VERIFICATION OF GAUSS THEOREM (GT)

Thiis is a *Mathematica*-based analytical companion to numerical MATLAB study.

1 Theory

1.1 General

Define total flux Φ through a closed surface

$$\Phi = \oint \vec{E} \cdot d\vec{A} \tag{1}$$

then, GT states

$$\Phi = 4\pi k Q \tag{2}$$

Here Q is the total charge enclosed *inside* the surface; otherwise the actual location of the charge, size and shape of the surface and all *outside* charges do not matter.

The electric field of a single charge q is given by

$$\vec{E} = kq\vec{r}/r^3 \tag{3}$$

where $\vec{r}$ runs from the charge towards the observation point. To verify (not prove!) the GT for a single charge one needs to evaluate the flux Φ via the integral and compare it with $4\pi kQ$, where $Q = q$ for the charge inside the surface and $Q = 0$ if the charge is outside.

1.2 Specific geometry and parameters

Consider $kq = 1$ and the surface in the shape of a cube centered at the origin, with side 2. The six faces of the cube have normals

$$\vec{n}^x = (\pm 1, 0, 0)\,,\ \vec{n}^y = (0, \pm 1, 0)\,,\ \vec{n}^z = (0, 0, \pm 1) \tag{4}$$

The charge is located at $(h, 0, 0)$ so that

$$\vec{r} = (x - h, y, z)$$

To find the flux through a given face of the cube we note

$$d\vec{A} = \vec{n} dA \tag{5}$$

(with $\vec{n}$ being the corresponding normal) and introduce

$$\phi = \vec{E} \cdot \vec{n} \tag{6}$$

for each of the faces, evaluating a corresponding integral $\int \phi \, dA$. The total flux is represented as a sum of 3 contributions

$$\Phi = \Phi^x + \Phi^y + \Phi^z$$

each corresponding to a net flux through *both* faces with a normal parallel (antiparallel) to an indicated axis. For example, ϕ^z evaluated on the upper or lower faces with $z = \pm 1$ is given by

$$\phi^z = \frac{1}{(1 + (-h + x)^2 + y^2)^{3/2}}$$

This gives

$$\Phi^z = 2 \int_{-1}^{1} dx \int_{-1}^{1} dy \, \phi^z = 4 \left(-\mathrm{ArcTan}\left[\frac{-1 + h}{\sqrt{2 + (-1 + h)^2}} \right] + \mathrm{ArcTan}\left[\frac{1 + h}{\sqrt{2 + (1 + h)^2}} \right] \right) \tag{7}$$

From symmetry, Φ^y is the same.

Next, for the faces at $x = \pm 1$

$$\phi^x_{1,2} = \pm \frac{\pm 1 - h}{((\pm 1 - h)^2 + y^2 + z^2)^{3/2}}$$

and

$$\Phi^x = \int_{-1}^{1} dz \int_{-1}^{1} dy \; (\phi^x_1 + \phi^x_2) = \tag{8}$$

$$-4\mathrm{ArcCot}\left[(-1 + h)\sqrt{3 + (-2 + h)h} \right] + 4\mathrm{ArcCot}\left[(1 + h)\sqrt{3 + h(2 + h)} \right]$$

Note that the total Φ appears as a function of h. However, for any $|h| < 1$ it evaluates to 4π as expected, while for $h > 1$ (charge outside the cube) $\Phi = 0$. Actual verification is done analytically for $h = 0$ (charge at the center), $h = 1/2$, $h = 2$, and numerically for a random $0 < h < 1$.

2 Mathematica input (bold) and output

$r = \{x - h, y, z\}$;
$e = r/(r.r)$^(3/2)

$\left\{\frac{-h+x}{((-h+x)^2+y^2+z^2)^{3/2}}, \frac{y}{((-h+x)^2+y^2+z^2)^{3/2}}, \frac{z}{((-h+x)^2+y^2+z^2)^{3/2}}\right\}$

nz = {0, 0, Sign[z]};
phiz = e.nz/.$z \to 1$

$\frac{1}{(1+(-h+x)^2+y^2)^{3/2}}$

Phiz =
2Integrate[phiz, {x, −1, 1},
{y, −1, 1},
Assumptions → h^2 < 1]//
FullSimplify

$4\left(-\text{ArcTan}\left[\frac{-1+h}{\sqrt{2+(-1+h)^2}}\right] + \text{ArcTan}\left[\frac{1+h}{\sqrt{2+(1+h)^2}}\right]\right)$

Phiy = Phiz;
Clear[nx]; nx = {Sign[x], 0, 0}

$\{\text{Sign}[x], 0, 0\}$

phix1 = e.nx/.$x \to 1$

$\frac{1-h}{((1-h)^2+y^2+z^2)^{3/2}}$

phix2 = e.nx/.$x \to -1$

$-\frac{-1-h}{((-1-h)^2+y^2+z^2)^{3/2}}$

Phix =
Integrate[(phix1 + phix2),
{z, −1, 1}, {y, −1, 1},
Assumptions → h^2 < 1]//
FullSimplify

$-4\text{ArcCot}\left[(-1+h)\sqrt{3+(-2+h)h}\right] + 4\text{ArcCot}\left[(1+h)\sqrt{3+h(2+h)}\right]$

Phi = Phix + Phiy + Phiz//FullSimplify

$-4\text{ArcCot}\left[(-1+h)\sqrt{3+(-2+h)h}\right]+4\text{ArcCot}\left[(1+h)\sqrt{3+h(2+h)}\right]-8\text{ArcTan}\left[\frac{-1+h}{\sqrt{2+(-1+h)^2}}\right]+ 8\text{ArcTan}\left[\frac{1+h}{\sqrt{2+(1+h)^2}}\right]$

Phi/.$h \to 0$

4π

Phi/.$h \to 1/2$//FullSimplify

4π

Phi/.$h \to$ Random[]

12.5664

% == 4Pi//Simplify

True

Phi/.$h \to 2$//FullSimplify

0

(*since outside*)