In[ ]:= (*Gauss*)

(*we first define electric field of a point charge,
$\overrightarrow{E=k_e q}\ \vec{r}/r^3$*)

In[30]:=
```
efield[rvec_, q_] := ke q rvec / Norm[rvec] ^3
```

In[31]:=
```
rvec := {x, y, z}
```

In[ ]:=

In[ ]:=

In[32]:=
```
efield[rvec, q]
```

Out[32]=

$$\left\{\frac{\text{ke q x}}{\left(\text{Abs}[x]^2+\text{Abs}[y]^2+\text{Abs}[z]^2\right)^{3/2}}, \frac{\text{ke q y}}{\left(\text{Abs}[x]^2+\text{Abs}[y]^2+\text{Abs}[z]^2\right)^{3/2}}, \frac{\text{ke q z}}{\left(\text{Abs}[x]^2+\text{Abs}[y]^2+\text{Abs}[z]^2\right)^{3/2}}\right\}$$

In[34]:=
```
Clear[q];
ef = 1 / (ke q)
  FullSimplify[efield[rvec, q], Assumptions → {x ∈ Reals, y ∈ Reals, , z ∈ Reals}]
```

Out[34]=

$$\left\{\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}$$

(*which is just $\vec{r}/r^3$; thus the dimensionless Gauss integral is expected to be
 4Pi *)

In[36]:=
```
efx05 = ef /. x → x - 0.5
```

Out[36]=

$$\left\{\frac{-0.5+x}{\left((-0.5+x)^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left((-0.5+x)^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left((-0.5+x)^2+y^2+z^2\right)^{3/2}}\right\}$$

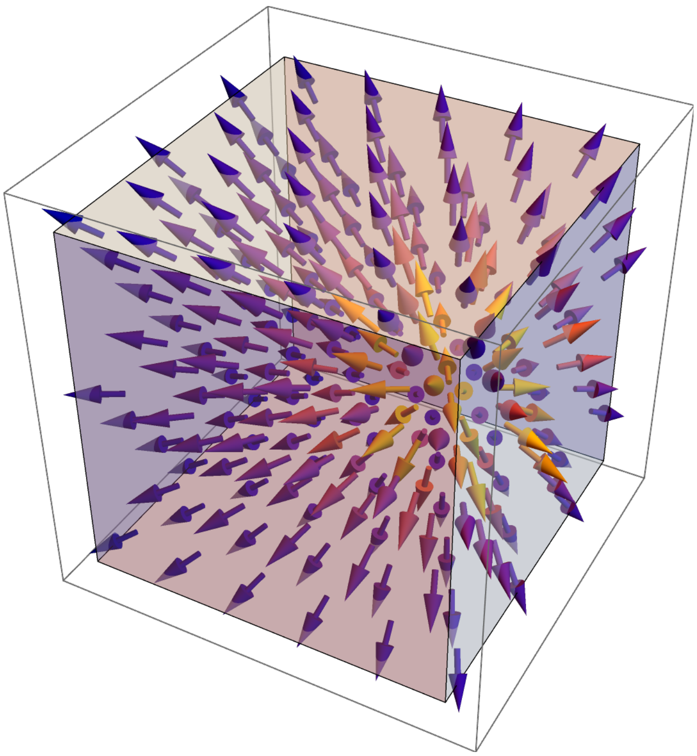
In[ ]:=

In[ ]:=

In[ ]:=

In[38]:=
```
Show[Graphics3D[{{Opacity[.4], Cube[2]}}],
 VectorPlot3D[efx05, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]]
```

Out[38]=

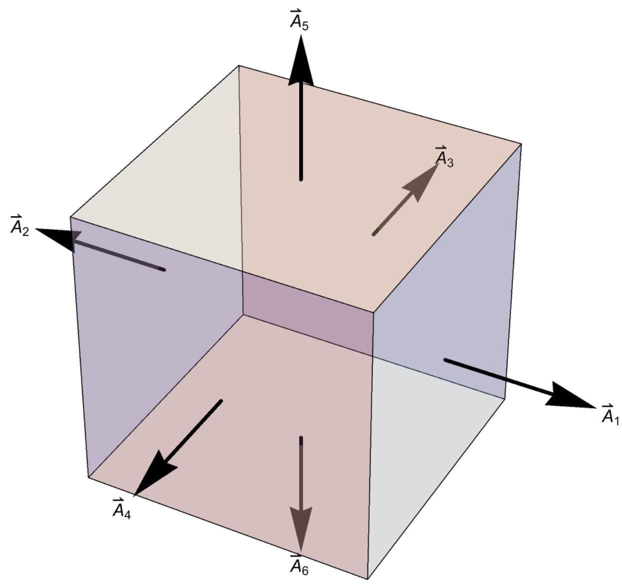


In[ ]:=

```
Clear[A1, A2, A3, A4, A5, A6];(*vector areas; for a unit cube coincide with normals*)
A1 = {1, 0, 0};(*right face*)
A2 = -A1;(*left face*)
A3 = {0, 1, 0};(*back face*)
A4 = -A3 ;(*front face*)
A5 = {0, 0, 1};(*top face*)
A6 = -A5;(*bottom face*)
```

In[41]:=
```
Graphics3D[{{Opacity[.3], Cube[2]}, Thick, Arrow[{A1, 2 A1}],
  Text["A⃗1", 2.1 A1], Arrow[{A2, 2 A2}], Text["A⃗2", 2.1 A2], Arrow[{A3, 2 A3}],
  Text["A⃗3", 2.1 A3], Arrow[{A4, 2 A4}], Text["A⃗4", 2.1 A4], Arrow[{A5, 2 A5}],
  Text["A⃗5", 2.1 A5], Arrow[{A6, 2 A6}], Text["A⃗6", 2.1 A6]}, Boxed → False]
```

Out[41]=




```
(*to shorten notations and make them closer to those used in MATLAB, we introduce*)
int := Integrate;
nint := NIntegrate
```

```
(*start with charge at the center; *)
```

In[46]:=
```
A1.ef /.
 x → 1 (*projection of the field near the front face on the direction of its normal*)
```

Out[46]=

$$\frac{1}{\left(1+y^2+z^2\right)^{3/2}}$$

In[47]:=
```
phi1 = int[A1.ef /. x → 1, {y, -1, 1}, {z, -1, 1}]
```

Out[47]=

$$\frac{2\pi}{3}$$

In[48]:=
```
phi2 = int[A2.ef /. x → -1, {y, -1, 1}, {z, -1, 1}]
```
Out[48]=

$$\frac{2\pi}{3}$$

```
(*etc., the rest check together*)
```

In[∘]:=
```
{int[A3.ef /. y → 1, {x, -1, 1}, {z, -1, 1}], int[A4.ef /. y → -1, {x, -1, 1}, {z, -1, 1}],
 int[A5.ef /. z → 1, {x, -1, 1}, {y, -1, 1}], int[A6.ef /. z → -1, {x, -1, 1}, {y, -1, 1}]}
```
Out[∘]=

$$\left\{\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right\}$$

```
phi1 + phi2 + 4 * 2 Pi / 3 (*total flux*)
```
Out[∘]=

$$4\pi$$

```
(*as expected*)
```

In[∘]:=

In[∘]:=

In[∘]:=

In[∘]:=
```
(*now we shift the charge to x=1/2*)
```

In[49]:=
```
efx05 = ef /. x → x - 1 / 2
```
Out[49]=

$$\left\{\frac{-\frac{1}{2}+x}{\left(\left(-\frac{1}{2}+x\right)^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left(\left(-\frac{1}{2}+x\right)^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left(\left(-\frac{1}{2}+x\right)^2+y^2+z^2\right)^{3/2}}\right\}$$

In[50]:=
```
phi1 = int[A1.efx05 /. x → 1, {y, -1, 1}, {z, -1, 1}]
```
Out[50]=

$$\pi + \text{ArcTan}\left[\frac{336}{527}\right]$$

In[51]:=
```
phi2 = int[A2.efx05 /. x → -1, {y, -1, 1}, {z, -1, 1}]
```
Out[51]=

$$2\,\text{ArcTan}\left[\frac{24\sqrt{17}}{137}\right]$$

```
(*etc., we will not show all long analytical expressions,
also expressed through varios ArcTan *)
```

In[52]:=
```
phi3 = int[A3.efx05 /. y → 1, {x, -1, 1}, {z, -1, 1}];
phi4 = int[A4.efx05 /. y → -1, {x, -1, 1}, {z, -1, 1}];
phi5 = int[A5.efx05 /. z → 1, {x, -1, 1}, {y, -1, 1}];
phi6 = int[A6.efx05 /. z → -1, {x, -1, 1}, {y, -1, 1}];
```

In[53]:=
```
(*they are all the same, e.g.:*)phi3 - phi4
```

Out[53]=

0

In[ ]:=
```
(*now total flux*)
```

In[54]:=
```
phi1 + phi2 + 4 phi3
```

Out[54]=

$$\pi + \text{ArcTan}\left[\frac{336}{527}\right] + 2\,\text{ArcTan}\left[\frac{24\sqrt{17}}{137}\right] + 8\,\text{ArcTan}\left[\frac{1}{24}\left(13 + 5\sqrt{17}\right)\right]$$

In[55]:=
```
% // Simplify(*does not work analytically*)
```

Out[55]=

$$\pi + \text{ArcTan}\left[\frac{336}{527}\right] + 2\,\text{ArcTan}\left[\frac{24\sqrt{17}}{137}\right] + 8\,\text{ArcTan}\left[\frac{1}{24}\left(13 + 5\sqrt{17}\right)\right]$$

In[56]:=
```
% - 4. Pi(*4. forces a switch to Numerics*)
```

Out[56]=

8.88178×10^{-16}

In[57]:=
```
(*which is a machine zero*)
```

```
(*Extra: we now try to verify if outside charge indeed
   contributes zero flux. We select x=2 for the location of the charge:*)
```

In[60]:=
```
efx2 = ef /. x → x - 2
```

Out[60]=

$$\left\{\frac{-2+x}{\left((-2+x)^2+y^2+z^2\right)^{3/2}}, \frac{y}{\left((-2+x)^2+y^2+z^2\right)^{3/2}}, \frac{z}{\left((-2+x)^2+y^2+z^2\right)^{3/2}}\right\}$$

In[61]:=
```
phi1 = int[A1.efx2 /. x → 1, {y, -1, 1}, {z, -1, 1}]
```

Out[61]=

$$-\frac{2\pi}{3}$$

```
(*note the "-"*)
```

In[62]:=
```
phi2 = int[A2.efx2 /. x → -1, {y, -1, 1}, {z, -1, 1}]
```

Out[62]=

$$2\,\text{ArcTan}\left[\frac{3\sqrt{11}}{49}\right]$$

In[63]:=
```
phi3 = int[A3.efx2 /. y → 1, {x, -1, 1}, {z, -1, 1}];
phi4 = int[A4.efx2 /. y → -1, {x, -1, 1}, {z, -1, 1}];
phi5 = int[A5.efx2 /. z → 1, {x, -1, 1}, {y, -1, 1}];
phi6 = int[A6.efx2 /. z → -1, {x, -1, 1}, {y, -1, 1}];
```

In[64]:=
```
(*again, they should be the same*){phi3, phi4, phi5, phi6}
```

Out[64]=

$$\left\{-\frac{5\pi}{6}+2\,\text{ArcTan}\left[10+3\sqrt{11}\right],\ -\frac{5\pi}{6}+2\,\text{ArcTan}\left[10+3\sqrt{11}\right],\ -\frac{5\pi}{6}+2\,\text{ArcTan}\left[10+3\sqrt{11}\right],\ -\frac{5\pi}{6}+2\,\text{ArcTan}\left[10+3\sqrt{11}\right]\right\}$$

In[65]:=
```
phi1 + phi2 + 4 * phi3
```

Out[65]=

$$-\frac{2\pi}{3}+2\,\text{ArcTan}\left[\frac{3\sqrt{11}}{49}\right]+4\left(-\frac{5\pi}{6}+2\,\text{ArcTan}\left[10+3\sqrt{11}\right]\right)$$

In[66]:=
```
% // N
```

Out[66]=

$$-2.22045\times10^{-16}$$

In[67]:=
```
Chop[%]
```

Out[67]=

0

```
(*Yes! A machine zero*)

(*good, but since pure analytics seems to get unmanageable,
we switch to numerical integration. *)
```

In[69]:=
```
(*General; get rid of 4Pi, so the theoretical value is 1;
the function below does the same integrations as before, only without interruptions,
computes the total flux, and gives the error compared to exact value*)
testCube[x0_, y0_, z0_] := {Clear[f, phiTot];
  f = 1 / (4 Pi) ef /. {x → x - x0, y → y - y0, z → z - z0};
  phi1 = nint[(A1.f /. x → 1), {y, -1, 1}, {z, -1, 1}];
  phi2 = nint[(A2.f /. x → -1), {y, -1, 1}, {z, -1, 1}];
  phi3 = nint[A3.f /. y → 1, {x, -1, 1}, {z, -1, 1}];
  phi4 = nint[A4.f /. y → -1, {x, -1, 1}, {z, -1, 1}];
  phi5 = nint[A5.f /. z → 1, {x, -1, 1}, {y, -1, 1}];
  phi6 = nint[A6.f /. z → -1, {x, -1, 1}, {y, -1, 1}];
  phiTot = phi1 + phi2 + phi3 + phi4 + phi5 + phi6;
  phiTot - 1}
```

```
(*we first double check the two cases which we verified analytically, and then other*)
```

In[71]:=
```
{testCube[0, 0, 0], testCube[.5, 0, 0]}
```

Out[71]=

$\{\{-1.9264 \times 10^{-9}\}, \{-4.82647 \times 10^{-9}\}\}$

In[113]:=
```
testCube[.5, .5, 0]
```

Out[113]=

$\{-1.77243 \times 10^{-8}\}$

In[112]:=

In[74]:=
```
(*now a random test*)
x0 = RandomReal[];
y0 = RandomReal[];
z0 = RandomReal[];
testCube[x0, y0, z0];
Print["x0=", x0, " ", "y0=", y0, " ", "z0=", z0, "  ", "error=", phiTot - 1];
```

x0=0.473341 y0=0.912875 z0=0.644184 error= -2.79505×10^{-8}

x0=0.693657 y0=0.00268512 z0=0.372206 error= -6.23508×10^{-8}

In[75]:=
```
(*Extra: test for a charge outside the cube. Note: error in this case is |phiTot|,
which shouldl be close to zero*)
```

```
(*AAdvanced. To rpeat a random test maaanyy times we can use, e.g. the "Do" loop:*)
```

In[77]:=
```
Do[{x0 = RandomReal[{-1, 1}];
  y0 = RandomReal[{-1, 1}];
  z0 = RandomReal[{-1, 1}];
  testCube[x0, y0, z0];
  Print["x0=", x0, " ", "y0=", y0, " ", "z0=", z0, "  ", "error=", phiTot - 1];}
 , {10}]
```

x0=0.179191 y0=0.32641 z0=0.764337 error=-6.74868×10^{-9}

x0=0.758191 y0=0.617569 z0=0.207494 error=-1.4149×10^{-8}

x0=0.861654 y0=-0.910783 z0=-0.636172 error=-5.70543×10^{-8}

x0=0.191913 y0=0.442232 z0=0.165241 error=-7.03791×10^{-9}

x0=0.223338 y0=-0.899776 z0=0.438585 error=-3.66678×10^{-8}

x0=-0.674594 y0=0.771644 z0=0.196429 error=-7.55091×10^{-9}

x0=-0.615067 y0=-0.575982 z0=0.49445 error=-3.28598×10^{-8}

x0=-0.760207 y0=0.875595 z0=0.715869 error=-2.50768×10^{-8}

x0=0.569451 y0=-0.18703 z0=-0.0940902 error=3.1311×10^{-9}

NIntegrate: Numerical integration converging too slowly; suspect one of the following: singularity, value of the integration is 0, highly oscillatory integrand, or WorkingPrecision too small.

x0=0.985526 y0=-0.631114 z0=-0.0470422 error=-4.00439×10^{-8}

In[119]:=

In[120]:=

```
(*Advanced. Analytics for r>\*)
```

In[ ]:=

Out[118]=

$Aborted

In[109]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[ ]:=

In[78]:=

In[104]:=

In[106]:=

In[107]:=

In[79]:=

In[80]:=

In[81]:=

In[82]:=

In[83]:=

In[84]:=

In[85]:=

In[86]:=

In[87]:=

In[89]:=

In[90]:=

In[91]:=

In[92]:=

In[93]:=

In[94]:=

In[95]:=

In[96]:=

In[97]:=

In[98]:=

In[ ]:=

In[99]:=

In[100]:=

In[102]:=

In[ ]:=

In[ ]:=

In[ ]:=