In[1]:= (*note: to simplifynotations all periodic functions of x are expected to have a period of 2 Pi,
as Ch. 11.1 by KR. If a function has a period "2L" as in his later sections,
one can always switch to a new variable xnew =Pi x/L and get the original periodicity*)

In[2]:=
(*first define colors for future graphics*)

In[3]:= red = RGBColor[1, 0, 0]; green = RGBColor[0, 1, 0]; blue = RGBColor[0, 0, 1]; black = RGBColor[0, 0, 0];

In[4]:= fapp[x_, nmax_] := a0 + Sum[*a*[*n*] Cos[*n x*] + *b*[*n*] Sin[*n x*], {*n*, nmax}]

In[5]:= (*Note, unlike formal expression which have infinity for the upper limit, we have a finite nmax,
so that an approximation to the original function f is considered – "trigonometric polynomials" (KR. 11.6). However,
accuracy can be made arbitrary high with increase of nmax. *)

In[6]:= Clear[a0, *a*, *b*];

In[7]:= $a0 := \frac{\int_{-\pi}^{\pi} f[x]\, dx}{2\pi}$

In[8]:= $a[n_] := \frac{\int_{-\pi}^{\pi} f[x]\, Cos[n\, x]\, dx}{\pi}$

In[9]:= $b[n_] := \frac{\int_{-\pi}^{\pi} f[x]\, Sin[n\, x]\, dx}{\pi}$

In[10]:= (*above are Euler formulas; "/;" represents the condition "If" *)

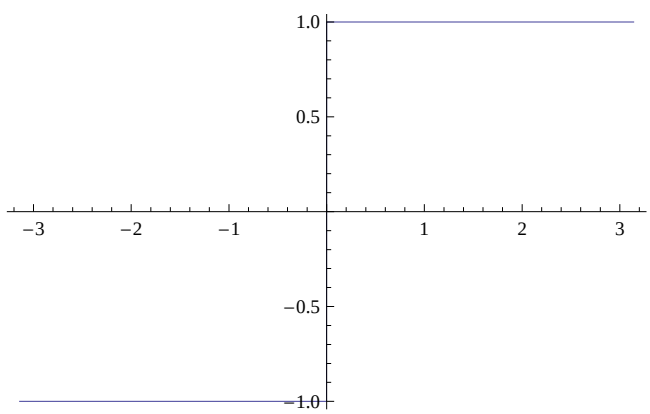
In[11]:= (*Example 1*)

In[12]:=

In[13]:= Clear[*f*]; *f*[x_] := If[(*x* > −Pi && *x* < 0), −1, 1]

In[14]:=

In[15]:= Plot[*f*[*x*], {*x*, −Pi, Pi}]

Out[15]=


In[16]:= a0

Out[16]= 0

In[17]:= *a*[*n*]

Out[17]= 0

In[18]:= **b[n]**

Out[18]= $$\frac{2 - 2\,\mathrm{Cos}[n\,\pi]}{n\,\pi}$$

In[19]:= **Table[{n, b[n]}, {n, 5}] // TableForm**

Out[19]//TableForm=

1	$\frac{4}{\pi}$
2	0
3	$\frac{4}{3\,\pi}$
4	0
5	$\frac{4}{5\,\pi}$

In[20]:= **f3 = fapp[x, 3]**

Out[20]= $$\frac{4\,\mathrm{Sin}[x]}{\pi} + \frac{4\,\mathrm{Sin}[3\,x]}{3\,\pi}$$

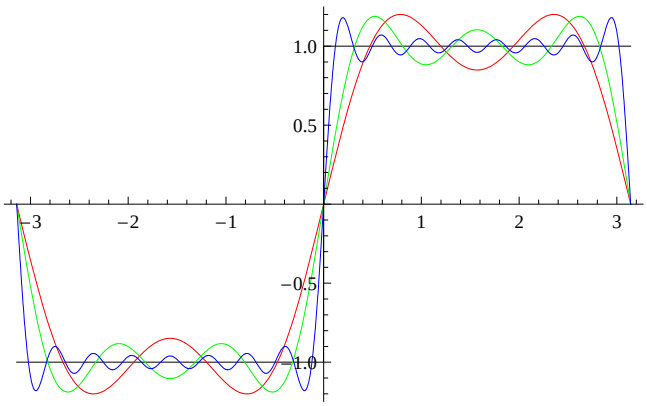
In[21]:= **f5 = fapp[x, 5]**

Out[21]= $$\frac{4\,\mathrm{Sin}[x]}{\pi} + \frac{4\,\mathrm{Sin}[3\,x]}{3\,\pi} + \frac{4\,\mathrm{Sin}[5\,x]}{5\,\pi}$$

In[22]:= **f15 = fapp[x, 15];**

In[23]:= **Plot[{f[x], f3, f5, f15}, {x, −Pi, Pi}, PlotStyle → {black, red, green, blue}]**

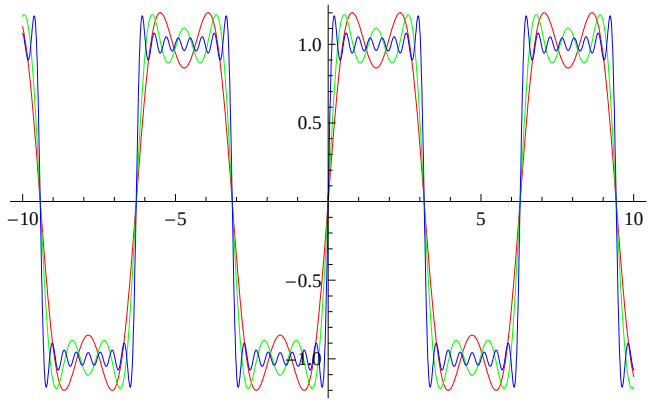
Out[23]=



In[24]:= **(∗note: the sum approaches 0 at + or − Pi − this is an example of Theorem 2, p. 484 in KR.∗)**

In[25]:= **Plot[{f3, f5, f15}, {x, −10, 10}, PlotStyle → {red, green, blue}]**

Out[25]=



In[26]:= **(∗Note: apprixomations extend the function periodically,**
regardless if the original f[x] was defined only on [−Pi,Pi] or was indeed periodic and defined for all x ∗)

In[27]:= **FullSimplify[b[n], n ∈ Integers]**

Out[27]= $-\frac{2\left(-1+(-1)^{n}\right)}{n\,\pi}$

In[28]:= **FullSimplify[%, (n / 2) ∈ Integers] (∗n even∗)**

Out[28]= 0

In[29]:= **FullSimplify[b[n], ((n + 1) / 2) ∈ Integers](∗n odd∗)**

Out[29]= $\frac{4}{n\,\pi}$

In[30]:= **(∗note: we had a[n]=0 since f[x] is an odd function; alternatively for an even function all b[n] would be zero − see 11.3 in KR.∗)**

In[31]:= **(∗ORTHOGONALITY∗)**

In[32]:= **Clear[f, m, n];**

In[33]:= **{a[n], b[n]}**

Out[33]= $\left\{\frac{\int_{-\pi}^{\pi}\mathrm{Cos}[n\,x]\,f[x]\,dx}{\pi},\ \frac{\int_{-\pi}^{\pi}f[x]\,\mathrm{Sin}[n\,x]\,dx}{\pi}\right\}$

In[34]:= **f[x_] = Cos[m x];**

In[35]:=

In[36]:=

In[37]:= **a[n]**

Out[37]= $(2\,m\,\mathrm{Cos}[n\,\pi]\,\mathrm{Sin}[m\,\pi]-2\,n\,\mathrm{Cos}[m\,\pi]\,\mathrm{Sin}[n\,\pi])\,/\,\left(\left(m^2-n^2\right)\pi\right)$

In[38]:= **Limit[%, m → n]**

Out[38]= $1+\frac{\mathrm{Sin}[2\,n\,\pi]}{2\,n\,\pi}$

In[39]:= **FullSimplify[%, $n \in$ Integers]**

Out[39]= 1

In[40]:= **FullSimplify[a[n], $n \in$ Integers && $m \in$ Integers]**

Out[40]= 0

In[41]:= **(∗i.e. cos(m x) and cos(n x) are orthogonal for m ≠ n ∗)**

In[42]:= **FullSimplify[b[n], $n \in$ Integers && $m \in$ Integers]**

Out[42]= 0

In[43]:= **(∗i.e. cos(m x) and sin(n x) are orthogonal for all m,n∗)**

In[44]:= **Clear[f, m, n];**

In[45]:= **f[x_] = Sin[m x];**

In[46]:= **b[n]**

Out[46]= $(2\,n\,\mathrm{Cos}[n\,\pi]\,\mathrm{Sin}[m\,\pi] - 2\,m\,\mathrm{Cos}[m\,\pi]\,\mathrm{Sin}[n\,\pi]) \big/ \left((m^2 - n^2)\,\pi\right)$

In[47]:= **Limit[%, m → n]**

Out[47]= $1 - \dfrac{\mathrm{Sin}[2\,n\,\pi]}{2\,n\,\pi}$

In[48]:= **FullSimplify[%, $n \in$ Integers]**

Out[48]= 1

In[49]:= **FullSimplify[b[n], $n \in$ Integers && $m \in$ Integers]**

Out[49]= 0

In[50]:= **(∗i.e. sin(m x) and sin(n x) are orthogonal for m ≠ n ∗)**

In[51]:=

(∗COMPLEX FOURIER − 11.4 in KR.∗)

In[52]:= **i = I (∗to have more familiar notations∗);**

In[53]:= **int := Integrate[#, {x, −Pi, Pi}] &**

In[54]:= **(∗Orthogonality:∗)**

In[55]:= **int[Exp[i n x] Exp[−i m x]]**

Out[55]= $\dfrac{2\,\mathrm{Sin}[(m - n)\,\pi]}{m - n}$

In[56]:= **Limit[%, m → n]**

Out[56]= $2\,\pi$

In[57]:= **FullSimplify[%%, $n \in$ Integers && $m \in$ Integers]**

Out[57]= 0

In[58]:= **(*i.e. the exponentials with different m and n are orthogonal. Note that in the integral of 2 functions one is taken as complex conjugate*)**

In[59]:= **Clear[f, c]; c[n_] := int[f[x] Exp[$-i\,n\,x$]] / (2 Pi)**

In[60]:= **fcomp[x_, nmax_] := Sum[c[n] Exp[$i\,n\,x$], {n, $-$nmax, nmax}]**

In[61]:= **Clear[f]; f[x_] := If[($x > -$Pi && $x < 0$), -1, 1](*old example*)**

In[62]:= **c[n]**

Out[62]= $$\frac{i\,(-1+\text{Cos}[n\,\pi])}{n\,\pi}$$

In[63]:= **c[n] Exp[$i\,n\,x$] + c[$-n$] Exp[$-i\,n\,x$]**

Out[63]= $$-\frac{i\,e^{-i\,n\,x}\,(-1+\text{Cos}[n\,\pi])}{n\,\pi}+\frac{i\,e^{i\,n\,x}\,(-1+\text{Cos}[n\,\pi])}{n\,\pi}$$

In[64]:= **FullSimplify[%, $n \in$ Integers]**

Out[64]= $$\frac{i\,\left(-1+(-1)^{n}\right)e^{-i\,n\,x}\left(-1+e^{2\,i\,n\,x}\right)}{n\,\pi}$$

In[65]:= **FullSimplify[%, $n/2 \in$ Integers]**

Out[65]= 0

In[66]:= **FullSimplify[%%, $(n+1)/2 \in$ Integers]**

Out[66]= $$\frac{4\,\text{Sin}[n\,x]}{n\,\pi}$$

In[67]:= **(*i.e. return to the original real−value series*)**

In[68]:= **(*example 1 from p. 498 in KR.*)**

In[69]:= **Clear[f]; f[x_] = Exp[x];**

In[70]:= **c[n]**

Out[70]= $$\frac{i\,\text{Sinh}[\pi-i\,n\,\pi]}{(i+n)\,\pi}$$

In[71]:= **FullSimplify[%, $n \in$ Integers]**

Out[71]= $$\frac{i\,(-1)^{n}\,\text{Sinh}[\pi]}{(i+n)\,\pi}$$

In[72]:= **(*which is equivalent to KR.*)**

In[73]:= **f2 = fcomp[x, 2]**

Out[73]= $$\frac{-e^{-\pi}+e^{\pi}}{2\,\pi}-\frac{\left(\frac{1}{2}-\frac{i}{2}\right)e^{-i\,x}\,\text{Sinh}[\pi]}{\pi}-\frac{\left(\frac{1}{2}+\frac{i}{2}\right)e^{i\,x}\,\text{Sinh}[\pi]}{\pi}+\frac{\left(\frac{1}{5}-\frac{2\,i}{5}\right)e^{-2\,i\,x}\,\text{Sinh}[\pi]}{\pi}+\frac{\left(\frac{1}{5}+\frac{2\,i}{5}\right)e^{2\,i\,x}\,\text{Sinh}[\pi]}{\pi}$$

In[74]:= **FullSimplify[%, $x \in$ Reals]**

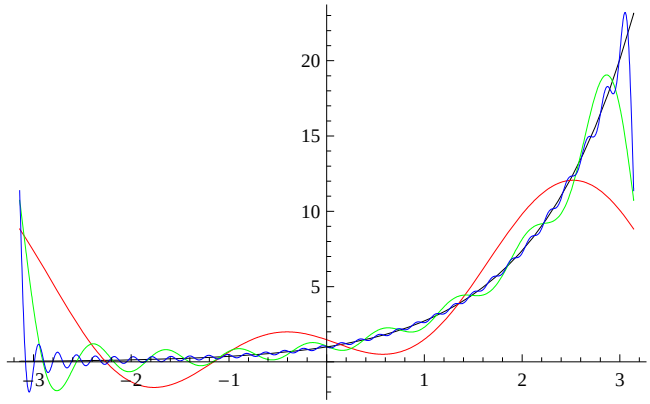
Out[74]= $\frac{1}{5\pi}(5 + 2\,\mathrm{Cos}[2\,x] + 5\,\mathrm{Sin}[x] - \mathrm{Cos}[x]\,(5 + 8\,\mathrm{Sin}[x]))\,\mathrm{Sinh}[\pi]$

In[75]:= **(∗i.e a trigonometric polynomial∗)**

In[76]:= **f8 = fcomp[*x*, 8]; f32 = fcomp[*x*, 32];**

In[77]:= **Plot[{*f*[*x*], f2, f8, f32}, {*x*, −Pi, Pi}, PlotStyle → {black, red, green, blue}]**

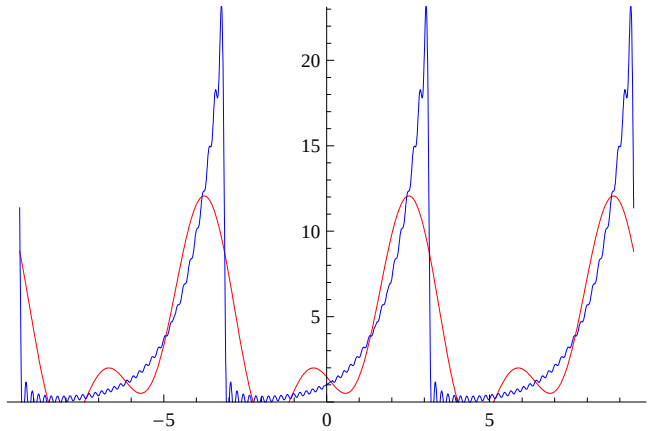
Out[77]=



In[78]:=

In[79]:= **Plot[{f2, f32}, {*x*, −3 Pi, 3 Pi}, PlotStyle → {red, blue}, PlotRange → {0, *E* ^ Pi}]**

Out[79]=



In[80]:= **(∗again, note that the function is periodically extended∗)**

In[81]:=

(∗FOURIER INTEGRAL∗)

In[82]:= **? ∗Fourier∗**

▼ **System`**

Fourier
FourierCosTransform
FourierDCT
FourierDST
FourierParameters
FourierSinTransform
FourierTransform
InverseFourier
InverseFourierCosTransform
InverseFourierSinTransform
InverseFourierTransform

In[83]:= $(*\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt* - FourierTransform*)$

In[84]:= $(*\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\, \cos(\omega t) dt* - \mathbf{FourierCosTransform}; *)$

In[85]:= $(*\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\, \sin(\omega t) dt - FourierSinTransform; *)$

In[86]:= **(∗similarly for inverse see also p. 519 in KR∗)**

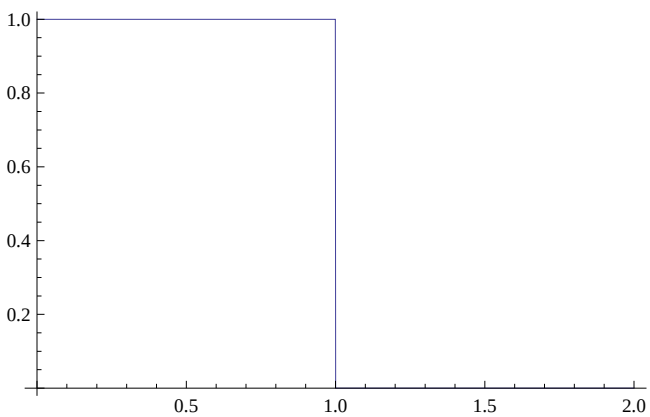
In[87]:=

(∗Example 1, p.515 a=k=1∗)

In[88]:= **Clear[f]; f [x_] = (1 + Sign[1 − x]) / 2;**

In[89]:= **Plot[f [x], {x, 0, 2}]**

Out[89]=



In[90]:= **FourierCosTransform[$f[x]$, x, om] // FullSimplify**

Out[90]= $$\frac{\sqrt{\frac{2}{\pi}}\ \mathrm{Sin}[\mathrm{om}]}{\mathrm{om}}$$

In[91]:= **(∗some difficulty for Mathematica since a discontinuous f[x]; however DiracDelta's cancel and get the following∗)**

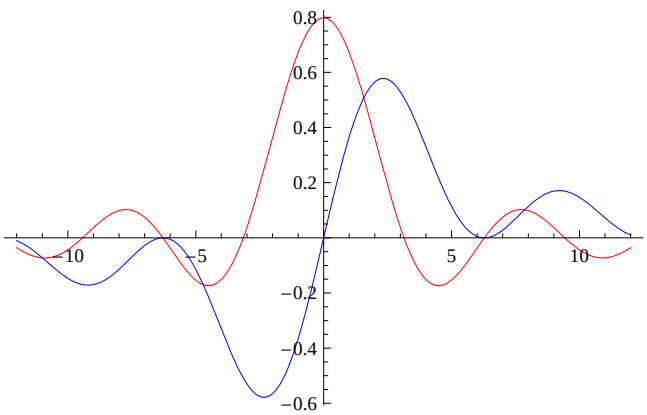
In[92]:= **fcos =** $$\frac{\sqrt{\frac{2}{\pi}}\ \mathrm{Sin}[\mathrm{om}]}{\mathrm{om}};$$

In[93]:= **fsin = FourierSinTransform[$f[x]$, x, om]**

Out[93]= $$\frac{4 - 4\,\mathrm{Cos}[\mathrm{om}]}{2\,\mathrm{om}\sqrt{2\,\pi}}$$

In[94]:= **Plot[{fcos, fsin}, {om, −12, 12}, PlotStyle → {red, blue}]**

Out[94]=



In[95]:= **fexp = FourierTransform[$f[x]$, x, om]**

Out[95]= $$-\frac{i\,e^{i\,\mathrm{om}}}{\mathrm{om}\sqrt{2\,\pi}} + \sqrt{\frac{\pi}{2}}\ \mathrm{DiracDelta}[\mathrm{om}]$$

In[96]:= **(∗again, DiracDelta is an artifact∗)**

In[97]:= **InverseFourierTransform[fexp, om, x](∗ok∗)**

Out[97]= $$\frac{1}{2}\,(1 - \mathrm{Sign}[-1 + x])$$

In[98]:= **(∗Example 2∗)**

In[99]:= **FourierCosTransform[E ^ − x, x, om]**

Out[99]= $$\frac{\sqrt{\frac{2}{\pi}}}{1 + \mathrm{om}^2}$$

In[100]:= **FourierSinTransform[E ^ − x, x, om]**

Out[100]= $$\frac{\mathrm{om}\sqrt{\frac{2}{\pi}}}{1 + \mathrm{om}^2}$$

In[101]:= **(∗Fourier Transform of Derivatives − theorem 1 on p.516 and example 3 on p.517∗)**

In[102]:= **Clear[*f*]; *f*[x_] = *E*^−*x***

Out[102]= e^{-x}

In[103]:= **FourierCosTransform[*D*[*f*[*x*], *x*], *x*, om] − om ∗ FourierSinTransform[*f*[*x*], *x*, om] + Sqrt[2] *f*[0] / Sqrt[Pi] // Simplify**

Out[103]= 0

In[104]:= **FourierSinTransform[*D*[*f*[*x*], *x*], *x*, om] + om ∗ FourierCosTransform[*f*[*x*], *x*, om] // Simplify**

Out[104]= 0

In[105]:= **Clear[*f*]; *f*[x_] = Exp[−*x*^2]**

Out[105]= e^{-x^2}

In[106]:=

In[107]:= **FourierTransform[*D*[*f*[*x*], *x*], *x*, om] + *i* om ∗ FourierTransform[*f*[*x*], *x*, om] // Simplify**

Out[107]= 0

In[108]:= `(*above is theorem 3 and example 3 from p. 521; in more detail:*)`

In[109]:= **FourierTransform[*E*^(−*x*^2), *x*, om]**

Out[109]= $\frac{e^{-\frac{om^2}{4}}}{\sqrt{2}}$

In[110]:= ***D*[*E*^(−*x*^2), *x*]**

Out[110]= $-2\, e^{-x^2}\, x$

In[111]:= **FourierTransform[%, *x*, om] / (−*i* om)**

Out[111]= $\frac{e^{-\frac{om^2}{4}}}{\sqrt{2}}$

In[112]:= **(∗same thing∗)**

In[113]:=

In[114]:=

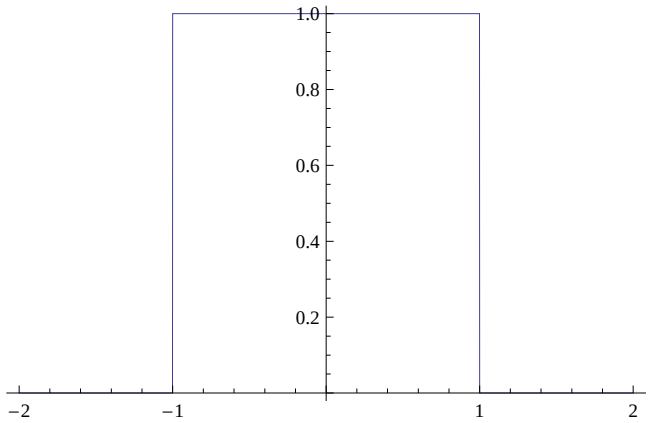
In[115]:= **ft := FourierTransform[#, *x*, om] &**

In[116]:= **ift := InverseFourierTransform[#, om, *x*] &**

In[117]:= **Clear[*f*]; *f*[x_] = 1 / 2 (Sign[1 + *x*] − Sign[*x* − 1]);**

In[118]:= **Plot[f[x], {x, −2, 2}]**

Out[118]=



In[119]:= **ft[f[x]]**

Out[119]= $\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[\text{om}]}{\text{om}}$

In[120]:= **FullSimplify[%, om ∈ Reals] (∗see p. 531 in KR∗)**

Out[120]= $\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[\text{om}]}{\text{om}}$

In[121]:= **f[x_] = 1 / 2 (Sign[x + b] − Sign[x − b]);**

In[122]:=

Out[123]= $Aborted

In[124]:=

In[125]:=

In[126]:=

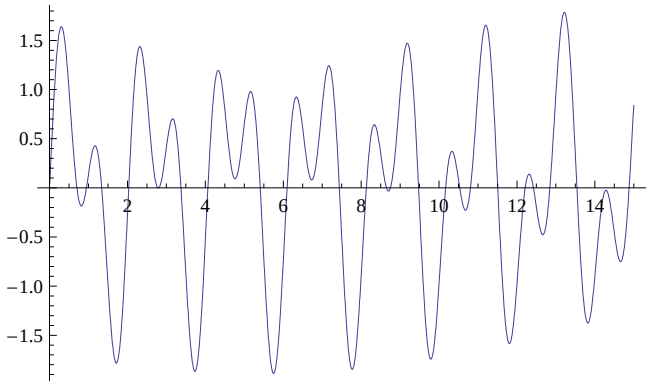
(∗EXTRACTING FREQUENCIES FROM A COMPLICATED SIGNAL∗)

In[127]:= **f[x_] = Exp[−.001 x] ∗ (Sin[3 x] + .9 Sin[2 Pi x])**

Out[127]= $e^{-0.001\,x}\,(\text{Sin}[3\,x] + 0.9\,\text{Sin}[2\,\pi\,x])$

In[128]:= **Plot[$f[x]$, {x, 0, 15}]**

Out[128]=



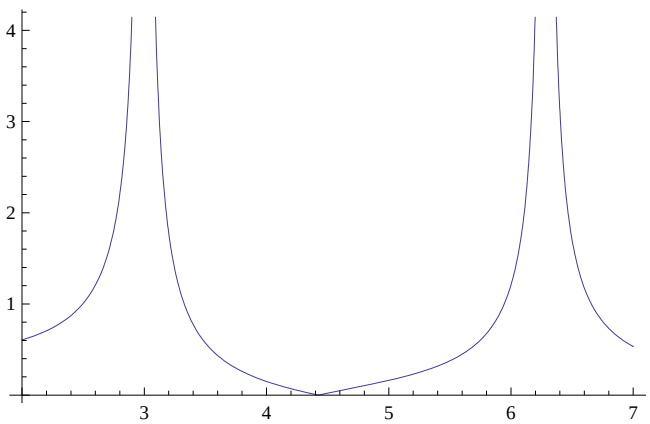
In[129]:= **ftt = FourierCosTransform[$f[x]$, x, om] // Chop**

Out[129]=

$$\frac{\frac{446.49-11.3097\,\text{om}^2}{1558.55-78.9568\,\text{om}^2+1.\,\text{om}^4}-\frac{6.\,(-3.+\text{om})\,(3.+\text{om})}{81.-18.\,\text{om}^2+1.\,\text{om}^4}}{\sqrt{2\,\pi}}$$

In[130]:= **Plot[Abs[ftt], {om, 2, 7}]**

Out[130]=

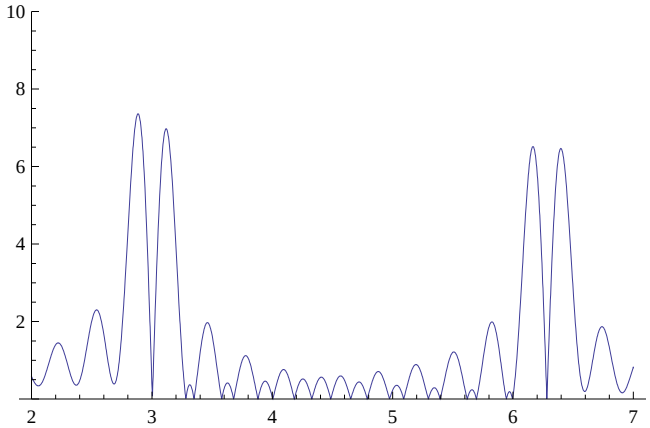


In[131]:= **myft := Integrate[# Cos[om x], {x, 0, L}] &**

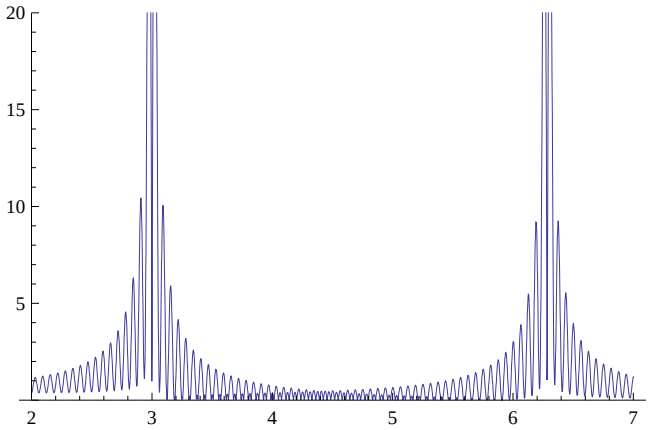
In[135]:= **L = 20; myftt = myft[$f[x]$] // Chop;**

In[136]:= **Plot[Abs[myftt], {om, 2, 7}, PlotRange → {0, 10}]**

Out[136]=



In[137]:= ***L* = 100; myftt = myft[*f*[*x*]] // Chop;**

In[138]:= **Plot[Abs[myftt], {om, 2, 7}, PlotRange → {0, 20}]**

Out[138]=