum_i \vec{r}_i \times \vec{p}_i \tag{127}$$

C. Rotating symmetric solid

1. *Angular velocity as a vector*

Direct $\vec{\omega}$ allong the axis of rotation using the right-hand rule.

Example. Find $\vec{\omega}$ for the spinning Earth.

Solution: Direction - from South to North pole. Magnitude:

$$\omega = \frac{2\pi \, rad}{24 \cdot 3600 \, s} \simeq \ldots \frac{rad}{s}$$

If axis of rotation is also an axis of symmetry for the body

$$\boxed{\mathcal{L} = \sum_i m_i r_i^2 \omega = I\omega} \tag{128}$$

or

$$\vec{\mathcal{L}} = I\vec{\omega}$$

D. 2nd Law for rotation in terms of $\vec{\mathcal{L}}$

Start with

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Then

$$\vec{\tau} = \frac{d\vec{\mathcal{L}}}{dt} \tag{129}$$

XIX. CONSERVATION OF ANGULAR MOMENTUM

Start with

$$\vec{\tau} = \frac{d\vec{\mathcal{L}}}{dt}$$

If $\vec{\tau} = 0$ (no net external torque)

$$\vec{\mathcal{L}} = const \tag{130}$$

valid everywhere (from molecules and below, to stars and beyond).

For a closed *mechanical* system, thus

$$E = const\,,\ \vec{P} = const\,,\ \vec{\mathcal{L}} = const$$

For *any* closed system (with friction, inelastic collisions, break up of material, chemical or nuclear reactions, etc.)

$$E \neq const\,,\ \vec{P} = const\,,\ \vec{\mathcal{L}} = const$$

A. Examples

1. Free particle

$$\vec{\mathcal{L}}(t) = \vec{r}(t) \times m\vec{v}$$

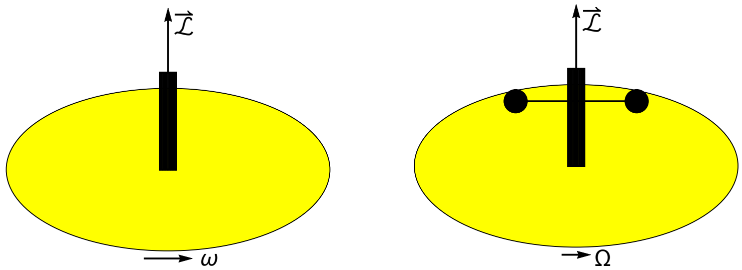
with $\vec{v} = const$ and

$$\vec{r}(t) = \vec{r}_0 + \vec{v}t$$

Thus, from $\vec{v} \times \vec{v} = 0$

$$\vec{\mathcal{L}}(t) = (\vec{r}_0 + \vec{v}t) \times m\vec{v} = \vec{r}_0 \times m\vec{v} = \vec{\mathcal{L}}(0)$$

2. Student on a rotating platform



Conservation of angular momentum $\mathcal{L}$. Left: a student stands on a platform with dumbbells close to the axis of rotation (not shown); their contribution to rotational inertia I is negligible. Right: the student extends his arms with the dumbbells increasing the rotational inertia. In order to conserve the angular momentum the angular velocity ω changes to a smaller value Ω.

Let I be rotational inertia of student+platform, and

$$I' \simeq I + 2Mr^2$$

the rotational inertia of student+platform+extended arms with dumbbells (r is about the length of an arm). Then,

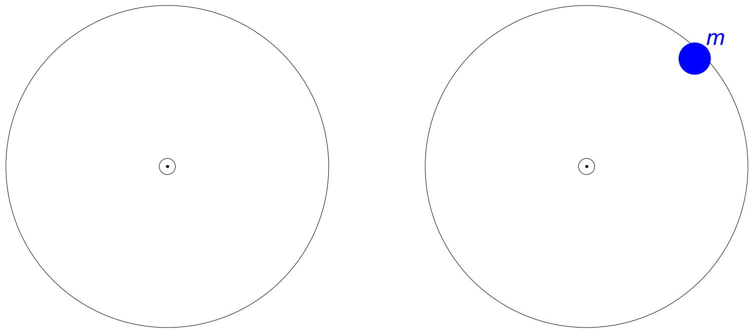
$$\mathcal{L} = I\omega \; = \; I'\Omega$$

or

$$\Omega = \omega \frac{I}{I'} = \omega \frac{1}{1 + 2Mr^2/I}$$

3. Chewing gum on a disk

An $m = 5\,g$ object is dropped onto a uniform disc of rotational inertia $I = 2 \cdot 10^{-4}\,kg \cdot m^2$ rotating freely at 33.3 revolutions per minute. The object adheres to the surface of the disc at distance $r = 5\,cm$ from its center. What is the angular momentum? What is the final angular velocity of the disk? How much energy is lost?



Solution.

$$\mathcal{L} = I\omega$$

Similarly to previous example, from conservation of angular momentum

$$\mathcal{L} = I\omega \;=\; I'\omega'$$

New rotational inertia

$$I' = I + mr^2$$

Thus

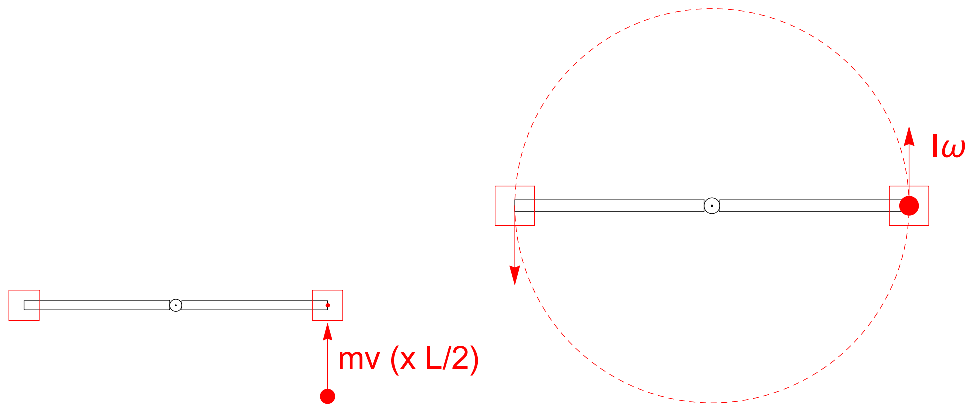
$$\omega' = \omega\frac{I}{I'} = \omega\frac{I}{I + mr^2} = \omega\frac{1}{1 + mr^2/I}$$

Energy:

$$K_i = \frac{1}{2}I\omega^2\,,\; K_f = \frac{1}{2}I'(\omega')^2$$

$$K_i - K_f = \ldots > 0$$

To measure the speed of a fast bullet a rod (mass M_{rod}) with length L and with two wooden blocks with the masses M at each end, is used. The whole system can rotate in a horizontal plane about a vertical axis through its center. The rod is at rest when a small bullet of mass m and velocity v is fired into one of the blocks. The bullet remains stuck in the block after it hits. Immediately after the collision, the whole system rotates with angular velocity ω. Find v. (use $L = 2$ meters, $\omega = 0.5\,rad/s$ and $m = 5\,g$, $M = 1\,kg$, $M_{rod} = 4\,kg$).



Solution. Let $r = L/2 = 1$ meter be the distance from the pivot. Angular momentum: before collision

$$\mathcal{L} = \frac{L}{2}mv$$

(due to bullet); after:

$$\mathcal{L} = I\omega$$

with

$$I = I_0 + m(L/2)^2$$

Here I_0 is rotational inertia without the bullet:

$$I_0 = M_{rod}L^2/12 + 2M(L/2)^2$$

Due to conservation of angular momentum

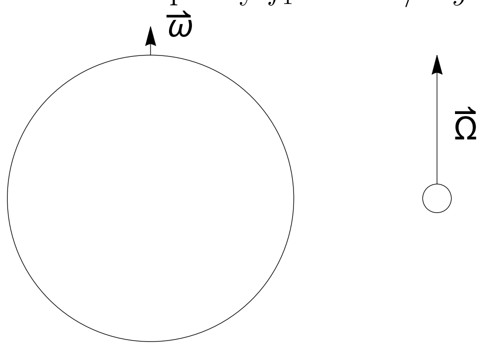
$$\frac{L}{2}mv = I\omega$$

and

$$v = 2I\omega/(mL)$$

5. Rotating star (white dwarf)

A uniform spherical star collapses to 0.3% of its former radius. If the star initially rotates with the frequency $f_1 = 1\, rev/day$ what would the new rotation frequency be?



$$\vec{\mathcal{L}} = \text{const} \;\Rightarrow\; I_1\vec{\omega} = I_2\vec{\Omega}$$

$$I_1 \approx \frac{2}{5}MR^2\,,\; I_2 \approx \frac{2}{5}Mr^2$$

$$\Omega \approx \omega\frac{R^2}{r^2} \;\Rightarrow\; f_2 = f_1\frac{R^2}{r^2} = \ldots \;-\; \text{large}$$

Kinetic energy:

$$K = \frac{1}{2}I\omega^2\,,\; \mathcal{L} = I\omega \;\Rightarrow\; K = \frac{\mathcal{L}^2}{2I}$$

$$\frac{K_2}{K_1} = \frac{I_1}{I_2} \approx \left(\frac{R}{r}\right)^2 \gg 1$$

Where does the huge increase in K come from?

We return to open systems with $\tau \neq 0$ (and thus $L \neq$ const). Note that the algebraic solution to many of the simple problems which involve simultaneous linear and rotational motion involve multiplication by a dimensionless

$$\boxed{\frac{1}{1+I/Mr^2}}$$

This include energy considerations where the speed in neglect of rotation $v = \sqrt{2gh}$ is transformed as

$$v^2 = 2gh\frac{1}{1+I/Mr^2}$$

and acceleration problems where the linear acceleration a_0 in neglect of rotation is transformed as

$$a = a_0\frac{1}{1+I/Mr^2}$$

Examples. Simple body rolling down incline. One has $I/Mr^2 =$ 1 (hoop) or 1/2 (disk) or 2/5 (solid sphere), etc. h is vertical displacement:

$$v^2 = 2gh\frac{1}{1+I/Mr^2}\,,\; a = g\sin\theta\frac{1}{1+I/Mr^2}$$

(Note that as usual $a = v^2/2l$ with $l = h/\sin\theta$ being the length of incline). Angular acceleration is then $\alpha = a/r$. Similarly, for a primitive yo-yo (disk)

$$v^2 = 2gh\frac{2}{3}\,,\; a = g\frac{2}{3}\,,\; \alpha = \frac{a}{r}$$

In more complex situations I does not have to be related to Mr^2. E.g., for a "bucket falling into a well", I is for the pulley while M is the mass of the bucket. Or, for a real yo-yo ("maxwell's wheel") r is the radius of the axis, much smaller than R of the heavy disk. The above formulas still work but a will be much smaller.

XX. EQUILIBRIUM

A. General conditions of equilibrium

$$\sum \vec{F}_i = 0 \tag{131}$$

$$\sum \vec{\tau}_i = 0 \tag{132}$$

Theorem. In equilibrium, torque can be calculated about *any* point.

Proof. Let $\vec{r}_i$ determine positions of particles in the system with respect to point O. Selecting another point as a reference is equivalent to a shift of every $\vec{r}_i$ by the same $\vec{r}_o$. Then,

$$\vec{\tau}_{new} = \sum_i (\vec{r}_i + \vec{r}_o) \times \vec{F}_i = \vec{\tau} + \vec{r}_o \times \sum_i \vec{F}_i = \vec{\tau}$$

B. Center of gravity

Theorem. For a *uniform* field $\vec{g}$ the center of gravity coinsides with the COM.

Proof. Torque due to gravity is

$$\tau_g = \sum_i \vec{r}_i \times m_i \vec{g} = \left(\sum_i m_i \vec{r}_i \right) \times \vec{g} = M\vec{R}_{CM} \times \vec{g} = \vec{R}_{CM} \times (M\vec{g})$$

C. Examples

1. Seesaw

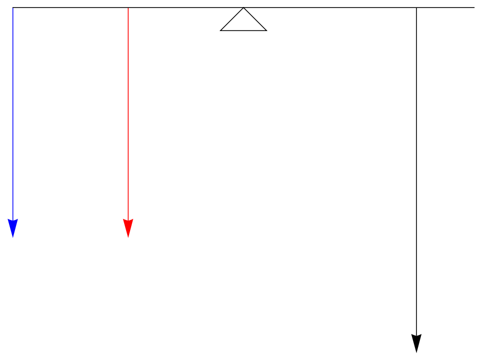


FIG. 33: Two twins, masses m and m (left) against their dad with mass M. Force of gravity on the seesaw and reaction of the fulcrum are not shown since they produce no torque.

If $2d$, d and D are distances from the fulcrum for each of the twins and the father,

$$mg \cdot 2d + mgd = MgD$$

or

$$3md = MD$$

Note that used only torque condition of equilibrium. If need reaction from the fulcrum $\vec{N}$ use the force condition

$$\vec{N} + (2m + M + M_{seesaw})\,\vec{g} = 0$$

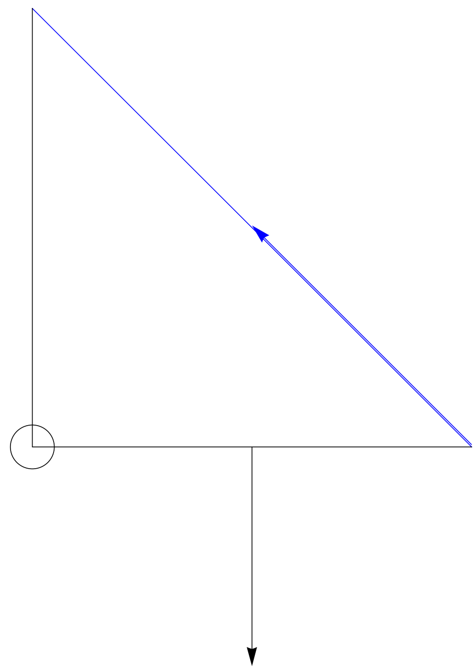


FIG. 34: Horizontal beam of mass M and length L supported by a blue cord making angle θ with horizontal. Only forces with non-zero torque about the pivot (tension $\vec{T}$ -blue- and gravity $M\vec{g}$ -black) are shown.

Cancellation of torques gives

$$TL\sin\theta = Mg\frac{L}{2}$$

or

$$T = Mg/2\sin\theta$$

Note: if you try to make the cord horizontal, it will snap ($T \to \infty$). For $\theta \to \pi/2$ one has $T \to Mg/2$, as expected. The force condition will allow to find reaction from the pivot:

$$\vec{R} + M\vec{g} + \vec{T} = 0$$

(and $-\vec{R}$ will be the force on the pivot).

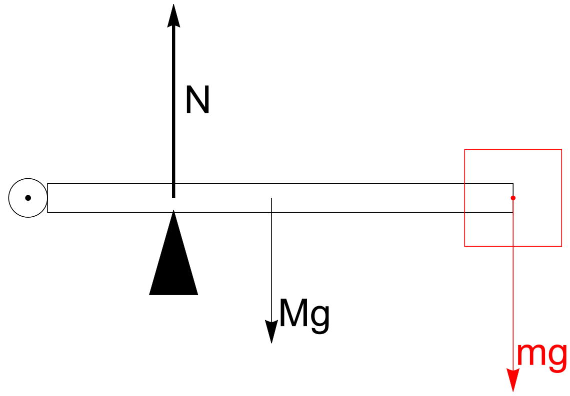


FIG. 35: Horizontal beam of mass M and length L with additional mass m at the end. The rod is pivoted at the left end, and there is a support at a distance $d < L/2$. Find the force N.

Consider torques about the pivot

$$N \cdot d = Mg\frac{L}{2} + mg \cdot L\,,\; N = \ldots$$

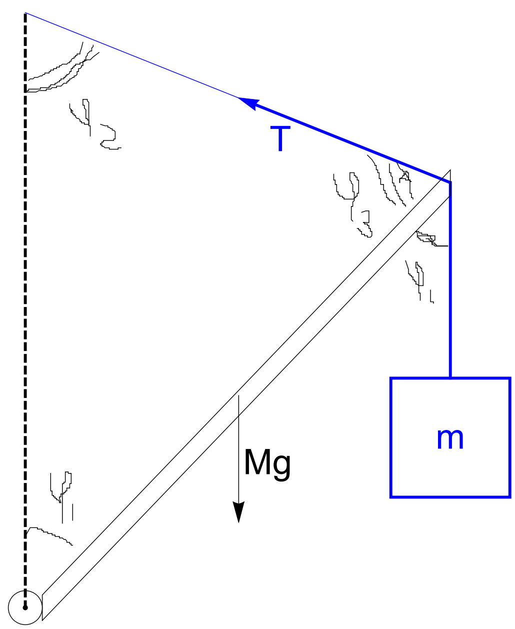
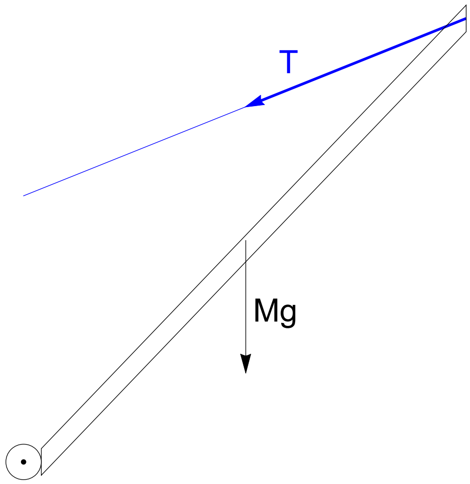


FIG. 36: A beam of mass M and length L with additional mass m at the end. The rod is pivoted at the left end and makes an angle ϕ_1 with vertical. There is a supporting cord which makes an angle ϕ_2 with vertical. Find the tension T.

$$TL\sin\phi_3 = Mg\frac{L}{2}\sin\phi_1 + mgL\sin\phi_1\,,\; \phi_3 = 180^o - \phi_1 - \phi_2$$

A beam with $M = 100\,kg$ makes an angle 45^o with horizontal, and is supported by a cable which makes 30^o with the beam. Find T.



$$TL\sin 30^o = Mg \cdot \frac{L}{2}\sin 45^o$$

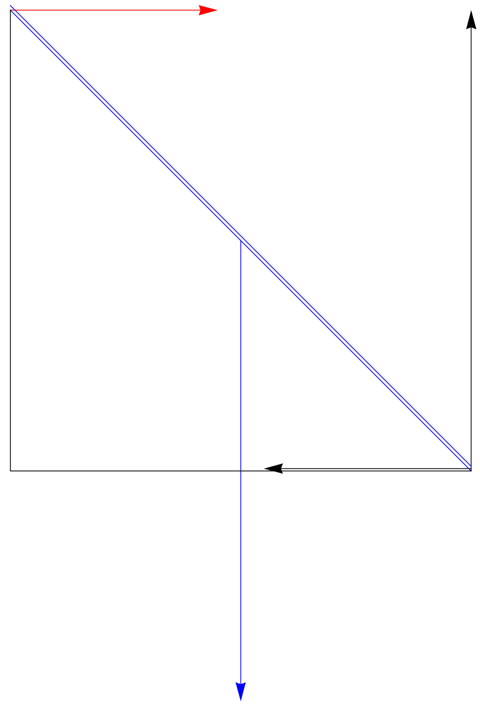


FIG. 37: Blue ladder of mass M and length L making angle θ with horizontal. Forces: $M\vec{g}$ (blue), wall reaction $\vec{F}$ (red), floor reaction $\vec{N}$ (vertical), friction $\vec{f}$ (horizontal).

Torques about the upper point:

$$NL\sin\left(\frac{\pi}{2}-\theta\right) - fL\sin\theta - Mg\frac{L}{2}\sin\left(\frac{\pi}{2}-\theta\right) = 0$$

But from force equilibrium (vertical) $N = Mg$ and (on the verge) $f = \mu N$. Thus,

$$\frac{1}{2}\cos\theta - \mu\sin\theta = 0$$

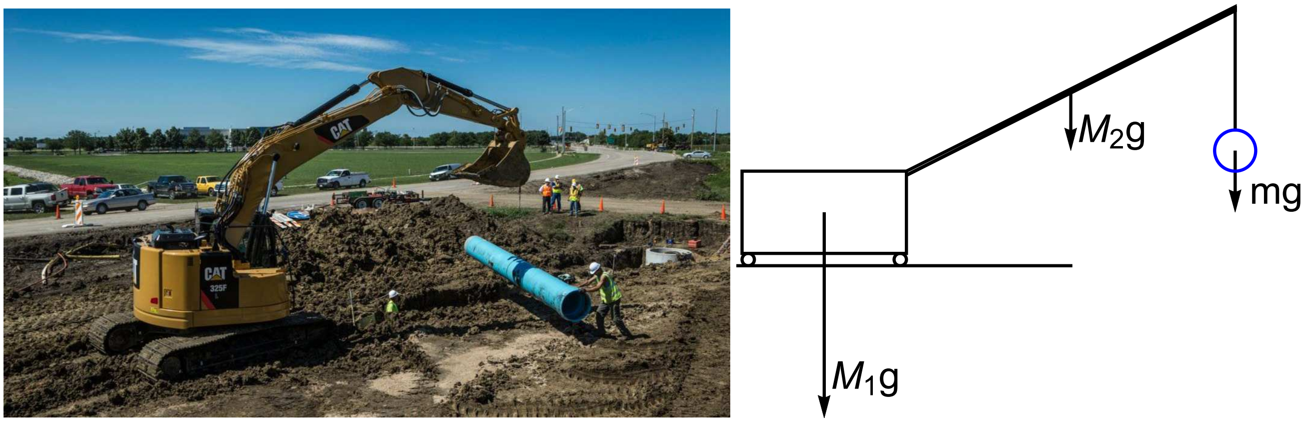


FIG. 38: An excavator of mass $M_1 = 50\,ton$ and length $l = 5\,m$ uses an extension arm with $M_2 = 1\,ton$ and $L = 10\,m$ to lift a load of mass m. The arm makes an angle θ with horizontal. Estimate the maximum safe m. (One ton is $1000\,kg$)

$$M_1 g \times \frac{l}{2} = M_2 g \times \frac{L}{2} \times \sin(90^o - \theta) + mgL\sin(90^o - \theta)$$
$$m = M_1 \frac{l}{2L} / \sin(90^o - \theta) - M_2 \frac{1}{2}$$

for $\theta = 0$

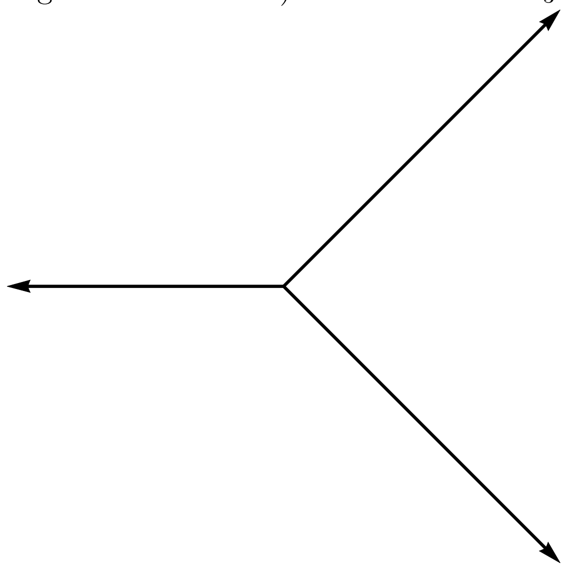
$$m = M_1 \frac{l}{2L} - M_2 \frac{1}{2} \simeq 50 \frac{5}{20} - \frac{1}{2} = 12\,ton$$

Equivalently, if right edge is taken as origin, for $\theta = 0$

$$X_{CM} = \frac{1}{M_1 + M_2 + m}\left(-\frac{l}{2}M_1 + \frac{L}{2}M_2 + Lm\right)$$

From $X_{CM} = 0$ we get the same result for m.

The captain of a large cargo barge needs to transport three pieces of heavy equipment with $M_1 = M_2 = 20\,ton$ and $M_3 = 40\,ton$. The masses M_1 and M_2 are placed at $\vec{r}_1 = 10\hat{\imath} + 10\hat{\jmath}$, $\vec{r}_2 = 10\hat{\imath} - 10\hat{\jmath}$ (in meters) with respect to the center of the barge (taken as the origin of coordinates). Where should M_3 be placed so that the barge remains balanced?



In order to balance the barge, the horizontal position of the CM should coincide with the center of the barge . Thus,

$$M_1\vec{r}_1 + M_2\vec{r}_2 + M_3\vec{r}_3 = 0 \Rightarrow \vec{r}_3 = -\frac{1}{M_3}\left(M_1\vec{r}_1 + M_2\vec{r}_2\right) =$$

$$= -\frac{1}{40} 20 \times 10 \times (\hat{\imath} + \hat{\jmath} + \hat{\imath} - \hat{\jmath}) = -10\hat{\imath}$$

XXI. FLUIDS

A. hydrostatics

1. Density

$$\rho = \frac{m}{V}\,,\ [\rho] = kg/m^3 \tag{133}$$

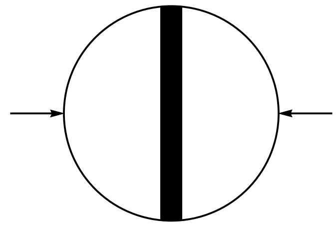
E.g. for water

$$\rho \simeq 1000\,kg/m^3 = 1\,g/cm^3$$

Example. Recently, a thief stole $1,000,000$ \$ worth of gold from an armour truck in NYC. Estimate the volume, if gold is worth about \$25 per gram, with density about 20 gram per cm^3.

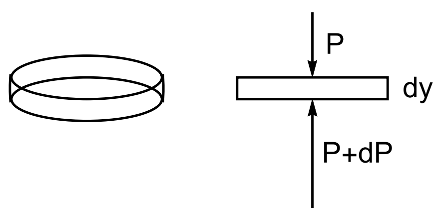
$$m = 10^6/25 = 4\cdot 10^4\,\mathrm{gram}\,;\ V = \frac{m}{\rho} = 2\cdot 10^3\,\mathrm{cm}^3 = 2\cdot 10^{-3}\,\mathrm{m}^3$$

$$\boxed{P = \frac{F_n}{A}},\ \text{Units: } [P] = Pa = \frac{N}{m^2}\,,\ 1\,atm \simeq 10^5\,Pa \simeq 760\,mm\,\text{Hg} \quad (134)$$



The Magdeburg hemispheres. When the rims were sealed and the air was pumped out, the sphere contained a vacuum and could not be pulled apart by teams of horses.

"Barostatic equation". Consider a cylindrical column of fluid and identify a thin layer (disk) with thickness $-dy$, area A and mass $dM = -\rho A\, dy$.



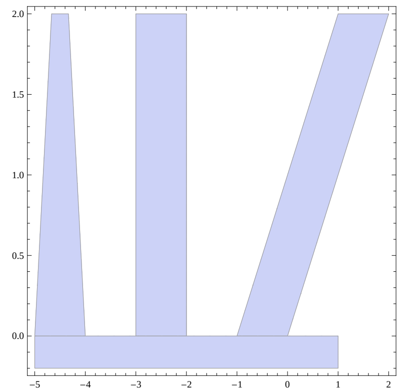
The layer is in equilibrium since the weight $dM\, g$ is balanced by an upward force due to difference in pressure:

$$F_{down} - F_{up} = A\, dp \;\Rightarrow\; A\, dp = -\rho A\, dy\, g$$

$$dP = -\rho g\, dy\,, \quad \text{for } \rho = const \text{ ("incompressible")} \; \boxed{|\Delta P| = \rho g y}$$

Examples.(a) Find the hight of a column of water with $\Delta P = 1\,atm$. (b) same for mercury with $\rho_{Hg} = 13.6\rho_{H_2O}$.

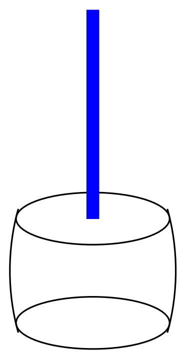
$$\Delta P = \rho g h \;\Rightarrow\; h = \frac{\Delta P}{\rho g},\ h_{H2O} = \frac{10^5}{10^3 \cdot 9.8} \simeq 10\,m\,,\ h_{Hg} = \frac{10^5}{13.6 \cdot 10^3 \cdot 9.8} \simeq 0.76\,m$$

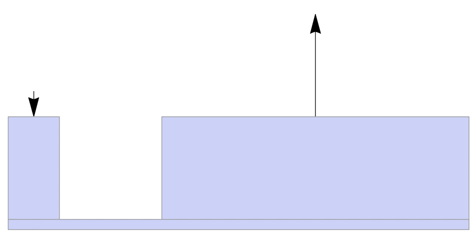


Pressure under any of the tubes is the same and is determined only by the level of the fluid.

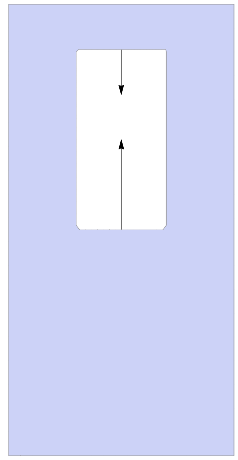
Pascal's principle: extra pressure δP is transferred to every point of the fluid

$$\delta P = const$$





Hydrolic lift (pistons not shown). Extra pressure $\delta P = f/A_1$ (due to a small force f applied to the narrow column) is transferred to the wider column, creating a bigger force $F = A_2 \delta P$.



The force on the upper surface is PA; the force on the lower is $(P + \rho g \Delta h)A$ (where ρ is the density of fluid!). The resultant

$$F_A = \rho g \Delta V$$

where ΔV is the "displaced volume".

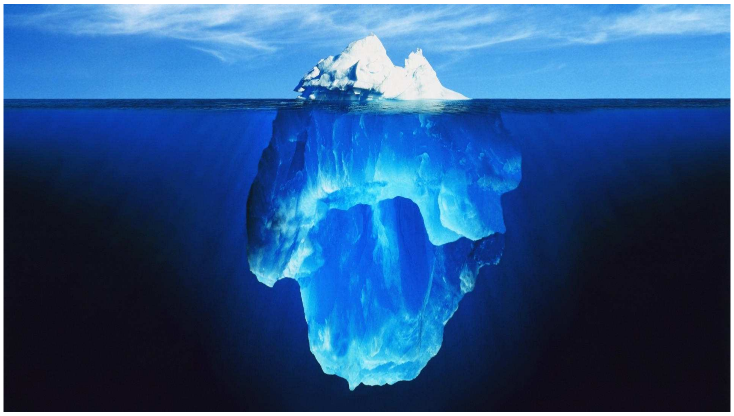
Example: find the density of a body ("crown") from its weight in air $\simeq mg = \rho_c V g$ and "weight" in fluid W.

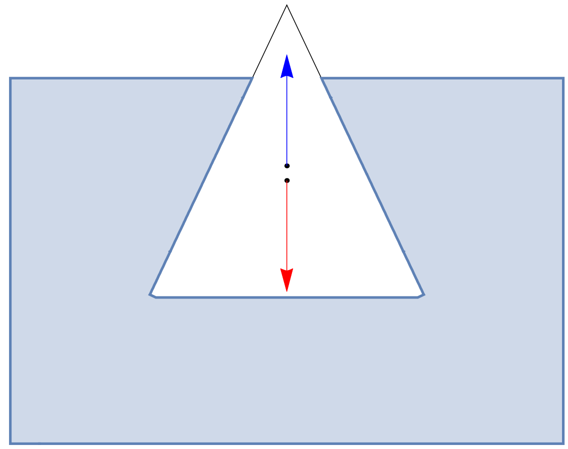
$$W = mg - F_A = mg\left(1 - \frac{F_A}{mg}\right)$$

$$mg = \rho_c V g\ ,\ F_A = \rho V g \ \Rightarrow \ \frac{W}{mg} = 1 - \frac{\rho}{\rho_c}$$

$$\rho_c = \rho \frac{mg}{mg - W}$$

Example: Iceberg, $\rho_{ice} < \rho$, V - volume of the iceberg, v - volume above water

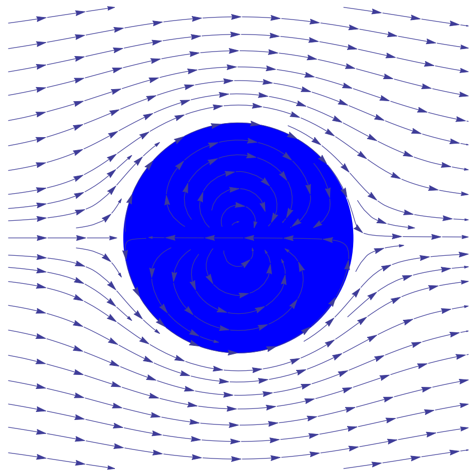




$$F_A = \rho(V - v)g \;=\; Mg = \rho_{ice} V g$$

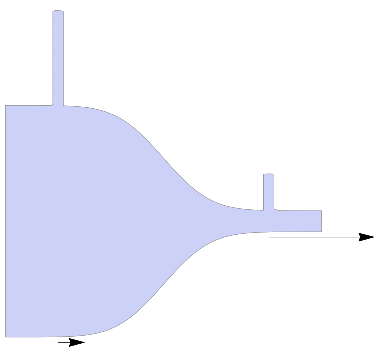
$$1 - \frac{v}{V} = \rho_{ice}/\rho\,,\; \frac{v}{V} = \frac{\rho - \rho_{ice}}{\rho}$$

B. Hydrodynamics



Laminar flow of ideal fluid (non-viscous and incompressible) around a cylinder

1. *Continuity equation and the Bernoulli principle*



$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 \ , \quad \rho \neq const \tag{135}$$

$$v_1 A_1 = v_2 A_2 \ , \quad \rho = const \tag{136}$$

Example. Crude oil is pumped through a pipe of radius $R = 0.1\,m$. The speed of the flow is $v = 2\,\mathrm{m/s}$. What is the volume flow speed, in L/s ?

$$\frac{dV}{dt} = \pi R^2 v = \pi \times 10^{-2} \times 2 \times 10^3 \, \frac{L}{s}$$

Example. A doctor finds a partial blockade in a vessel. The cross-sectional area available for blood flow near the blockage is $A_b = 0.1\,mm^2$. The flow rate dV/dt in the region of the partial blockage is $2500\,ml/hour$. In a nearby vessel that is clear, the area is $A_c = 0.5\,mm^2$. What is the blood flow rate, and what is the velocity v_c in m/s in the clear region?

$$dV/dt = 2500\,ml/h \ - \ \text{same!}$$

$$v_c = \frac{dV/dt}{A_c} = 2500 \frac{10^{-6}\,m^3}{3600\,s \times 0.5 \cdot 10^{-6}\,m^2} = \ldots$$

Bernoulli. Consider element of fluid bounded by flowlines and moving right with $m = \rho V$ and small volume $V = A\Delta x$. The energy of the element when it moves Δx is changed due to work by pressure difference

$$W = -\Delta P \cdot A\Delta x = -\Delta P V \tag{137}$$

(ignore gravity). The change in kinetic energy of the element

$$\Delta K = \Delta\left(\frac{1}{2}mv^2\right) = \frac{1}{2}V\Delta(\rho v^2) \tag{138}$$

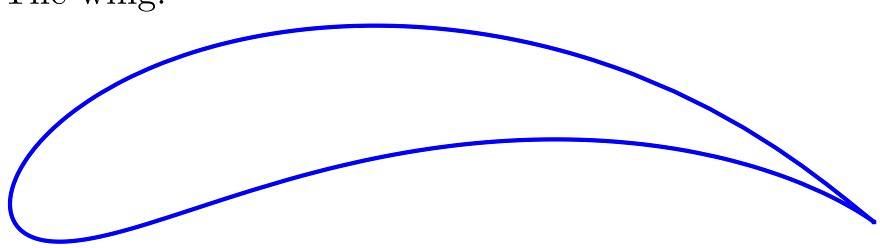
(note: $V = const$ (!)). From

$$W = \Delta K\,,\ \Delta P + \Delta\left(\frac{1}{2}\rho v^2\right) = 0$$

$$P + \frac{1}{2}\rho v^2 = const\,,\ (no\ gravity) \tag{139}$$

$$P + \rho g h + \frac{1}{2}\rho v^2 = const\,,\ (with\ gravity) \tag{140}$$

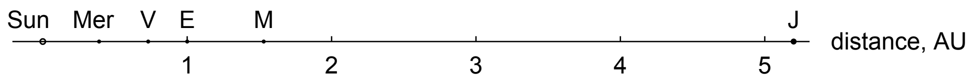
The wing:



XXII. GRAVITATION

A. Solar system

$1\,AU \simeq 150 \cdot 10^6\,km$, about the distance between Earth and Sun. Solar rad. - $0.005\,AU$. Distances: Mer - $0.39\,AU$, V - $0.73\,AU$, M - $1.53\,AU$, J - $5.2\,AU$, ...



Note how empty it is ...

B. Kepler's Laws

1. *Intro: Ellipse, etc.*

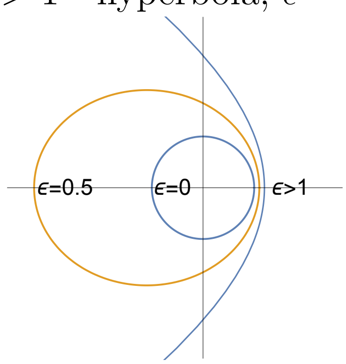
$$\text{Cartesian: } x^2/a^2 + y^2/b^2 = 1 \tag{141}$$

a - "semi-major axis", b - "semi-minor axis"; if $b = a$ - circle

$$\text{Polar: } x = r\cos\phi\,,\ y = r\sin\phi \tag{142}$$

$$r(\phi) = a(1-\epsilon^2)/(1+\epsilon\cos\phi) \tag{143}$$

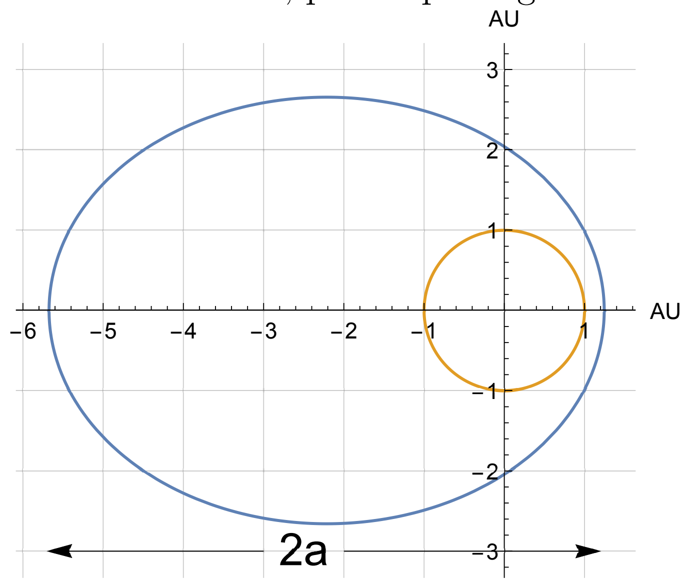
$\epsilon > 1$ - hyperbola, $\epsilon = 1$ - parabola ($a \to \infty$).



2. 1st law

Path of a planet - ellipse, with Sun in the focus.

(Justification is hard, and requires energy, momentum and angular momentum conservation, plus explicit gravitational force - Newton's law)



3. 2nd law

Sectorial areas during the same time intervals are same - see figure.

(Justification is easy, and requires ONLY angular momentum conservation - in class)

$$dS = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{1}{2m}L \cdot dt = const \cdot dt$$

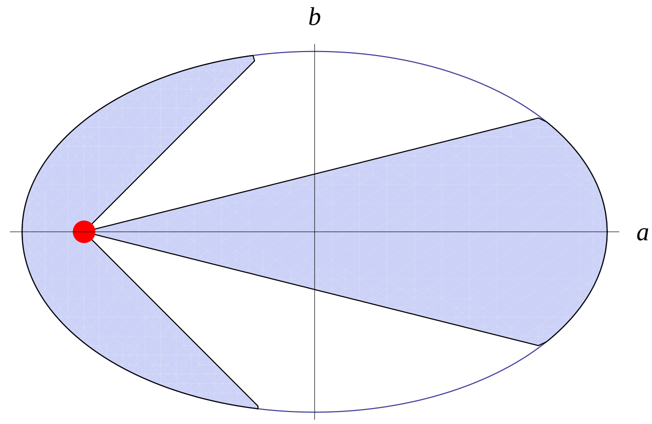


FIG. 39: The ellipse and Kepler's Law's:

a and b - semi-major and semi-minor axes; $f = \pm\sqrt{a^2 - b^2}$ - foci, $\epsilon = |f|/a$

1st law: path of a planet - ellipse; Sun in the focus (the other focus is empty!)

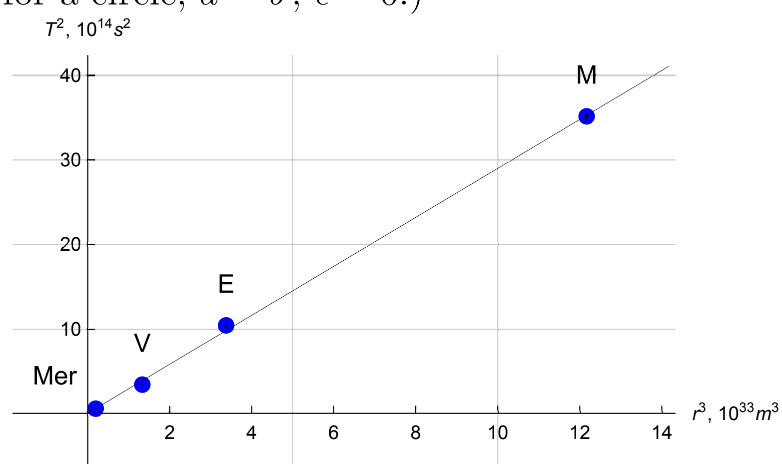
2nd law: sectorial speed is constant

3rd law: $T^2 \propto a^3$ (and b does not matter!)

4. 3rd law

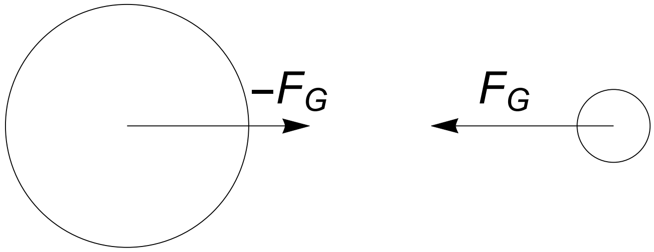
$$T^2 \propto a^3 \text{ , and } b \text{ does not matter!!!} \tag{144}$$

(Justification is hard, and requires energy, momentum and angular momentum conservation, plus explicit gravitational force; will derive explicitly only for a circle, $a = b$, $\epsilon = 0$.)



C. The Law of Gravitation

1. Force



$$\boxed{F = -G\frac{Mm}{r^2}}\,,\ G \approx 6.67 \cdot 10^{-11}\, N \cdot m^2/kg^2 \tag{145}$$

or in vector form

$$\vec{F} = -G\frac{Mm\vec{r}}{r^3} \tag{146}$$

Note: the Law is valid not only for point masses but also for spheres, r being the distance between centers (proof in phys121). Since Sun and planets are near-spherical, the Law is *very* accurate.

Example. In an attempt to replicate the Cavendish experiment, a 4- and 10-kg lead balls are placed at a distance 10 cm between their centers. Find the magnitude of forces which act on each of the balls.

$$F = G\frac{Mm}{r^2} = 6.67 * 10^{-11} * \frac{4 * 10}{0.1^2} \approx 2.7 * 10^{-7} \text{ (N)}$$

The forces on each of the balls are the same in magnitude (3rd Law of Newton !).

$$g = F/m = -G\frac{M}{r^2} \;-\; \text{"strength of gravitational field", } M \text{ - mass of the central body} \tag{147}$$

$$g_s = -G\frac{M}{R^2} \;-\; \text{acceleration on the surface, } R\text{-radius of the central body}$$

$$g(r) = g_s\left(\frac{R}{r}\right)^2$$

In vector form

$$\vec{g} = -G\frac{M\vec{r}}{r^3} = -\hat{r}g_s\left(\frac{R}{r}\right)^2 \;,\;\; \hat{r} = \frac{\vec{r}}{r} \tag{148}$$

Example. Find the mass of Earth (*Cavendish*).

$$M_E = |g_s|\frac{R_E^2}{G} \approx 9.8\frac{\left(6400\cdot 10^3\right)^2}{6.7\cdot 10^{-11}} \sim 6\cdot 10^{24}\,kg$$

Example. Find the strength of gravitational field of Earth at a distance $r = 5R_E$ from the center.

$$g = g_s\left(\frac{R_E}{r}\right)^2 \simeq g_s\frac{1}{25} \approx 0.4\,\frac{m}{s^2}$$

Example. Find g_V if $R_V \approx 0.4\,R_E$ and $M_V \approx 0.056\,M_E$

$$g \propto \frac{M}{R^2} \text{ thus } \frac{g_V}{g_E} = \frac{M_V}{M_E}\left(\frac{R_E}{R_V}\right)^2 \approx 0.4$$

M_S- mass of Sun, m_P- mass of planet, r-distance between them; v - orbital speed of the planet. Force of gravity is the centripetal force, thus

$$G\frac{M_S m_P}{r^2} = m_P\frac{v^2}{r}$$

$$v = \sqrt{GM_S/r} \tag{149}$$

$$T = 2\pi r/v = 2\pi r^{3/2}/\sqrt{GM_S}$$

$$T^2 = \left(4\pi^2/GM_S\right) r^3\,, \text{ with } M_S \simeq 1.99 \cdot 10^{30}\,kg \tag{150}$$

the 3rd law.

Example. Find v for earth around sun, and check if $T = 1\,year$.

$$v = \sqrt{6.7 \cdot 10^{-11}\, 2 \cdot 10^{30}/(150 \cdot 10^{9})} \approx 30\,\frac{km}{s}$$

$$T = \frac{2\pi r}{v} \approx 3 \cdot 10^7\, s \approx 1\,year$$

Example. Calculate the period of revolution around the Sum of an asteroid with radius of orbit $r_A = 600 \cdot 10^6\,km$ (assume a circular orbit).

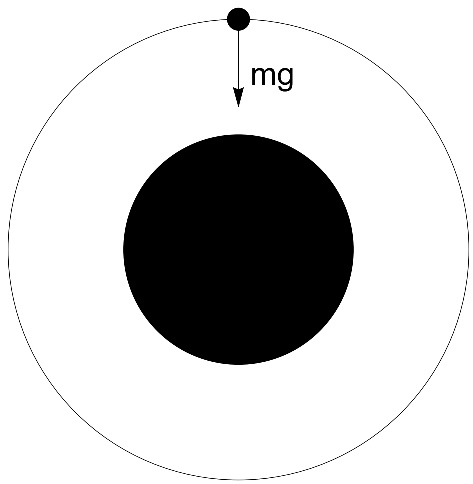
Solution. From third law of Kepler

$$T_A = \kappa r_A^{3/2}\,,\ \kappa = \frac{2\pi}{\sqrt{GM_s}}$$

The actual value of the coefficient κ is not needed if we note that for Earth

$$T_E = \kappa r_E^{3/2}\,, \text{ (with } r_E = 150 \cdot 10^6\,km \text{ and same } \kappa) \ \Rightarrow$$

$$\frac{T_A}{T_E} = \left(\frac{r_A}{r_E}\right)^{3/2} = \left(\frac{600}{150}\right)^{3/2} = 4^{3/2} = 8$$



$$\frac{v^2}{r} = g(r) = GM/r^2$$

$$v = \sqrt{GM/r}\,,\ T = \frac{2\pi r}{v} = 2\pi r^{3/2}/\sqrt{GM} \tag{151}$$

$$T^2 = \left(4\pi^2/GM\right) r^3 \tag{152}$$

which is similar to the 3rd law of Kepler, but M is the mass of the central body. (r is distance from *center* !). This is an accurate way to determine mass M of a planet *if* it has satellites.

Orbital speed:

$$v^2/r = g \;\Rightarrow\; v = \sqrt{gr}$$

$$\text{low orbit: } v \simeq \sqrt{g_s R} \approx 8\,\frac{km}{s} \text{ for Earth} \tag{153}$$

Example. A satellite has a speed of 4.0 km/s. Find its distance from the center of Earth; assume a circular orbit and use $M_{earth} \simeq 6.0 * 10^{24}$ kg.

(a): $v_{orb} = \sqrt{\dfrac{GM}{r}}$ thus $r = \dfrac{GM_{earth}}{v_{orb}^2} = \dfrac{6.67 * 10^{-11} * 6.0 * 10^{24}}{(4.0 * 10^3)^2} \approx 2.5{*}10^7\,\text{m} \approx 3.9{*}R_{earth}$

or (quick estimation): $\dfrac{v_{orb}}{8\,\text{km/s}} = \sqrt{\dfrac{R_{earth}}{r}}$ and $\dfrac{r}{R_{earth}} = \left(\dfrac{8\,\text{km/s}}{v_{orb}}\right)^2 \approx 4$

Example. Many asteroids are loose collections of rocks, held together only by gravity. Assuming an average density of rock $5 \cdot 10^3$ kg/m^3 and a near-spherical shape, estimate the maximum revolution frequency (minimal revolution period T) for such an asteroid.

Solution. The maximum possible revolution frequency is when the speed of outside rocks coincides with orbital speed (for faster speeds gravity is insufficient and the rocks will start flying off). If M is the mass of the asteroid and R its radius

$$\omega_{\max}^2 R = g_s = \frac{GM}{R^2} = \frac{G\frac{4}{3}\pi R^3 \rho}{R^2} = \frac{4}{3}\pi\rho G R \;\Rightarrow$$

$$\omega_{\max} = \sqrt{\frac{4}{3}\pi\rho G} = \sqrt{\frac{4}{3}\pi \times 5000 \times 6.7 \cdot 10^{-11}} \sim 10^{-3}\,\text{s}^{-1}\,, \;\Rightarrow$$

$$T_{\min} = \frac{2\pi}{\omega_{\max}} \sim 1.5\,\text{h}$$

Note: the actual size of the asteroid does not matter!

D. Energy

$$U = \int_r^\infty F(r')dr' = -GMm \int_r^\infty \frac{dr'}{(r')^2} \tag{154}$$

$$\tag{155}$$

$$\boxed{U = -G\frac{Mm}{r}} \tag{156}$$

Small elevations: $r = R + h$ with $h \ll R$:

$$\frac{1}{R+h} \simeq \frac{1}{R} - \frac{h}{R^2}$$

Since $|g_s| = GM/R^2$:

$$U \simeq \text{const} + m|g_s|h$$

Probe m in combined field of Sun and planet

$$U = -G\frac{M_S m}{r_{pS}} - G\frac{M_E m}{r_{pE}} \tag{157}$$

1. *Escape velocity and Black Holes*

$$E = K + U$$

$U(\infty) = 0$, $K(\infty) = E > 0$ for escape. Thus

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} \geq 0\,,\ v_{esc}^2/2 = GM/R$$

$$v_{esc} = \sqrt{2GM/R} = \sqrt{2g_sR} \approx 11\,km/s$$

for the surface of Earth.

Example. A missile m is launched vertically up from the surface with $v_0 = \frac{1}{2}v_{esc}$. Find h

$$E = \frac{1}{2}mv_0^2 + \left(-G\frac{Mm}{R}\right) = \frac{1}{2}m(v_0^2 - v_{esc}^2)$$

$$E = -G\frac{Mm}{R+h} = -G\frac{Mm}{R}\frac{R}{R+h} = -\frac{1}{2}mv_{esc}^2\frac{R}{R+h}$$

$$v_{esc}^2\frac{R}{R+h} = v_{esc}^2 - v_0^2 = \frac{3}{4}v_{esc}^2$$

$$\frac{R}{R+h} = \frac{3}{4}\,,\ 1 + \frac{h}{R} = \frac{4}{3}\,,\ h = R/3$$

Example. A missile m is launched vertically up from the surface with $v_0 = 2v_{esc}$. Find v_∞

$$E_I = \frac{1}{2}mv_0^2 + \left(-G\frac{Mm}{R}\right)$$

$$E_f = \frac{1}{2}mv_\infty^2 + 0\,,\ E_f = E_i \ \Rightarrow$$

$$v_\infty^2 = v_0^2 - \frac{2GM}{R} = v_0^2 - v_{esc}^2$$

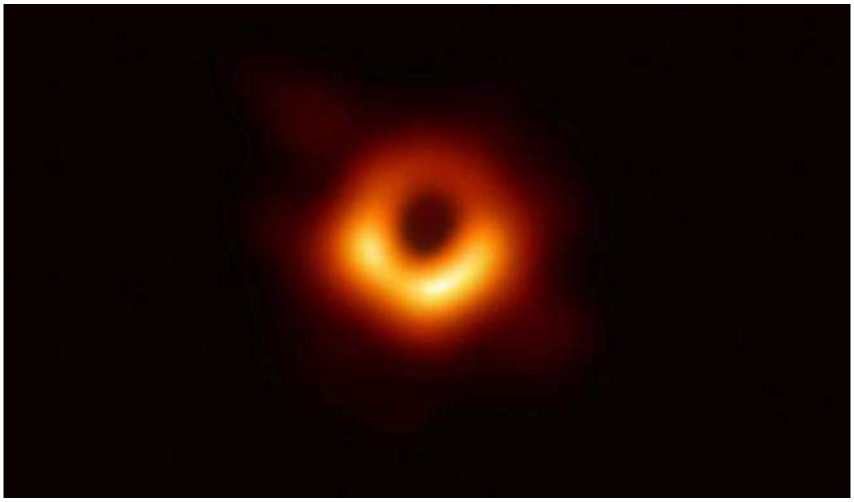
Black Hole

$$v_{esc} = \sqrt{2GM/R} = c \simeq 3 \cdot 10^5\, km/s \ \text{ (speed of light)}$$

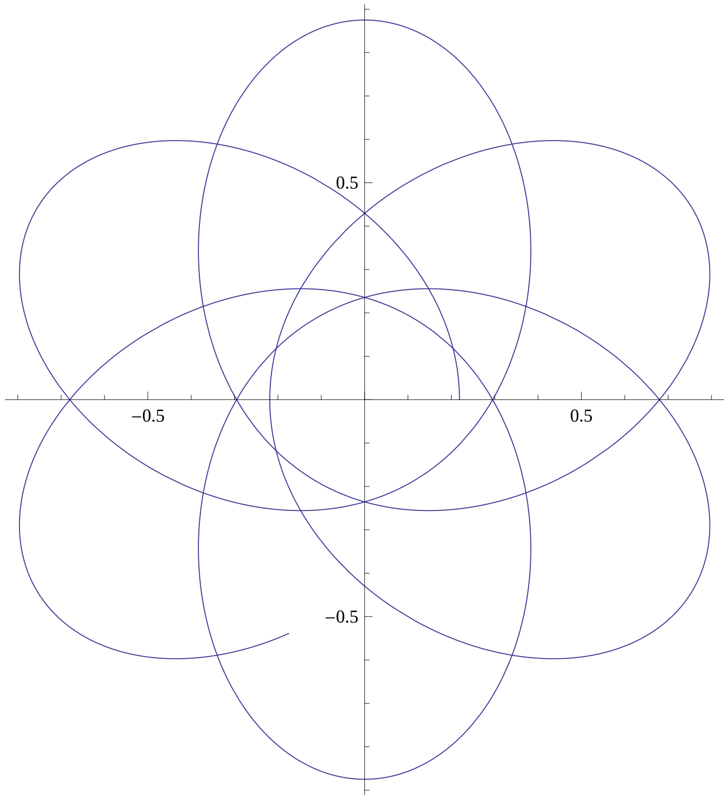
Radius:

$$R_B = \frac{2GM}{c^2}$$

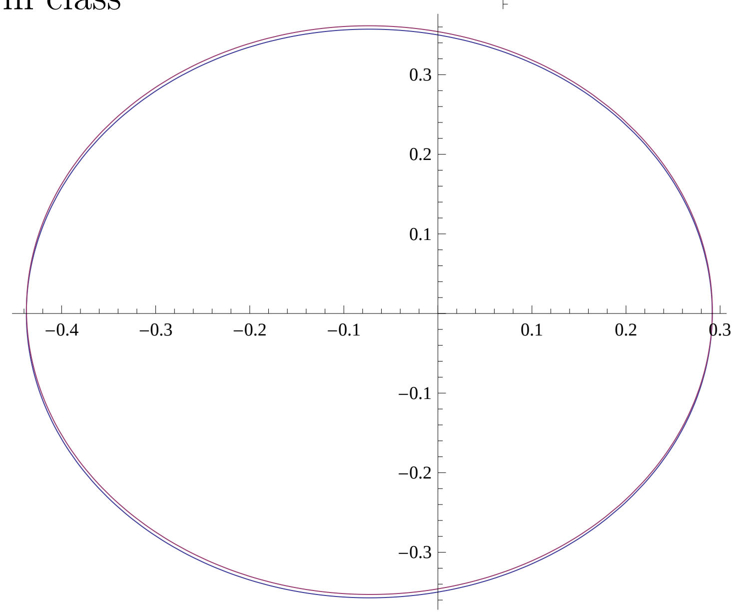
$$\text{For Sun: } R_B = \frac{2 \times 6.7 \cdot 10^{-11} \times 2 \cdot 10^{30}}{9 \cdot 10^{16}} \approx 3\, km$$



E. Advanced: Deviations from Kepler's and Newton's laws



in class



XXIII. OSCILLATIONS

A. Introduction: Math

1. $\sin(x)$, $\cos(x)$ *for small* x

$$\sin x \simeq x \tag{158}$$

error is about $-x^3/6$ and can be neglected for $x \ll 1$ [we will need this for a pendulum]

$$\cos x \simeq 1 - \frac{x^2}{2} \tag{159}$$

error is tiny for small x, about $x^4/24$.

2. *Differential equation* $\ddot{x} + x = 0$

The equation

$$\ddot{x}(t) + x(t) = 0 \tag{160}$$

has a general solution

$$x(t) = B\cos t + C\sin t$$

with arbitrary constants B, C. Can be checked by direct verification (note that $\ddot{x} = -x$). The values of B, C are determined by *initial conditions* $x(0)$ and $\dot{x}(0)$. Alternatively, one can combine sin and cos:

$$x(t) = A\cos(t + \phi)$$

with two constants $A = \sqrt{B^2 + C^2}$, and ϕ.

The equation

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{161}$$

is reduced to the above by replacing t with ωt. Thus,

$$x(t) = B\cos(\omega t) + C\sin(\omega t) \tag{162}$$

with

$$B = x(0)\,,\; C = \dot{x}(0)/\omega \tag{163}$$

Or,

$$x(t) = A\cos\left(\omega t + \phi\right)\,,\; A = \sqrt{B^2 + C^2} \tag{164}$$

with A known as the *amplitude* and ϕ the initial *phase*.

B. Spring pendulum

Hook's law:

$$F = -kx \tag{165}$$

and 2nd Newton's law

$$F = m\ddot{x} \tag{166}$$

give eq. (161) with

$$\omega = \sqrt{\frac{k}{m}} \tag{167}$$

(in radians per second). The oscillation frequency

$$f = \omega/2\pi \tag{168}$$

with units $1/s$, or Hz. Period of oscillations

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{169}$$

1. Energy

Kinetic:

$$K(t) = \frac{1}{2}m\left[\dot{x}(t)\right]^2$$

Potential:

$$U(t) = \frac{1}{2}kx(t)^2$$

Total:

$$E = K + U = const$$

C. Simple pendulum

Restoring force:

$$F = -mg\sin\theta \simeq -mg\theta$$

Tangential acceleration:

$$a = L\ddot{\theta}$$

Thus

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{170}$$

exactly like eq. (161). Thus, the same solution with $x \to \theta$ and

$$\omega^2 = \frac{g}{L} \tag{171}$$

or

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{172}$$

D. Physical pendulum

Rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

If l - distance from CM:

$$\tau = -mgl\sin\theta \approx -mgl\theta$$

for small amplitudes.

Thus,

$$\ddot{\theta} + \theta\frac{mgl}{I} = 0 \tag{173}$$

which is the same differential equation as before with $x(t) \quad \rightarrow \quad \theta(t)$. Thus, the same trigonometric solution with

$$\omega^2 = \frac{mgl}{I}\,,\; T = 2\pi\sqrt{\frac{I}{mgl}} \tag{174}$$

Example: uniform rod of length L, pivoted at a distance l from the center.

Solution: from the rotational inertia of a rod about the CM, $I_0 = \frac{1}{12}ML^2$ and the parallel axis theorem

$$I = I_0 + Ml^2 = M\left(L^2/12 + l^2\right)$$

Thus,

$$T = 2\pi\sqrt{\frac{I}{Mgl}} = 2\pi\sqrt{\frac{L^2/12 + l^2}{gl}}$$

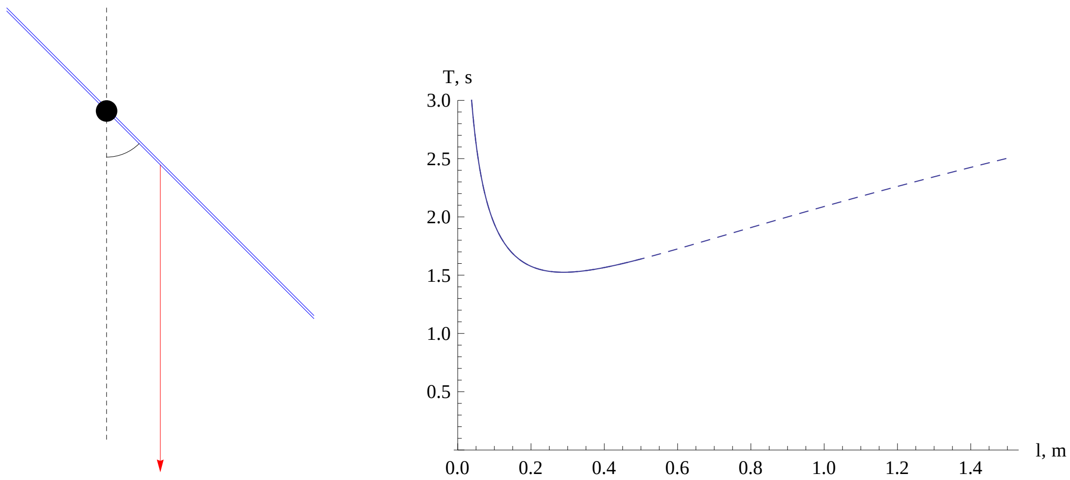


FIG. 40: Physical pendulum of mass M and length L making angle θ with vertical. Forces: $M\vec{g}$ (red), with torque $\tau = -Mgl\sin\theta$. Right: period of small oscillations T for $L = 1\,m$ as a function of the off-center distance l, with a minimum at $l \approx 29\,cm$; dashed line corresponds to the pivot outside of the rod (on a massless extension).

E. Torsional pendulum

Consider

$$\tau = -\kappa\theta$$

which is a torsional "Hook's law". Then, from rotational 2d law:

$$I\alpha \equiv I\ddot{\theta} = \tau$$

and

$$\ddot{\theta} + \theta\frac{\kappa}{I} = 0 \tag{175}$$

Thus,

$$\omega^2 = \frac{\kappa}{I}\,,\ T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{176}$$

F. Why are small oscillations so universal?

in class

G. Resonance

Add external driving to a spring pendulum:

$$F = -kx + F_0 \cos(\omega_d t)$$

Then

$$m\ddot{x} + kx = F_0 \cos(\omega_d t) \tag{177}$$

Look for a solution

$$x(t) = A \cos(\omega_d t)$$

where the amplitude A has to be found. Using

$$\ddot{x}(t) = -\omega_d^2 x(t)$$

one obtains

$$A\left(-m\omega_d^2 + k\right) = F_0$$

or, with $\omega_0 = \sqrt{k/m}$, the natural frequency

$$|A| = \frac{F_0/m}{|\omega_0^2 - \omega_d^2|} \tag{178}$$

Note INFINITY when $\omega_d = \omega_0$. This is the resonance!

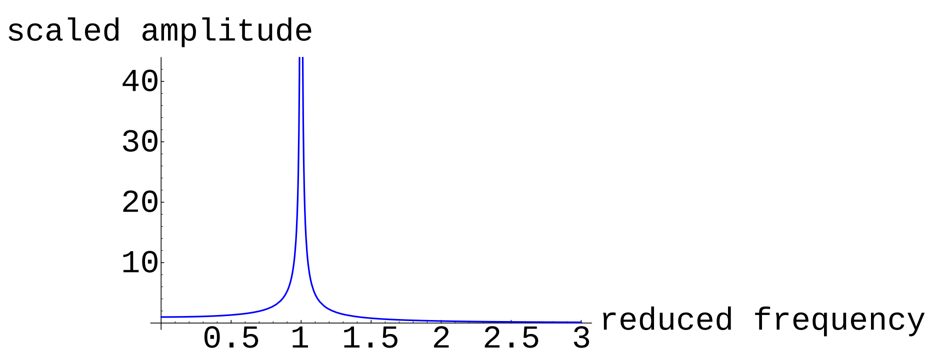


FIG. 41: Resonance. When the driving frequency ω_d is close to the natural frequency $\omega = \sqrt{k/m}$ there is an enormous increase in the amplitude.