Lecture Notes for math111: Calculus I.

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Introduction

$$y = f(x)$$

Limits & Continuity

Rates of change and tangents to curves

Average rate of change

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1}$$

Equation for the secant line:

$$Y = \frac{\Delta y}{\Delta x}(x - x_1) + f(x_1)$$

Tangent - secant in the *limit* $x_2 \to x_1$ (Or, $\Delta x \to 0$).
Slope of tangent

$$f'(x_1) \equiv \frac{\Delta y}{\Delta x},\ \Delta x \to 0 \tag{2}$$

(notation f' is not yet in the book, but will appear later...). Equation of tangent line:

$$Y = f'(x_1)(x - x_1) + f(x_1)$$

Example (Galileo): t instead of x or x_1, h instead of Δx.

$$y(t) = \frac{1}{2}at^2$$

where a is a constant ("acceleration"). Secant (finite h)

$$\Delta y = y(t+h) - y(t) = \frac{1}{2}a\left[(t+h)^2 - t^2\right] = \frac{1}{2}a\left(2th + h^2\right)$$

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{h} = a(t + h/2)$$

Tangent ($h \to 0$):

$$y'(t) = at$$

(known as "instantaneous velocity" or "instantaneous rate of change"). Returning to math notations

$$\left(x^2\right)' = 2x \tag{3}$$

Example: $y(x) = x^3$

Use $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ with $a = x+h$, $b = x$.
Slope of secant (A.R.C.), finite $h \equiv \Delta x$:

$$\Delta y = y(x+h) - y(x) = (x+h)^3 - x^3 = h\left[(x+h)^2 + x(x+h) + x^2\right] =$$

$$= h(3x^2 + 3xh + h^2)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{h} = 3x^2 + 3xh + h^2$$

Slope of tangent (I.R.C.) $h \to 0$:

$$\left(x^3\right)' = 3x^2 \qquad (4)$$

Example: $y(x) = \sqrt{x}$

$$A.R.C. = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

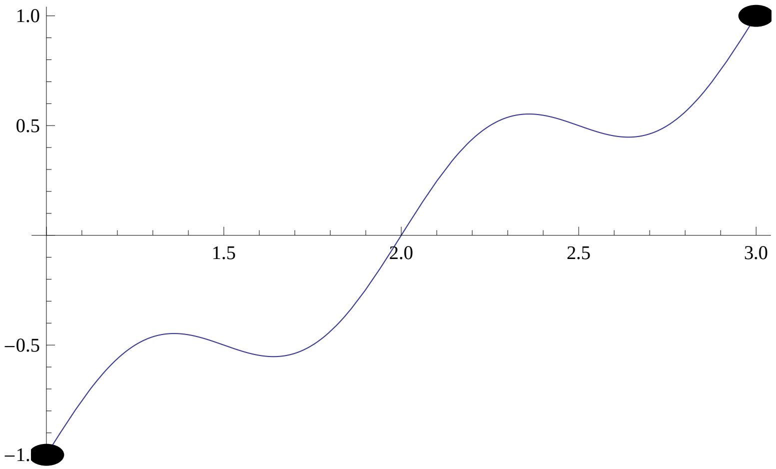
Now, no problem $h \to 0$:

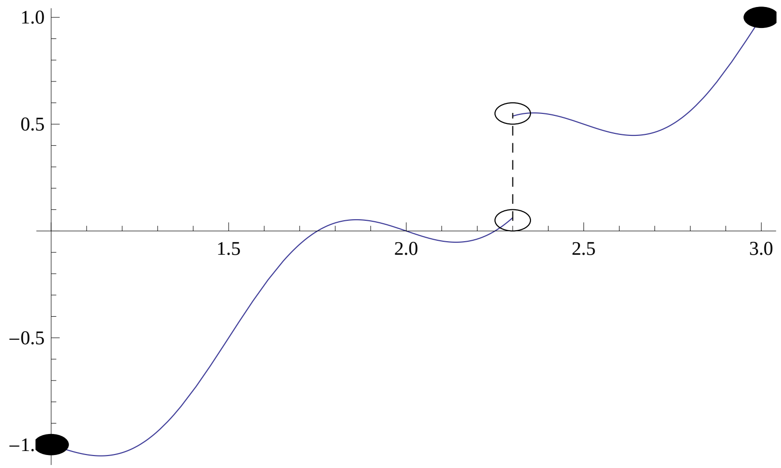
$$I.R.C. = 1/2\sqrt{x}$$

or

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}} \tag{5}$$

Continuity and Limits



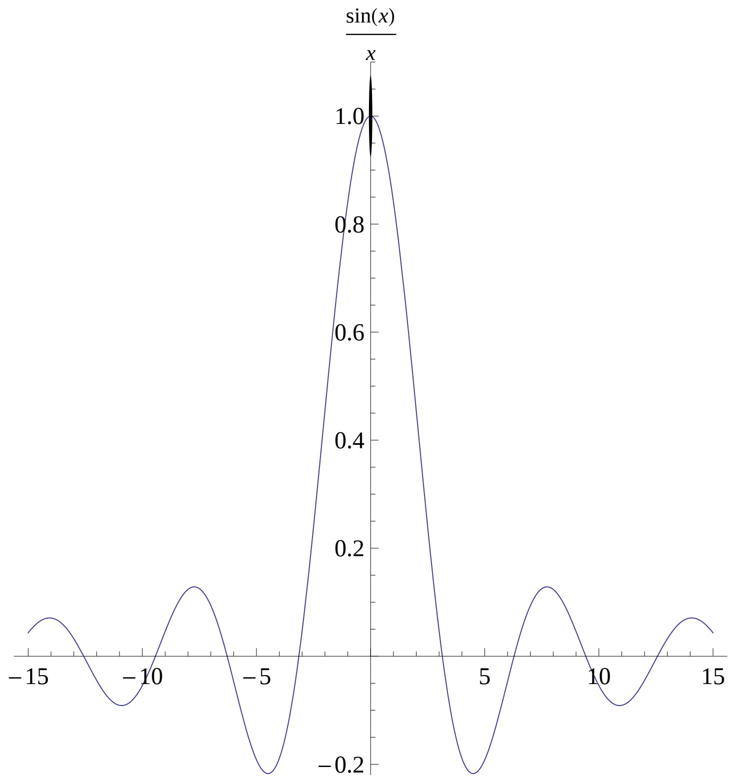


Continuous (left) and discontinuous (right) functions

Intermediate value Theorem 11, p.99

Limit of a function

Assume don't know $f(x_0)$ but know $f(x)$ for any x close to x_0 . Example:



$y(0)$ not defined. Can define $y(0) = a$, with any a. Why $a = 1$ is the "best"?

Limit:

$$\lim_{x \to x_0} f(x) = L$$

Example:

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} x + a = 2a$$

Functions with *no* limit as $x \to 0$:

step-function, $1/x$, $\sin(1/x)$

Limit Laws - Theorem 1, p.68.

If f, g have limits L & M as $x \to c$. Then,

$$\lim_{x\to c}(f \pm g) = L \pm M\,,\ \lim_{x\to c}(f \cdot g) = L \cdot M\,,\ etc. \tag{6}$$

$$\lim_{x\to c}(f^n) = L^n\,,\ \lim_{x\to c}(f^{1/n}) = L^{1/n} \tag{7}$$

"Dull functions" $\lim_{x\to c} f(x) = f(c)$. Example

$$\lim_{x\to -2}\sqrt{4x^2 - 3} = \sqrt{13}$$

Example: any polynomial $\lim_{x\to c} P_n(x) = P_n(c)$

Limits of rational functions $P(x)/Q(x)$ as $x \to c$.
If $Q(c) \neq 0$ - "dull", limit $R(c)/Q(c)$. If $Q(c) = 0$ check $P(c)$: if $P(c) \neq 0$ - no limit; If $P(c) = 0$ try to simplify.
Example:

$$\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 2)}{x(x-1)} = 4$$

Other functions with limits of type "0/0". Example:

$$\lim_{x \to 0} \frac{\sqrt{x^2 + a^2} - a}{x^2} = \lim_{x \to 0} \frac{\sqrt{x^2 + a^2} - a}{x^2} \cdot \frac{\sqrt{x^2 + a^2} + a}{\sqrt{x^2 + a^2} + a} =$$

$$= \lim_{x \to 0} 1/(\sqrt{x^2 + a^2} + a) = 1/2a$$

"Sandwich theorem" : Let $f(x) \leq g(x)$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L \; .$$

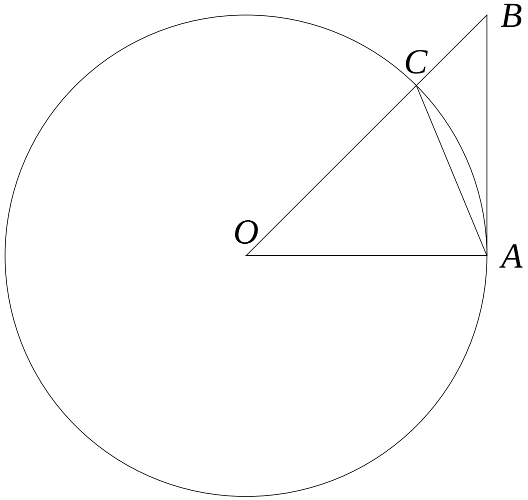
IF

$$f(x) \leq u(x) \leq g(x)$$

THEN

$$\lim_{x \to c} u(x) = L$$

Limit $(\sin x)/x$ *as* $x \to 0$



$\angle AOC = x$ (in radians !)

$S_{\triangle OAC} = \frac{1}{2}\sin x$

$S_{\text{sector}\, OAC} = \frac{x}{2}$

$S_{\triangle OAB} = \frac{1}{2}\tan x$

Thus,

$$\sin x < x < \tan x\,,\ 0 < x < \frac{\pi}{2} \qquad (8)$$

Divide by $\sin x$ and take the inverse, changing "$<$" to "$>$": $1 > \frac{\sin x}{x} > \cos x$. Since $\cos 0 = 1$, from "S.T."

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (9)$$

Examples (discussed in class):

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{10}$$

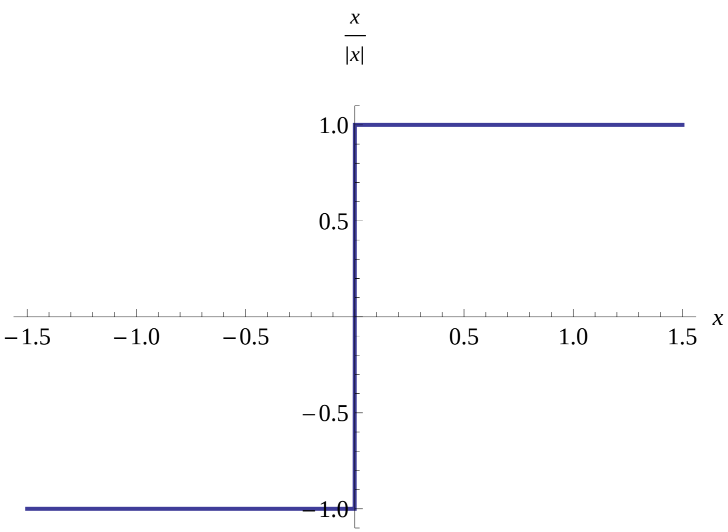
$$\lim_{x\to 0} \frac{\sin kx}{x} = k \tag{11}$$

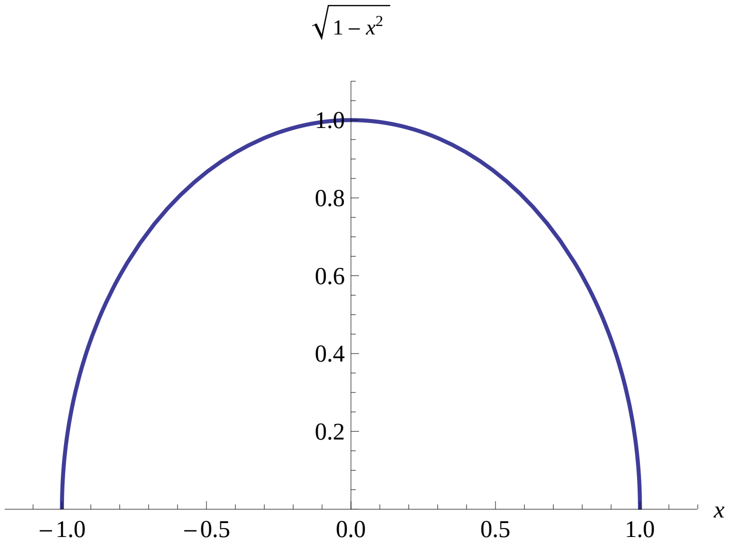
$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0 \tag{12}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \tag{13}$$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$$

One-sided limits (skip definition on p. 87)



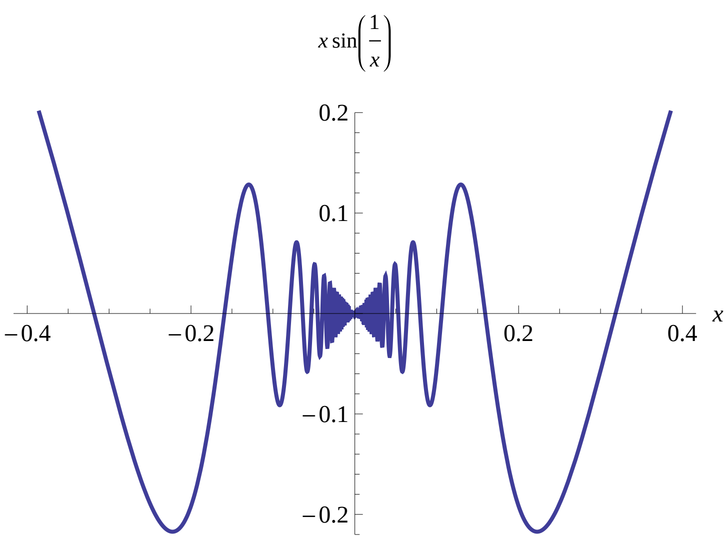


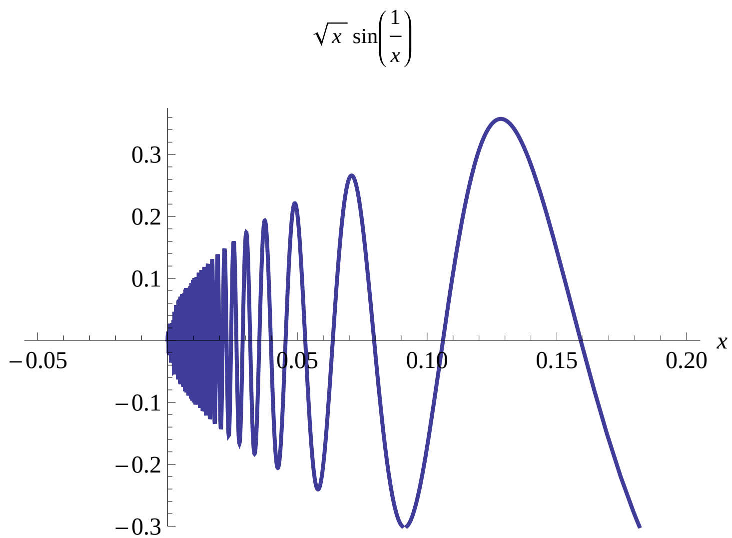
Functions with 1-sided limits.

$$\lim_{x\to 0^+}\frac{x}{|x|}=1\,,\quad \lim_{x\to 0^-}\frac{x}{|x|}=-1\,,\quad f(0)=?$$

$$\lim_{x\to -1^+}\sqrt{1-x^2}=0\,,\quad \lim_{x\to 1^-}\sqrt{1-x^2}=0$$

Theorem 6, p.86:
Limit of $f(x)$ for $x \to c$ exists if and only if L.H. and R.H. limits at $x \to c^-, c^+$ exist and have the same value (in which case, this also will be the value of the limit of f) .





Tricky functions with 2- or 1-sided limits.

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0\,, \quad \lim_{x\to 0^+} \sqrt{x}\sin\frac{1}{x} = 0$$

Continuity

at a point $x = c$ (p. 93, or Continuity Test, p. 94):

$$f(c) = \lim_{x \to c} f(x)$$

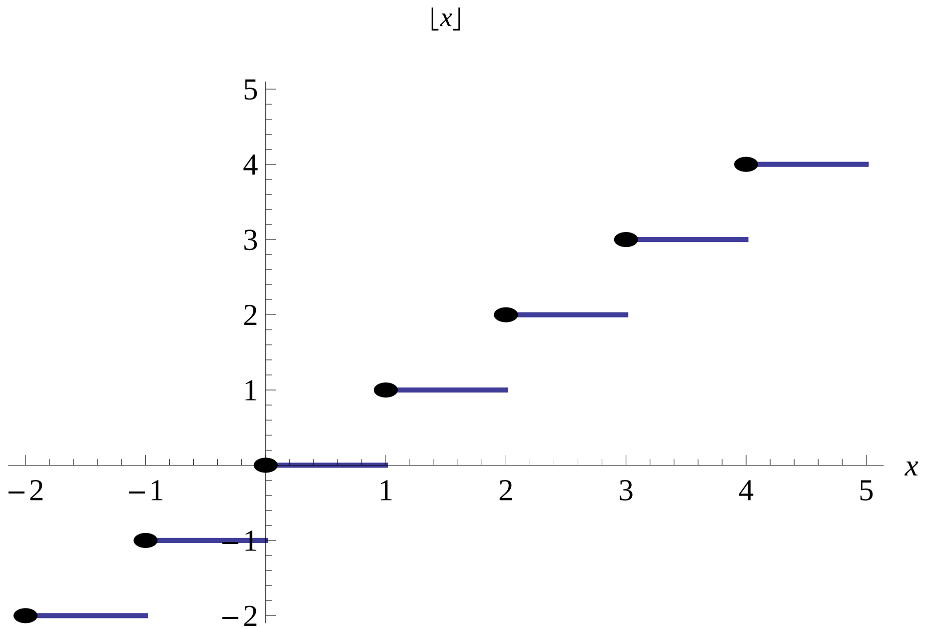
for an interior point (and similarly for an endpoint with corresponding one-sided limit).

Right-continuous at a point $x = c$:

$$f(c) = \lim_{x \to c^+} f(x)$$

Left-continuous at a point $x = c$:

$$f(c) = \lim_{x \to c^-} f(x)$$



right-continuous integer "floor" function

Continuous Function (CF)

CF on an interval
CF

Theorem 8, p. 95

$$f \pm g\,,\ etc.$$

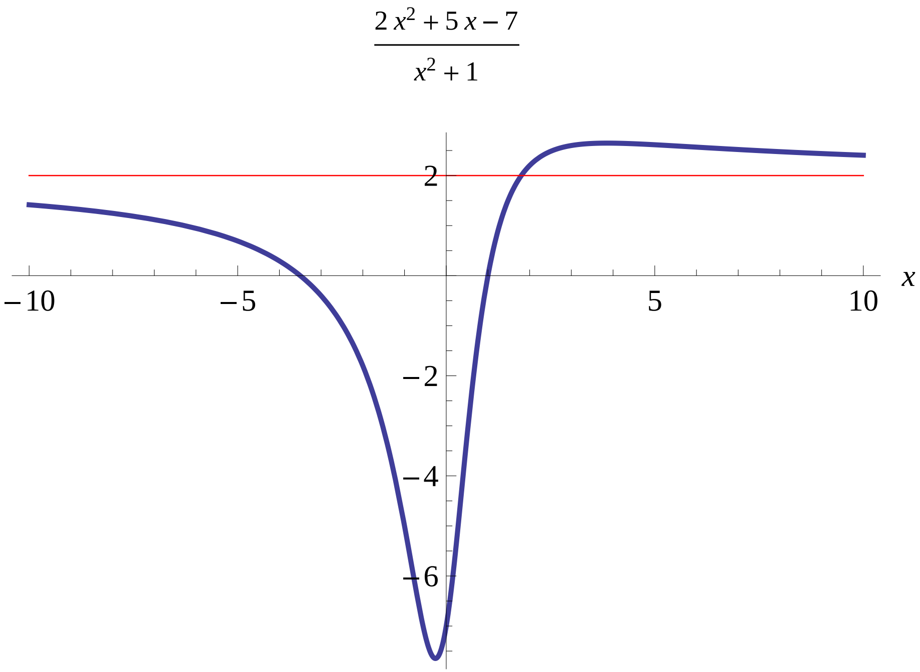
Inverse of a CF is a CF: e.g. e^x and $\ln x$

Composits of CF are CFs: e.g. $\sqrt{f(x)}\,, f(x) = 1 - x^2$
Theorems 9, 10 on p. 97

Continuous extension to a point: e.g. $F(x) = \sin x\,/x\,, x \neq 1; F(0) = 1$

Limits involving ∞

$x \to \pm\infty$ (skip p. 104)

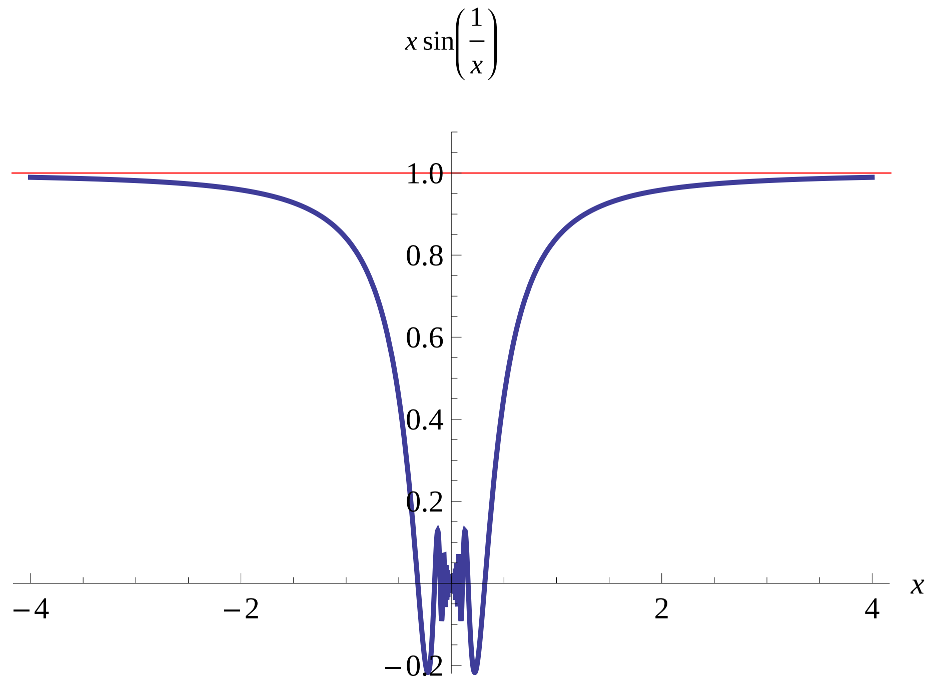


Limits of a rational function with the same power of denom. and numerator.

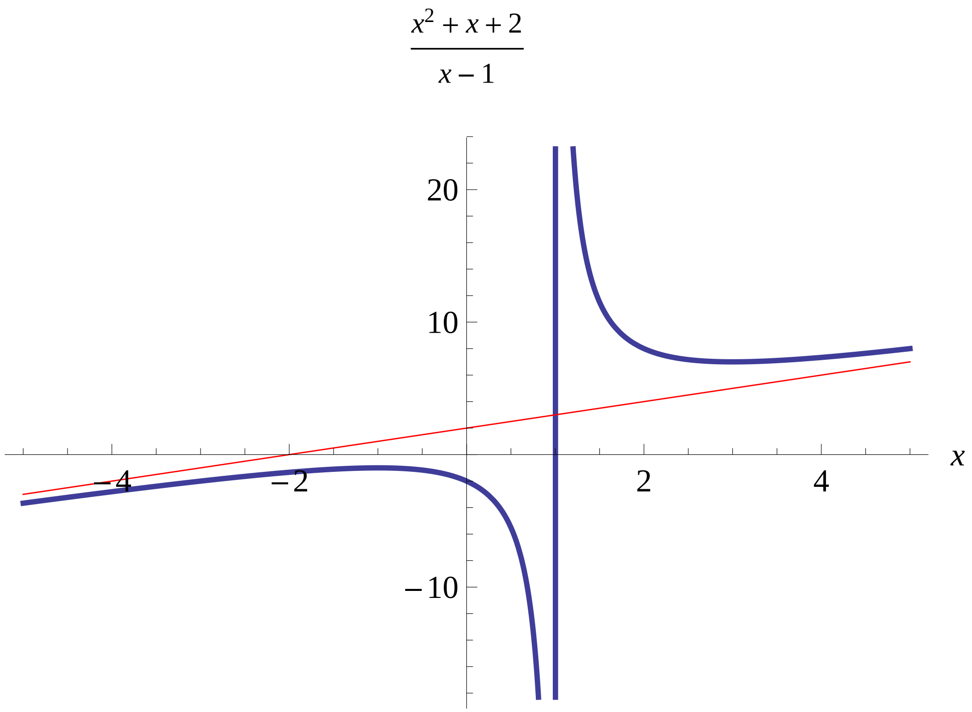
Horizontal asymptote(s) - lim(s) of $f(x)$ as $x \to \pm\infty$

Non-algebraic functions

$$\frac{x^3}{|x|^3+1}\,,\ e^x\,,\ x\sin\frac{1}{x}\,,\ \ etc.$$



Oblique and vertical asymptotes



Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \tag{14}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \tag{15}$$

If the limit exists, this is **derivative** at a point x, a.k.a:

- slope of the graph at x
- slope of the tangent to the graph at x

- rate of change of f with respect to x

Derivative as a function

x is not "frozen" anymore; can consider a new function

$$g(x) = f'(x)$$

Other notations

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \dots$$

$$g(a) = \left.\frac{df}{dx}\right|_{x=a}$$

Examples:

$$\frac{d}{dx}\frac{1}{x} = \lim_{z\to x}\frac{1/z - 1/x}{z - x} = \lim_{z\to x}\frac{x - z}{zx(z - x)} = -\frac{1}{x^2}$$

or

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \tag{16}$$

$$\frac{d}{dx}\frac{1}{x^2} = \lim_{z\to x}\frac{1/z^2 - 1/x^2}{z - x} = \lim_{z\to x}\frac{x^2 - z^2}{z^2x^2(z - x)} = -2\frac{1}{x^3}$$

$$\frac{d}{dx}\frac{x}{x-1} = \frac{d}{dx}\left(1 + \frac{1}{x-1}\right) = 0 - \frac{1}{(x-1)^2}$$

One-sided derivatives

From right:

$$f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x^+} \frac{f(z) - f(x)}{z - x}$$

From left:

$$f'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x^-} \frac{f(z) - f(x)}{z - x}$$

Example: $|x|' = 1$ for $x > 0$, $= -1$ for $x < 0$ and is not defined at $x = 0$ where only r.-h. or l.-h. derivatives exist. (This was a "corner"; other cases with no derivative may include "cusp" as $\sqrt{|x|}$, vertical tangent as $x^{1/3}$ and a discontinuity of the original function as $|x|/x$. Graphics in class.)

Theorem 3.1. If $f(x)$ has a derivative at $x = c$ it is continuous at $x = c$.

Consider identity

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} \cdot h$$

and take the limit of both sides as $h \to 0$.

Note: Converse is false, e.g. $|x|$.

Differentiation rules

$$(af(x) + bg(x))' = af' + bg' \tag{17}$$

Proof. Consider

$$\frac{af(z) + bg(z) - af(x) - bg(x)}{z - x} \equiv \frac{a[f(z) - f(x)] + b[g(z) - g(x)]}{z - x}$$

and take the limit $z \to x$.

$$(f(ax))' = a \left.\frac{df(z)}{dz}\right|_{z=ax} \tag{18}$$

Example: $(\sin(\omega x))' = \omega \cos(\omega x)$

$$[f(x) \cdot g(x)]' = f'g + fg' \tag{19}$$

Proof. Consider

$$\frac{f(z)g(z) - f(x)g(x)}{z - x} \equiv \frac{[f(z) - f(x)]g(z) + f(x)[g(z) - g(x)]}{z - x}$$

and take the limit $z \to x$.

Example:

$$(x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2} \tag{20}$$

Proof. Consider

$$\frac{1/f(z) - 1/f(x)}{z - x} \equiv \frac{f(x) - f(z)}{f(z)f(x)(z - x)}$$

and take the limit $z \to x$.

Example: $(1/x)' = -1/x^2$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2} \tag{21}$$

Derivatives of elementary functions

$$(x^n)' = nx^{n-1}\,, \text{ any } n \tag{22}$$

$$(\sin x)' = \cos x\,,\ (\cos x)' = -\sin x \tag{23}$$

$$(e^x)' = e^x\,,\ (a^x)' = \left(e^{x \ln a}\right)' = a^x \cdot \ln a \tag{24}$$

$$(\ln x)' = 1/x \tag{25}$$

Trigonometric functions

Earlier: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$ and

$$(\tan x)' = 1/\cos^2 x \tag{26}$$

Other basic examples: $1/\cos x$, $1/\sin x$, $\cot x$, ...

Simple harmonic motion:

$$y(t) = A\sin(\omega t)\ , \quad \frac{dy}{dt} = A\omega\cos(\omega t) \tag{27}$$

$$\frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t) \equiv -\omega^2 y(t) \tag{28}$$

Chain rule

$$(f(g(x)))' = \frac{df}{dg} \cdot \frac{dg}{dx} \equiv (f(g))' \cdot (g(x))' \qquad (29)$$

''*Proof*''. Let $u = g(z)$, $v = g(x)$

$$(f(g(x)))' = \lim_{z \to x} \frac{f(u) - f(v)}{z - x} = \lim_{z \to x} \frac{f(u) - f(v)}{u - v} \cdot \frac{u - v}{z - x} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Simple example: $(f(ax))' = af'(ax)$

$$(u(x)^n)' = nu^{n-1}\frac{du}{dx} \tag{30}$$

$$(\sin u(x))' = [\cos u(x)]\frac{du}{dx} \tag{31}$$

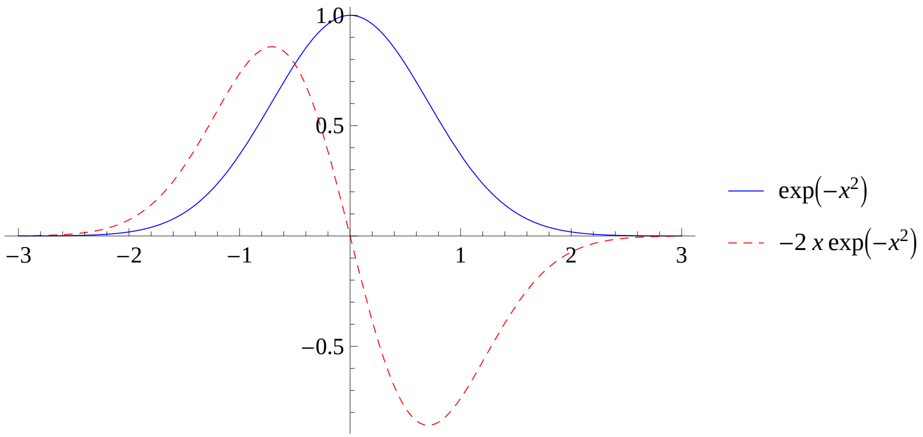
$$(e^u)' = e^u\frac{du}{dx} \tag{32}$$

$$(\ln u)' = \frac{1}{u}\frac{du}{dx} \tag{33}$$

$$\left(\sqrt{x^2+1}\right)' = \frac{1}{2\sqrt{x^2+1}} \cdot (2x)$$

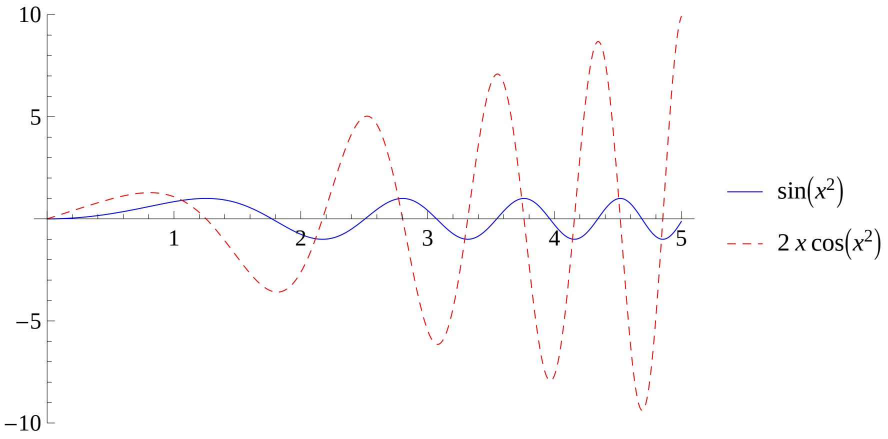
Example

$$\left(e^{-x^2}\right)' = -2xe^{-x^2}$$



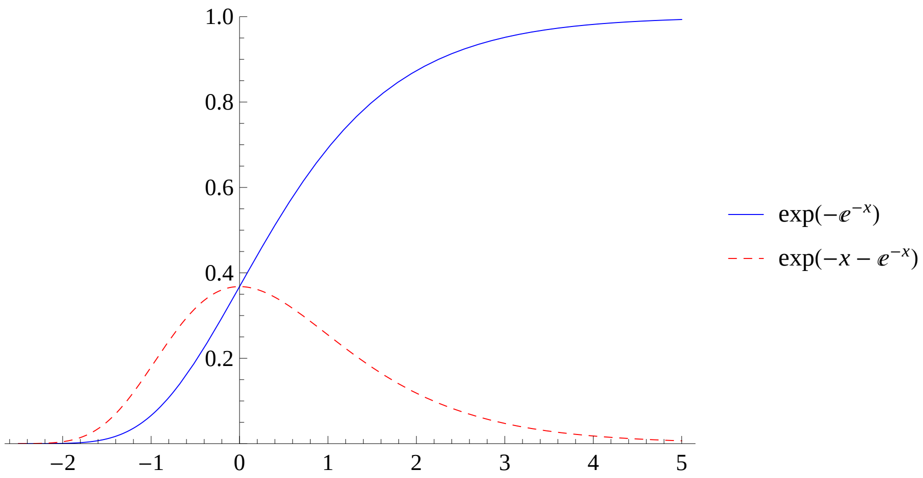
Example:

$$\left(\sin(x^2)\right)' = 2x\cos(x^2)$$



Example

$$\left(e^{-e^{-x}}\right)' = e^{-e^{-x}} \cdot \left(e^{-x}\right)$$



Implicit differentiation

Let $F(x, y) = 0$ where F is a simple known function (e.g. $F = x - y^2$ describing a parabola). Then

$$F'_x + F'_y \frac{dy}{dx} = 0 \tag{34}$$

and

$$\frac{dy}{dx} = -F'_x / F'_y$$

Tangent at $(x_0\,,\ y_0)$

$$Y = y_0 + \left.\frac{dy}{dx}\right|_{x=x_0} (x - x_0) \tag{35}$$

Normal:

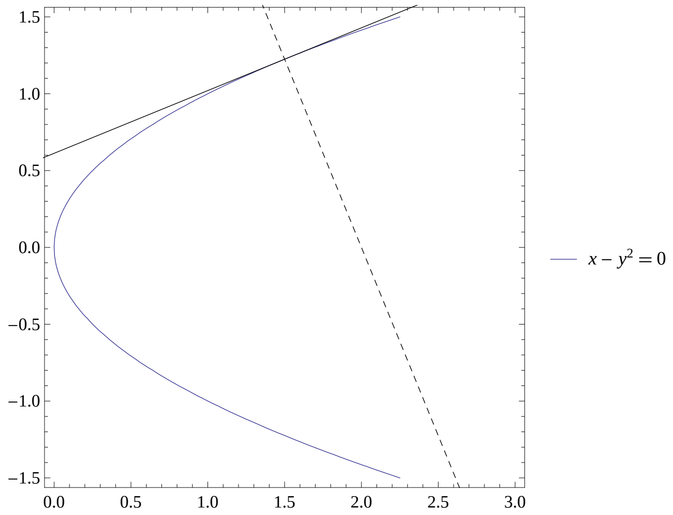
$$Y = y_0 - \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^{-1} (x - x_0) \tag{36}$$

Parabola:

$$y^2 = x \ , \ 2y\frac{dy}{dx} = 1 \ , \ \ \frac{dy}{dx} = \frac{1}{2y}$$

Tangent and normal:

$$Y = y_0 + \frac{1}{2y_0}(x - x_0) \ , \ \ Y = y_0 - (2y_0)(x - x_0)$$

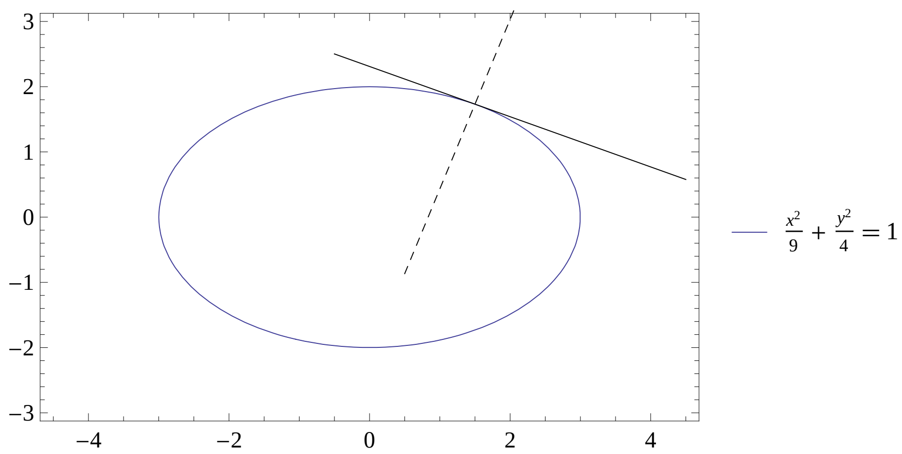


Ellipse:

$$(x/a)^2+(y/b)^2=1\ ,\ 2x/a^2+2y/b^2\frac{dy}{dx}=0\ ,\ \ \frac{dy}{dx}=-\frac{x}{y}\frac{b^2}{a^2}$$

Tangent and normal:

$$Y=y_0-\frac{x_0}{y_0}\frac{b^2}{a^2}(x-x_0)\ ,\ \ Y=y_0+\frac{y_0}{x_0}\frac{a^2}{b^2}(x-x_0)$$

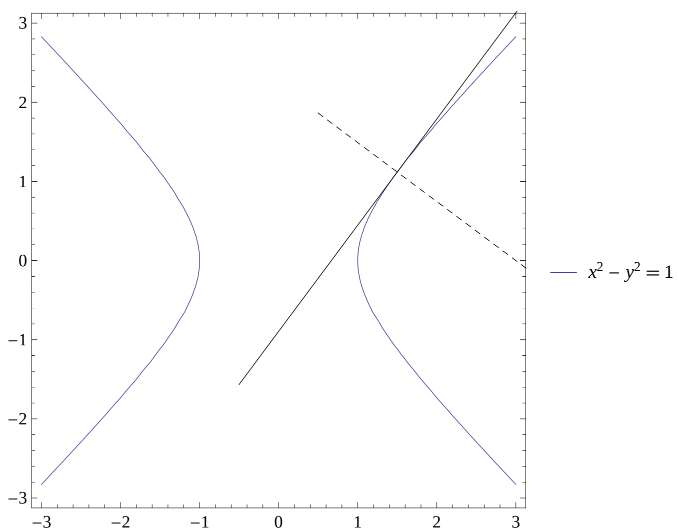


Hyperbola:

$$x^2 - y^2 = 1 \ , \ 2x - 2y\frac{dy}{dx} = 0 \ , \quad \frac{dy}{dx} = \frac{x}{y}$$

Tangent and normal:

$$Y = y_0 + \frac{x_0}{y_0}(x - x_0) \ , \quad Y = y_0 - \frac{y_0}{x_0}(x - x_0)$$



Inverse Functions

Consider $y = f(x)$ and solve for $x = F(y)$ (select only one branch, if several). Then, swap y and x. Then,

$$y = F(x)\,, \text{ with } (f(F(x))) = x$$

is the *inverse* function ($F \equiv f^{-1}$ in textbook).

Examples: $f = x^2$, $F = \sqrt{x}$, $f = e^x$, $F = \ln x$, $f = \sin x$, $F = \arcsin x$, etc.

Note: range of f becomes the domain of F, but the domain of f becomes range of F for monotonic f only. Examples: e^x (monotonic) and x^2 or $\sin x$ (non-monotonic).

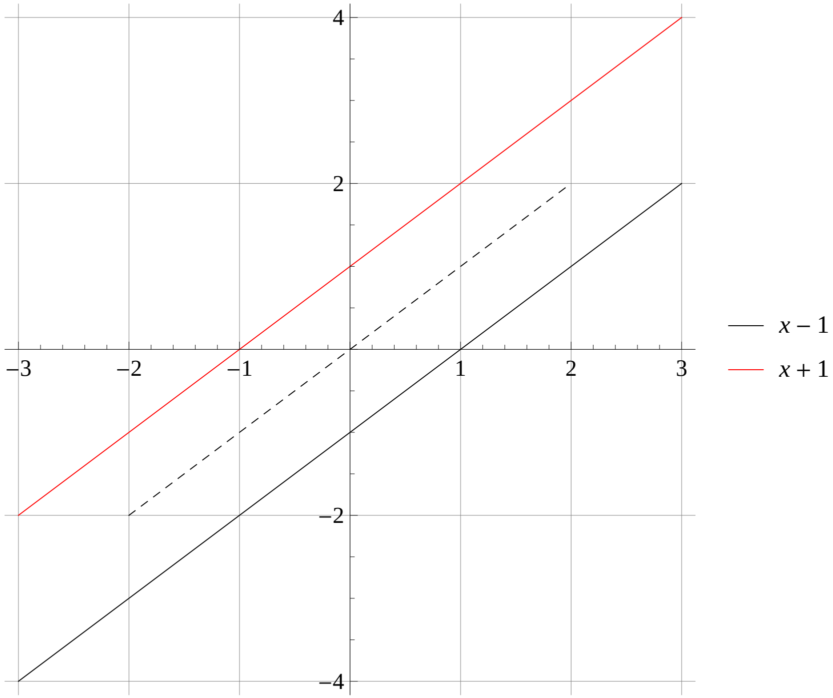
Note $(f(F(x))) = x$ for all x in the domain of F, while $F(f(x)) = x$ in the entire domain of f for monotonic f only. E.g.:

$e^{\ln x} = x$ for $0 < x < \infty$, and $\ln(e^x) = x$ for $-\infty < x < \infty$

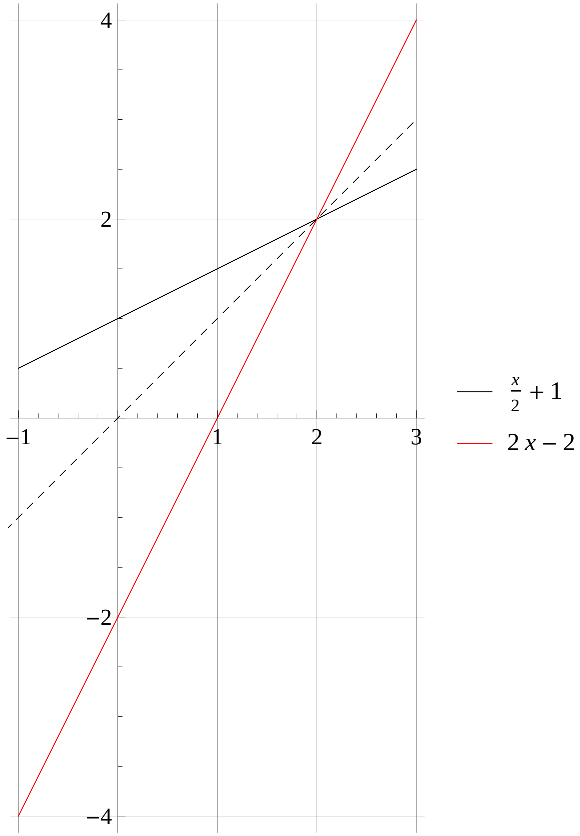
$$\sin(\arcsin x) = x \text{ for } -1 \leq x \leq 1\,, \text{ but}$$

$$\arcsin(\sin x) \neq x\,, \text{ for } |x| > \pi/2$$

Primitive example: $f = x - 1\,,\ F = x + 1$

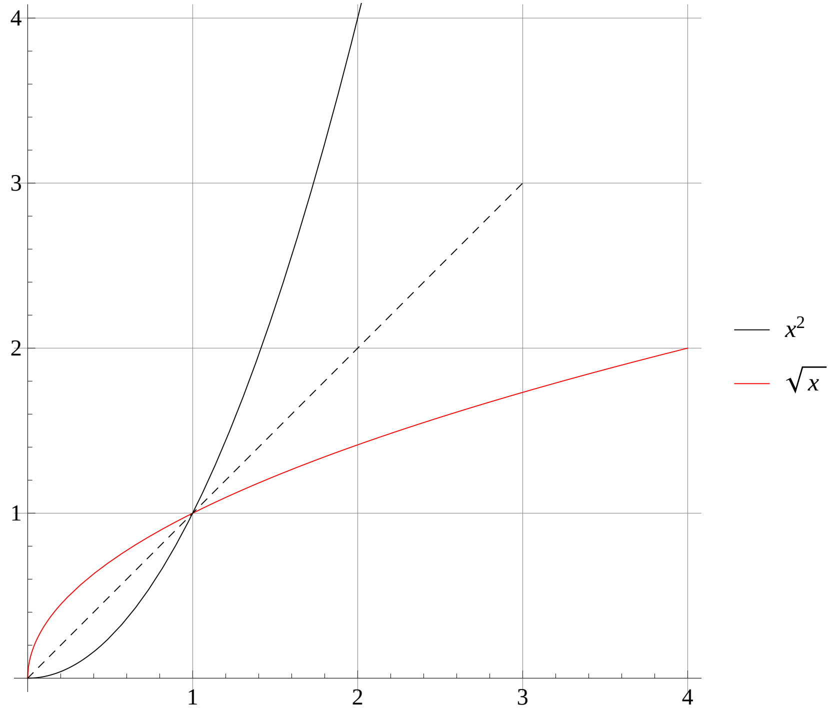


Primitive example: $f = x/2 + 1\,,\; F = 2x - 2$



$$f(F(x)) = \frac{1}{2} \cdot (2x - 2) + 1 = x$$

Example: $f = x^2$, $F = \sqrt{x}$



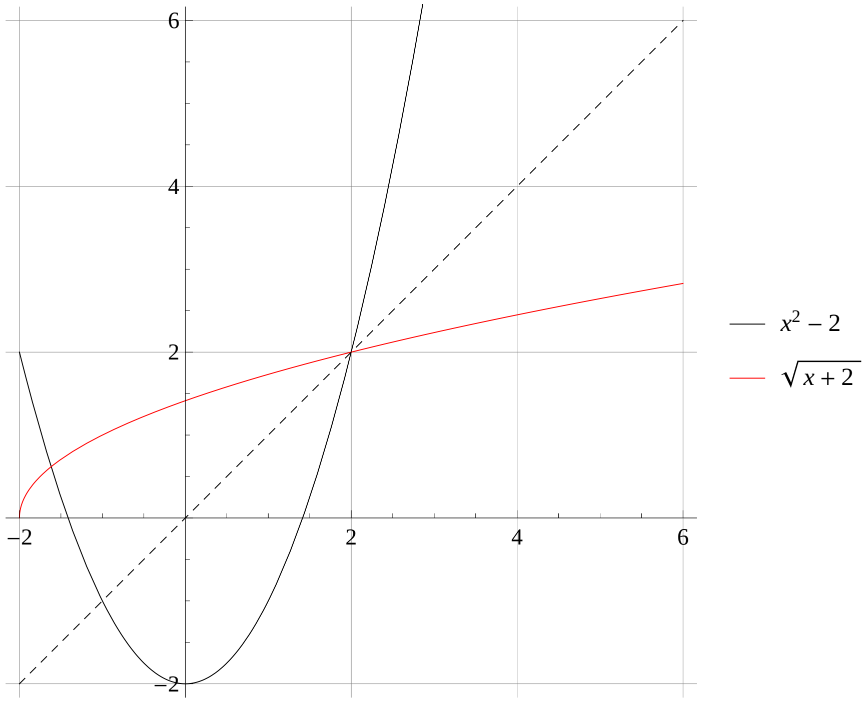
$(f)' = 2x$, $(F)' = \frac{1}{2\sqrt{x}}$

Consider $(x_0 = 2, y_0 = 4)$,

$(X_0 = 4, Y_0 = 2)$:

$(f)' = 4$, $(F)' = \frac{1}{4}$

Example: $f = x^2 - 2\,,\ F = \sqrt{x+2}$



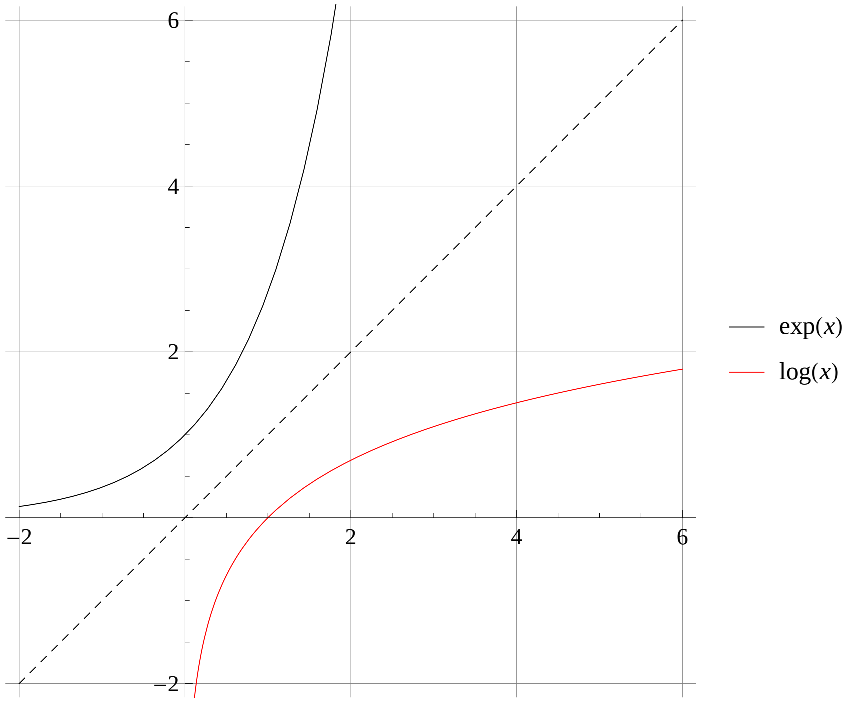
$(f)' = 2x\,,\ (F)' = \frac{1}{2\sqrt{x+2}}$

Consider $(x_0 = 3\,,\ y_0 = 7)\ ,$

$(X_0 = 7, Y_0 = 3)$:

$(f)' = 6\,,\ (F)' = \frac{1}{6}$

Example: $f = e^x$, $F = \ln x$



Logarithm

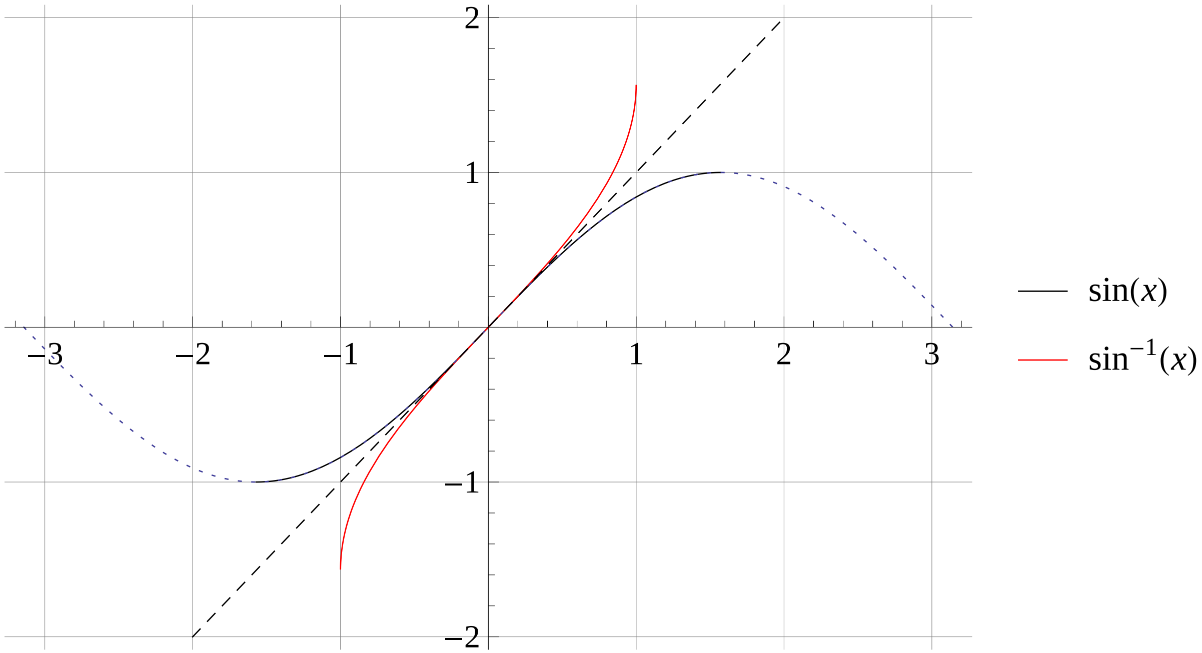
$$y = \ln x$$

$$e^y = x$$

$$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

Example: $f = \sin x$, $F = \arcsin x$



arcsin

$$y = \arcsin x$$

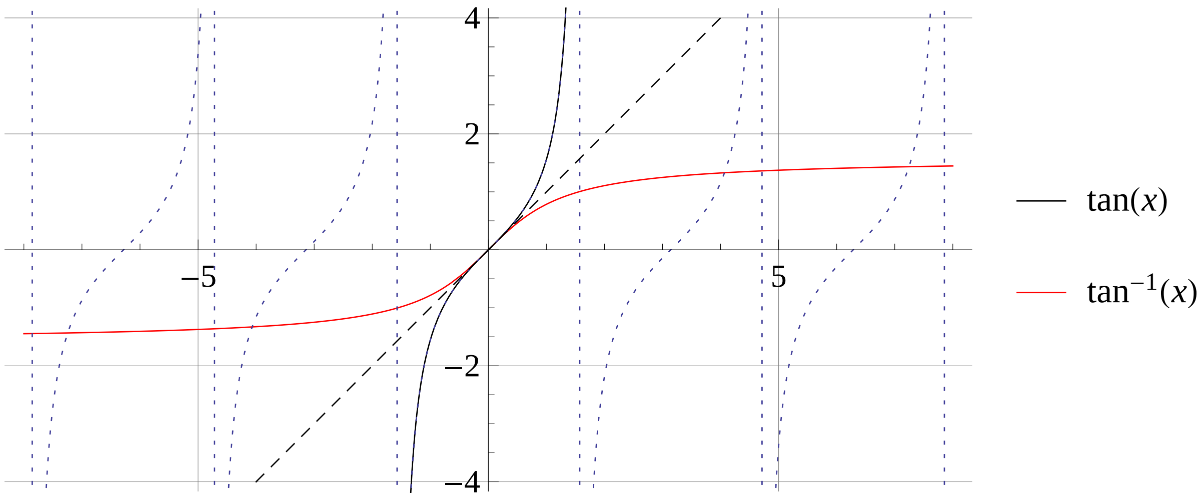
$$\sin y = x$$

$$(\cos y) \cdot y' = 1$$

$$y' = \frac{1}{\cos y} =$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Example: $f = \tan x$, $F = \arctan x$



arctan

$$y = \arctan x$$

$$\tan y = x$$

$$(1/\cos^2 y) \cdot y' = 1$$

$$y' = \frac{1}{1/\cos^2 y} =$$

$$= \cos^2 y = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2 + 1}$$

General

$$y = F(x)$$

$$f(y) = x$$

$$\frac{df}{dy} \cdot y' = 1$$

$$y' \equiv \frac{dF}{dx} = \left. \frac{1}{\frac{df}{dy}} \right|_{y=F(x)}$$

Logarithmic functions and differentiation

$$(\ln u(x))' = \frac{1}{u} \cdot (u)'$$

Examples

$$(\ln bx)' = \frac{b}{bx} = \frac{1}{x}$$

$$(\ln |x|)' = \frac{1}{|x|} \cdot (|x|)' = \frac{1}{x}, \; x \neq 0$$

Recall:

$$a^x = e^{x \ln a}\,,\ \log_a x = \frac{\ln x}{\ln a}\,,\ \ a > 0\,, a \neq 1 \qquad (37)$$

$$(a^x)' = \left(e^{x \ln a}\right)' = a^x \cdot \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(x^n)' = \left(e^{n \ln x}\right)' = \frac{n}{x} e^{n \ln x} = n x^{n-1}$$

$$(x^x)' = \left(e^{x \ln x}\right)' = e^{x \ln x}\left(\ln x + \frac{x}{x}\right) = x^x(\ln x + 1)$$

Remarkable limit:

$$\lim_{x \to 0} (1 + x)^{1/x} = e \tag{38}$$

Logarithmic differentiation

Let

$$y = \frac{P(x)Q(x)}{R(x)}$$

$$\ln y = \ln P + \ln Q - \ln R$$

$$y'/y = \frac{(P)'}{P} + \frac{(Q)'}{Q} - \frac{(R)'}{R}$$

$$y' = \frac{P(x)Q(x)}{R(x)} \cdot \left[\frac{(P)'}{P} + \frac{(Q)'}{Q} - \frac{(R)'}{R}\right]$$

Inverse trigonometric functions

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}\,, \; |x| < 1 \tag{39}$$

$$(\arctan x)' = \frac{1}{x^2+1}\,, \; -\infty < x < \infty \tag{40}$$

Now, $\arccos x = \frac{\pi}{2} - \arcsin x$ and $\cot^{-1} x = \frac{\pi}{2} - \arctan x$.
Thus

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}\,, \; |x| < 1 \tag{41}$$

$$\left(\cot^{-1} x\right)' = -\frac{1}{x^2+1}\,, \; -\infty < x < \infty \tag{42}$$

$$\sec^{-1}(x) = \arccos\frac{1}{x}\,, \;\; \csc^{-1}(x) = \arcsin\frac{1}{x}$$

$$\left(\sec^{-1}(x)\right)' = \left(\arccos\frac{1}{x}\right)' = -\frac{1}{\sqrt{1-(1/x)^2}} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}\,, \;\; |x| > 1$$

$$\left(\csc^{-1}(x)\right)' = \left(\arcsin\frac{1}{x}\right)' = \frac{1}{\sqrt{1-(1/x)^2}} \cdot \frac{-1}{x^2}$$

$$= -\frac{1}{|x|\sqrt{x^2-1}}\,, \;\; |x| > 1$$

Related rates

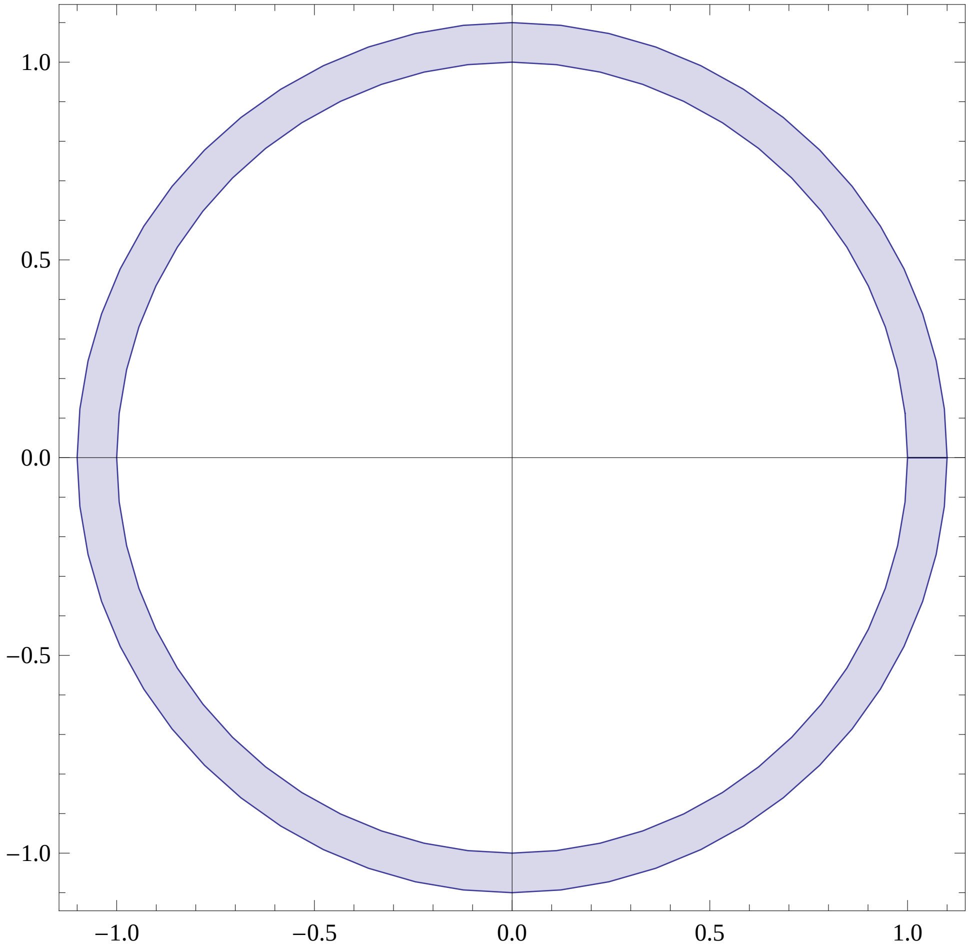
Suggested notations:
$\mathcal{V}$ - volume, A - area, L - length/distance, r , R - radii,
y Y, h, H - vertical position, x, X -horizontal position,
v, V velocity, t - time

Sphere:

$$\mathcal{V} = \frac{4}{3}\pi R^3\,,\ A = 4\pi R^2 \tag{43}$$

Cone:

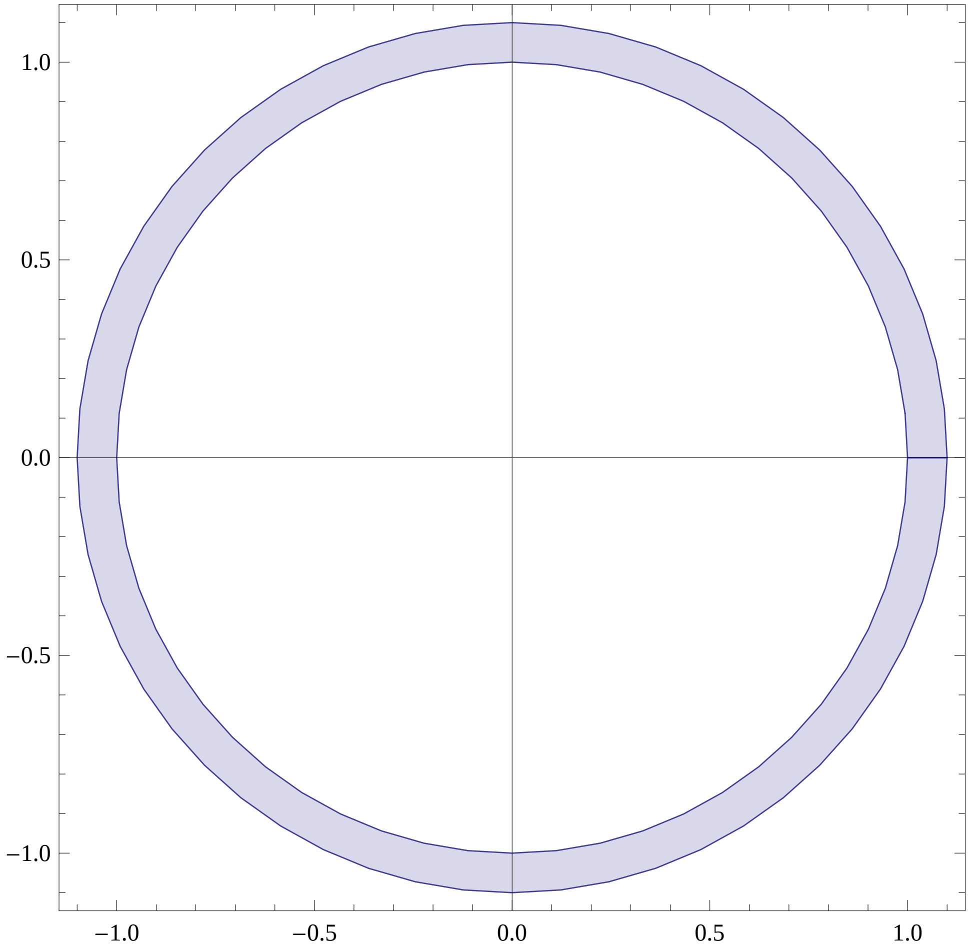
$$\mathcal{V} = \frac{1}{3}A_{base} \cdot h \tag{44}$$



Circle:

$A = \pi r^2$

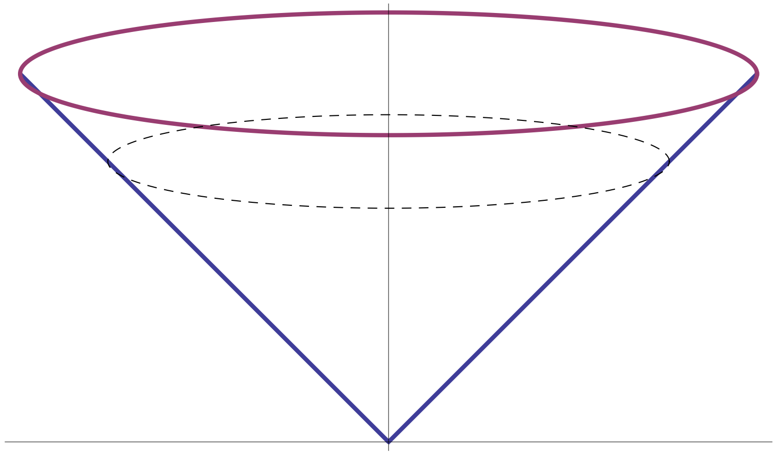
$dA/dt = 2\pi r \cdot dr/dt$:



Sphere:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} = A \cdot \frac{dr}{dt}$$



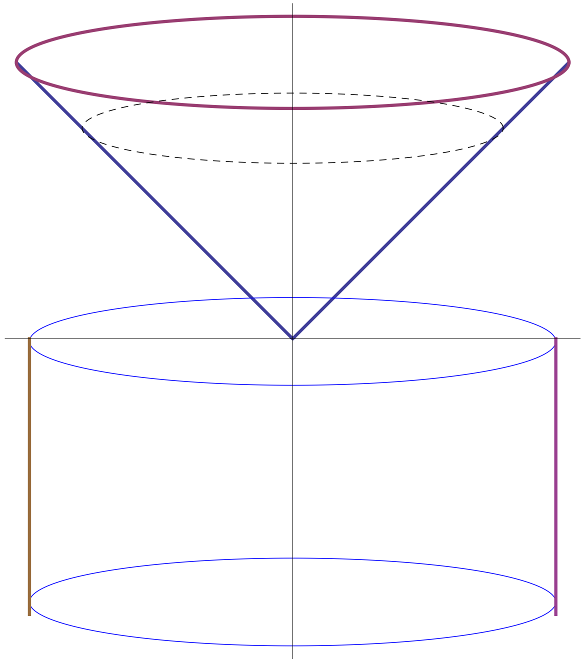
Cone:

$$V = \frac{1}{3}\pi r^2(h) \cdot h =$$

$$= \frac{1}{3}\pi \left(R(H)\frac{h}{H}\right)^2 \cdot h = \frac{V_0}{H^3} \cdot h^3$$

$$\frac{dV}{dt} = \frac{V_0}{H^3} \cdot 3h^2 \frac{dh}{dt} = A(h) \cdot \frac{dh}{dt}$$

"Coffee maker" :



$$V_1 + V_2 = const\,, \ \frac{dV_1}{dt} + \frac{dV_2}{dt} = 0$$

$$\frac{dV_1}{dt} = \pi r^2(h) \cdot \frac{dh}{dt}$$

$$\frac{dV_2}{dt} = \pi R^2 \cdot \frac{dY}{dt}$$

$$\frac{dY}{dt} = -\frac{1}{R^2} r^2(h) \cdot \frac{dh}{dt}$$

An $a(t) \times b(t)$ rectangle:

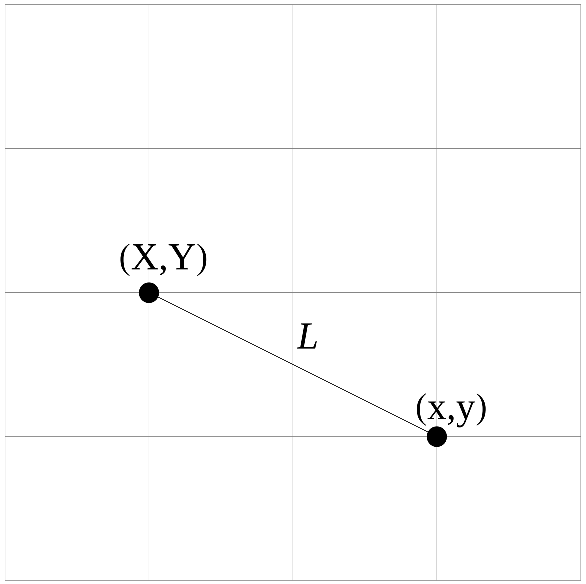
$$A = ab\,,\ P = 2(a+b)\,,\ D = \sqrt{a^2+b^2}$$

$$\frac{dA}{dt} = a\frac{db}{dt} + b\frac{da}{dt}$$

$$\frac{dP}{dt} = 2\left(\frac{da}{dt} + \frac{db}{dt}\right)$$

$$\frac{dD}{dt} = \frac{1}{2D}\frac{d}{dt}\left(a^2+b^2\right) = \frac{1}{D}\left(a\frac{da}{dt} + b\frac{db}{dt}\right)$$

Distance between two moving points:



$$L = \sqrt{(x - X)^2 + (y - Y)^2}$$

$$\frac{dL}{dt} = \frac{1}{2L}\frac{d}{dt}\Big((x - X)^2 + (y - Y)^2\Big) =$$

$$= \frac{1}{L}\left[(x - X)(\frac{dx}{dt} - \frac{dX}{dt}) + (y - Y)(\frac{dy}{dt} - \frac{dY}{dt})\right]$$

Police car:

$$X \equiv 0,\ dY/dt = -V\ ,\ y \equiv 0,\ dx/dt = v$$

$$\frac{dL}{dt} = \frac{1}{2L}\frac{d}{dt}\left(x^2 + Y^2\right) =$$

$$= \frac{1}{L}\left[x\frac{dx}{dt} + Y\frac{dY}{dt}\right] = \frac{1}{L}(xv - YV)$$

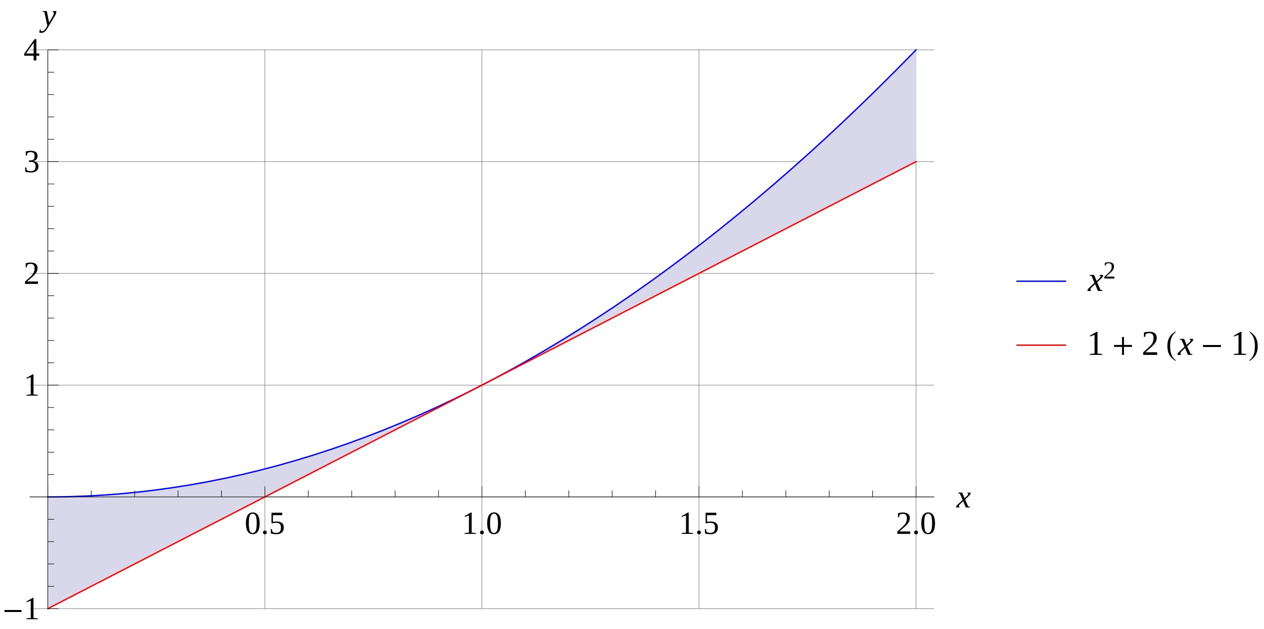
$$v = \frac{1}{x}(L \cdot dL/dt + YV)$$

Angle with horizontal ($X, Y = 0$, $L = \sqrt{x^2 + y^2} = x/\cos\theta$):

$$\tan\theta = y/x\,,\ \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{L^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

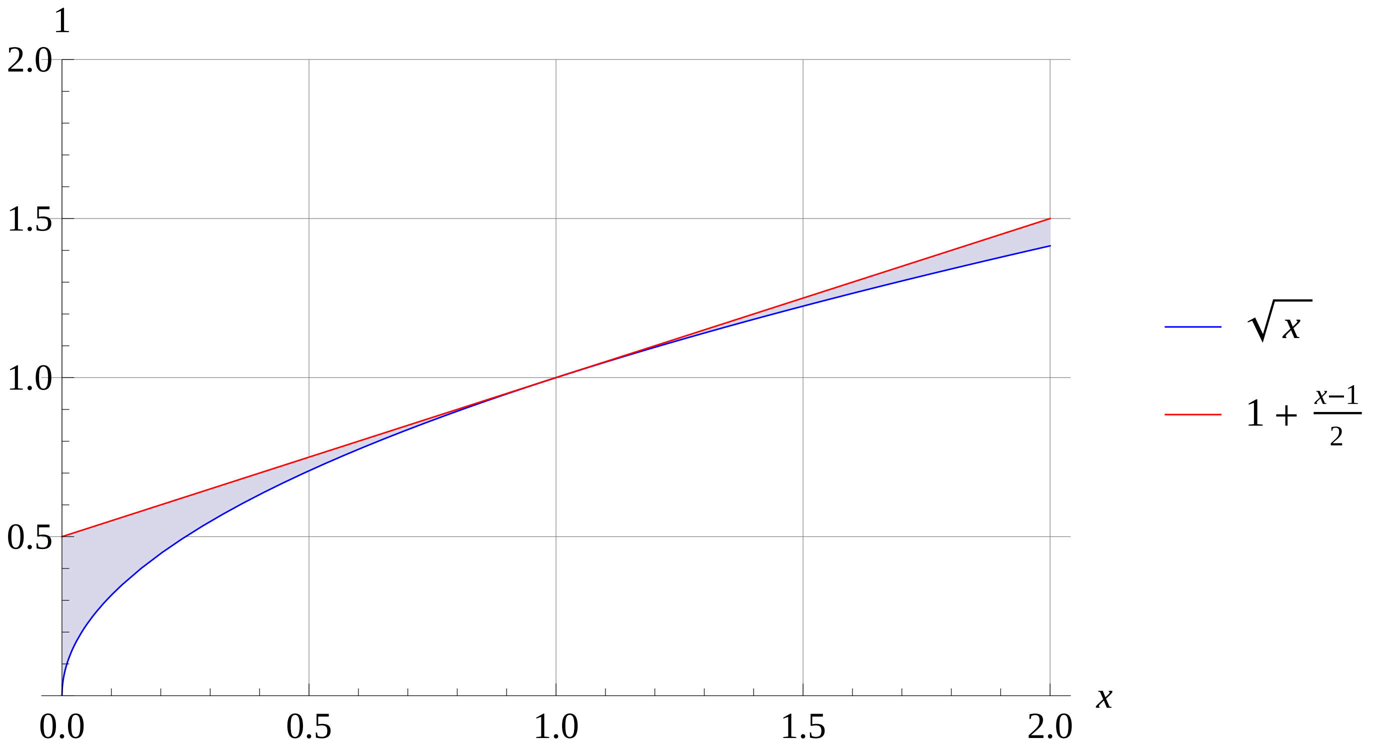
Balloon: $x = const\,,\ dy/dt = V\,.$
Aircraft: $y = const\,,\ dx/dt = -v.$



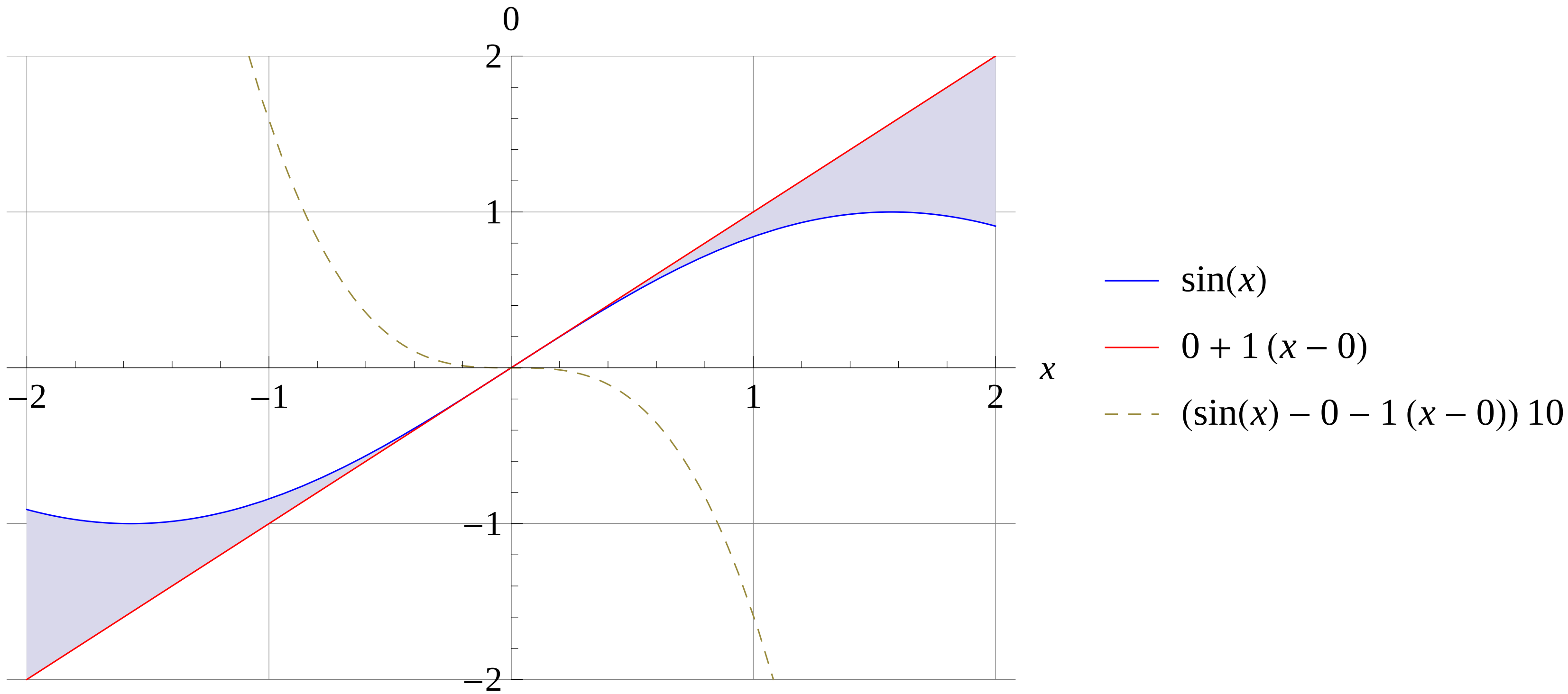
Tangent line and error (for a parabola):

$$Y(x) = y(a) + y'(a)(x - a)$$

$$y(x) - Y(x) = x^2 - a^2 - 2a(x - a) = (x - a)^2$$



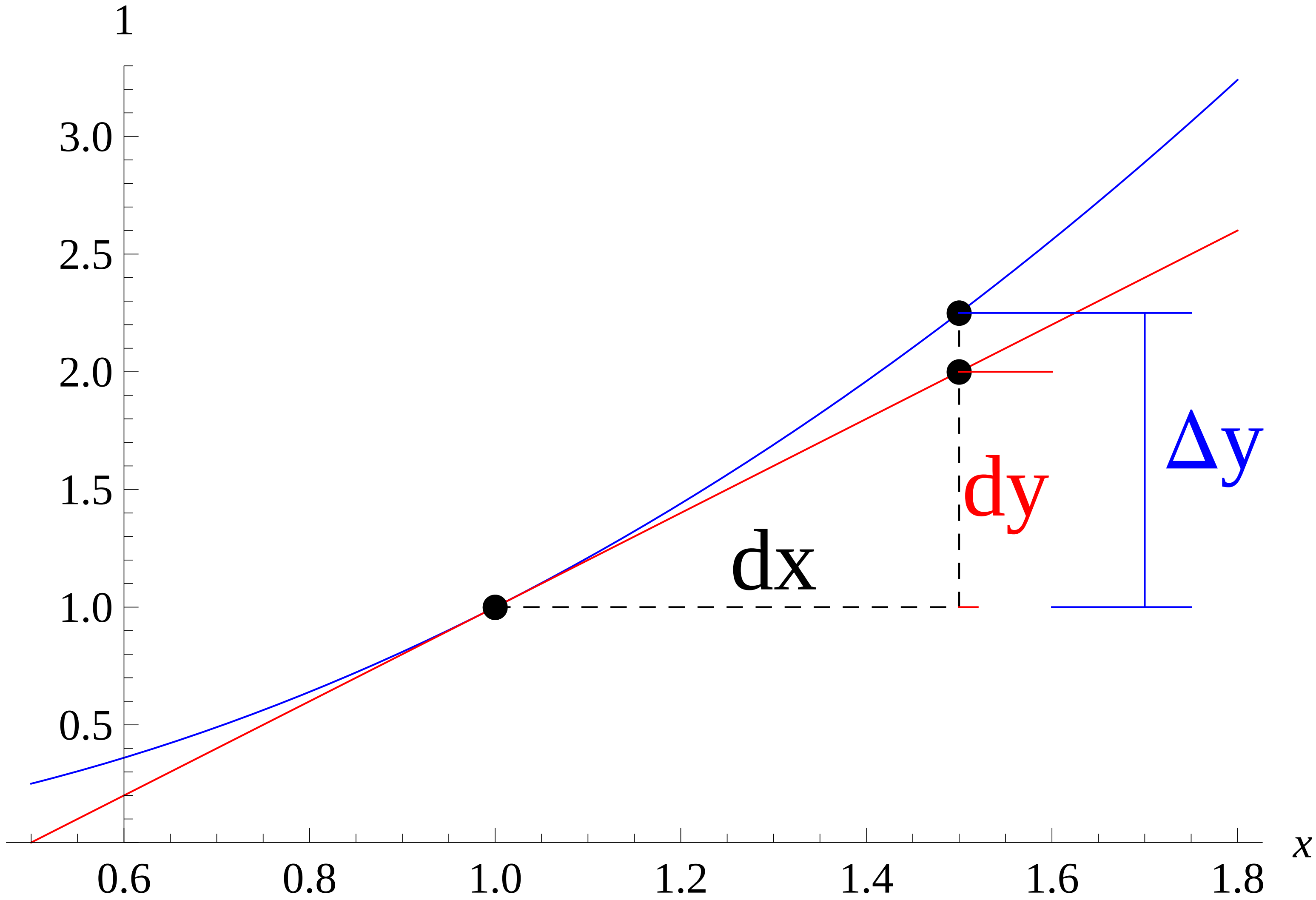
1
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
x
$\sqrt{x}$
$1 + \frac{x-1}{2}$


0
2
1
−1
−2
x
−2
−1
1
2
sin(x)
0 + 1 (x − 0)
(sin(x) − 0 − 1 (x − 0)) 10

Differential:

dx - independent variable

$$dy = y'(x)\,dx$$


1
3.0
2.5
2.0
1.5
1.0
0.5
0.6
0.8
1.0
1.2
1.4
1.6
1.8
x
dx
dy
Δy

$$f(x+dx) \simeq f(x) + df = f(x) + f'(x) \cdot dx$$

Examples (small x instead of dx; approximation only near $x = 0$):

$$(1+x)^2 \simeq 1 + 2x \tag{45}$$

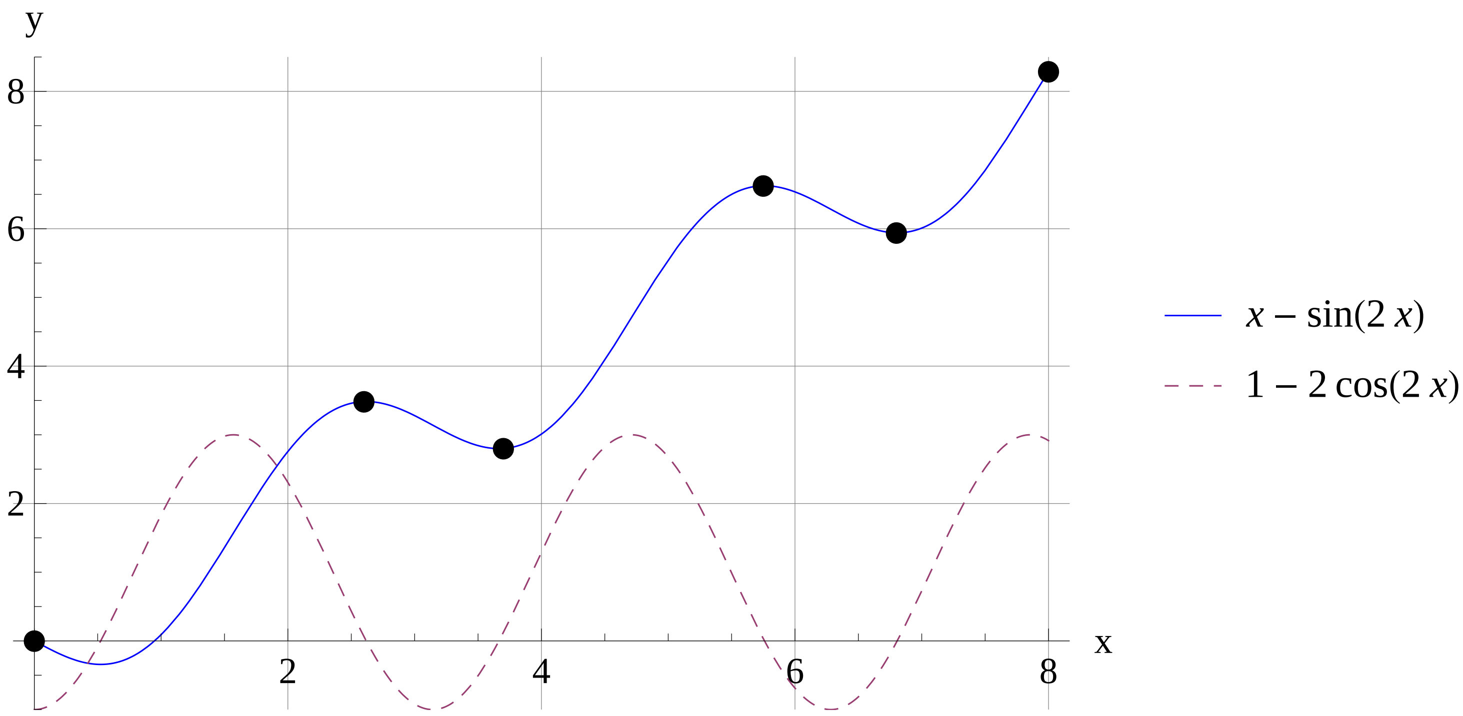
$$1/(1+x) \simeq 1 - x \tag{46}$$

$$\sqrt{1+x} \simeq 1 + x/2 \tag{47}$$

$$(1+x)^k \simeq 1 + kx \tag{48}$$

$$e^x \simeq 1 + x \tag{49}$$

$$\ln(1+x) \simeq x \tag{50}$$


y
8
6
4
2
2
4
6
8
x
$x - \sin(2\,x)$
$1 - 2\cos(2\,x)$

Absolute maximum:

$$f(c) = M\,,\; f(x) \leq M$$

for every x in the domain of f.

Theorem. On a closed domain $[a, b]$ any *continuous* $f(x)$ will have an absolute maximum.

Local maximum:

$$f(c) = M\,,\ f(x) \leq M$$

for every x on some open interval containing c.

Theorem. For any *differentiable* $f(x)$ a local maximum will have a zero derivative:

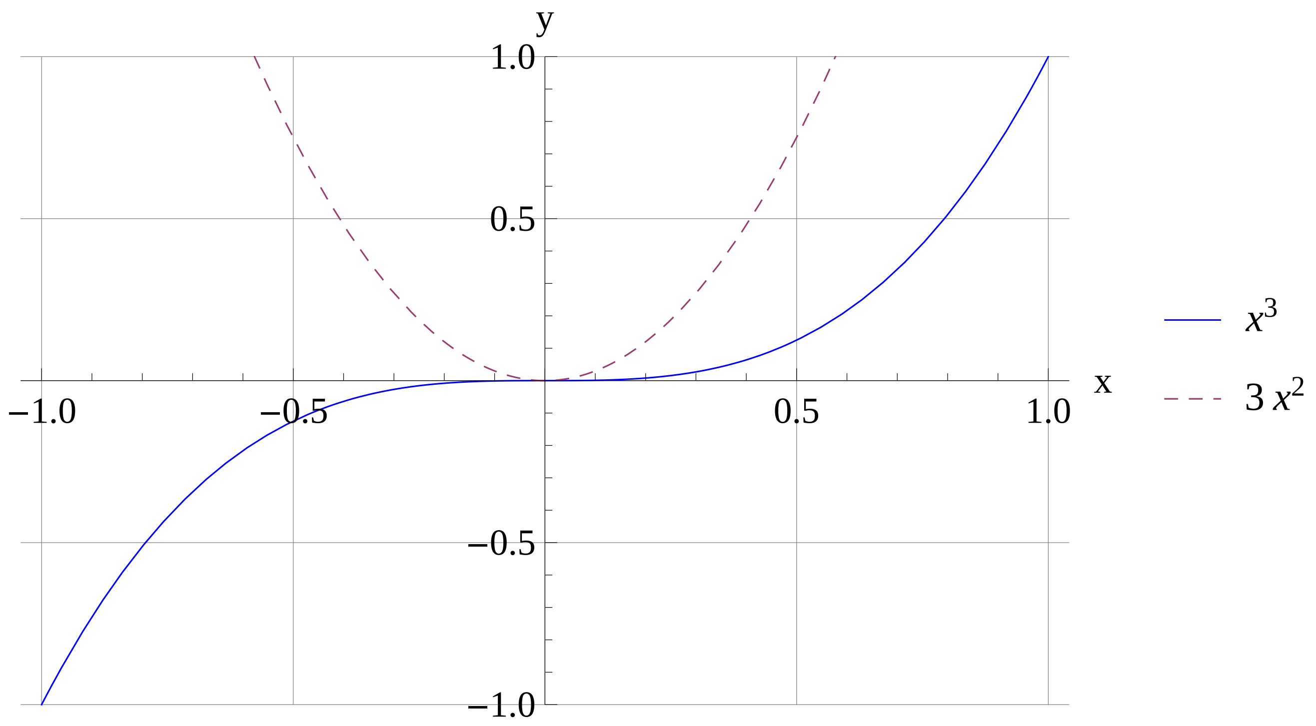
$$f(c) = M\,,\ (f(x))'_{x=c} = 0 \tag{51}$$

(but, the reverse can be false: e.g. x^3)

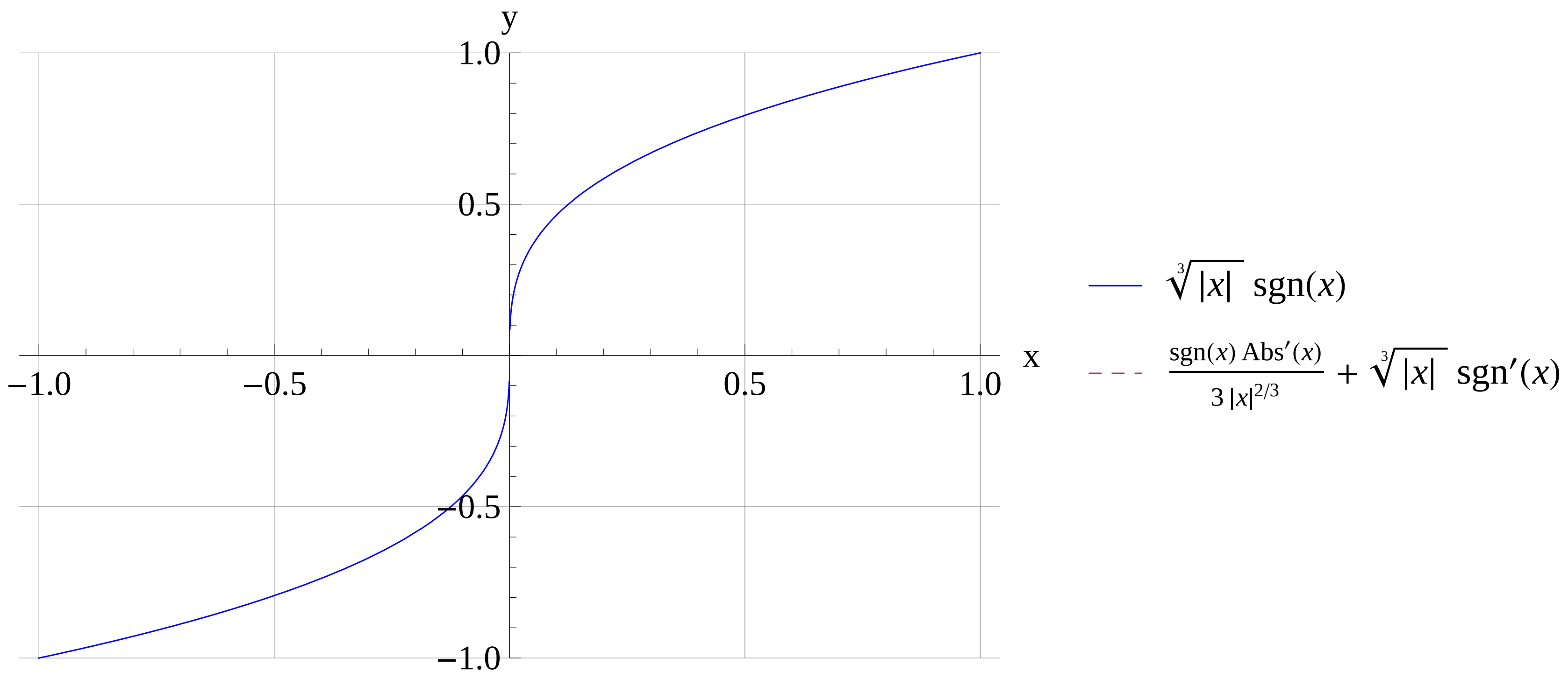
Critical and end points:

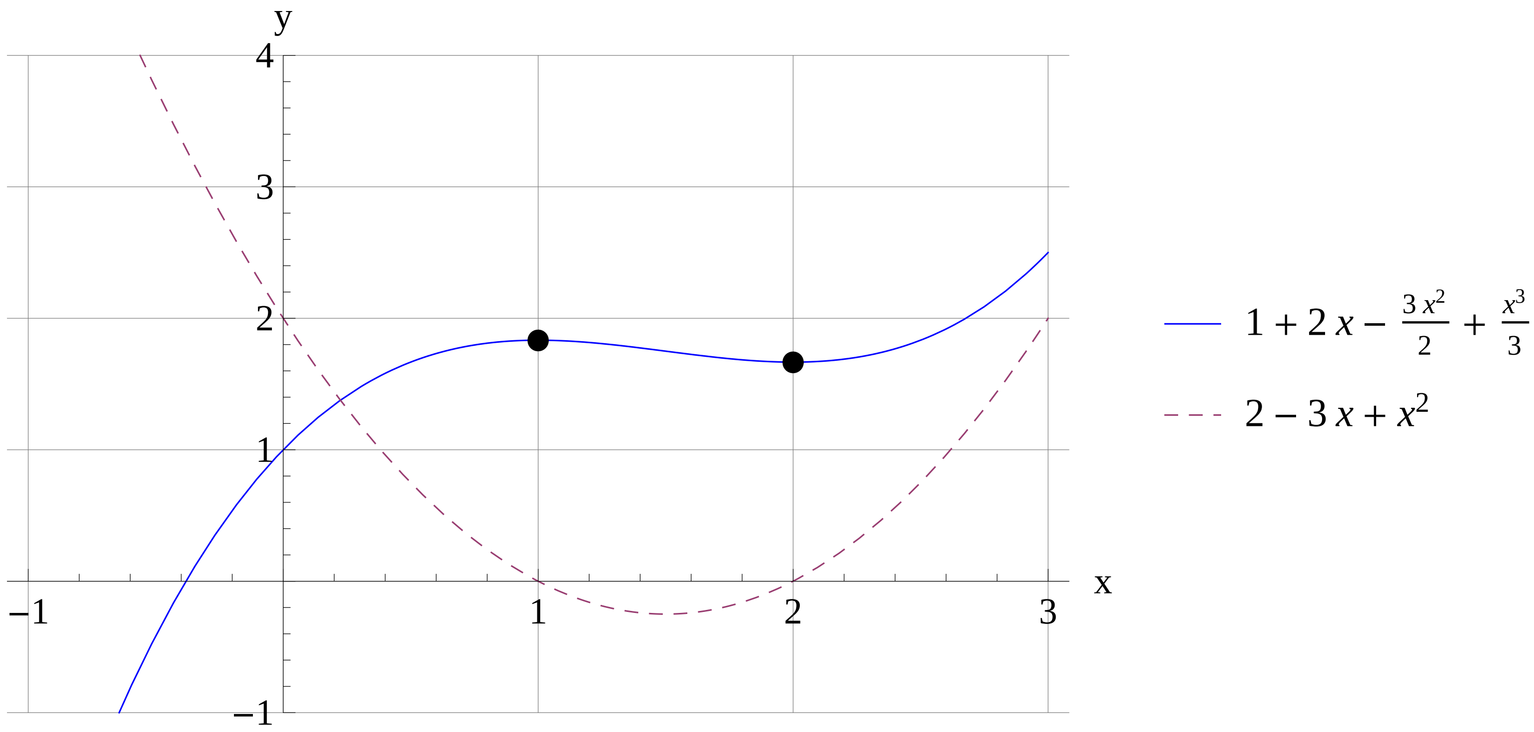
- endpoint(s)
- zero derivative
- no derivative (e.g $|x|$ or $\sqrt{|x|}$)

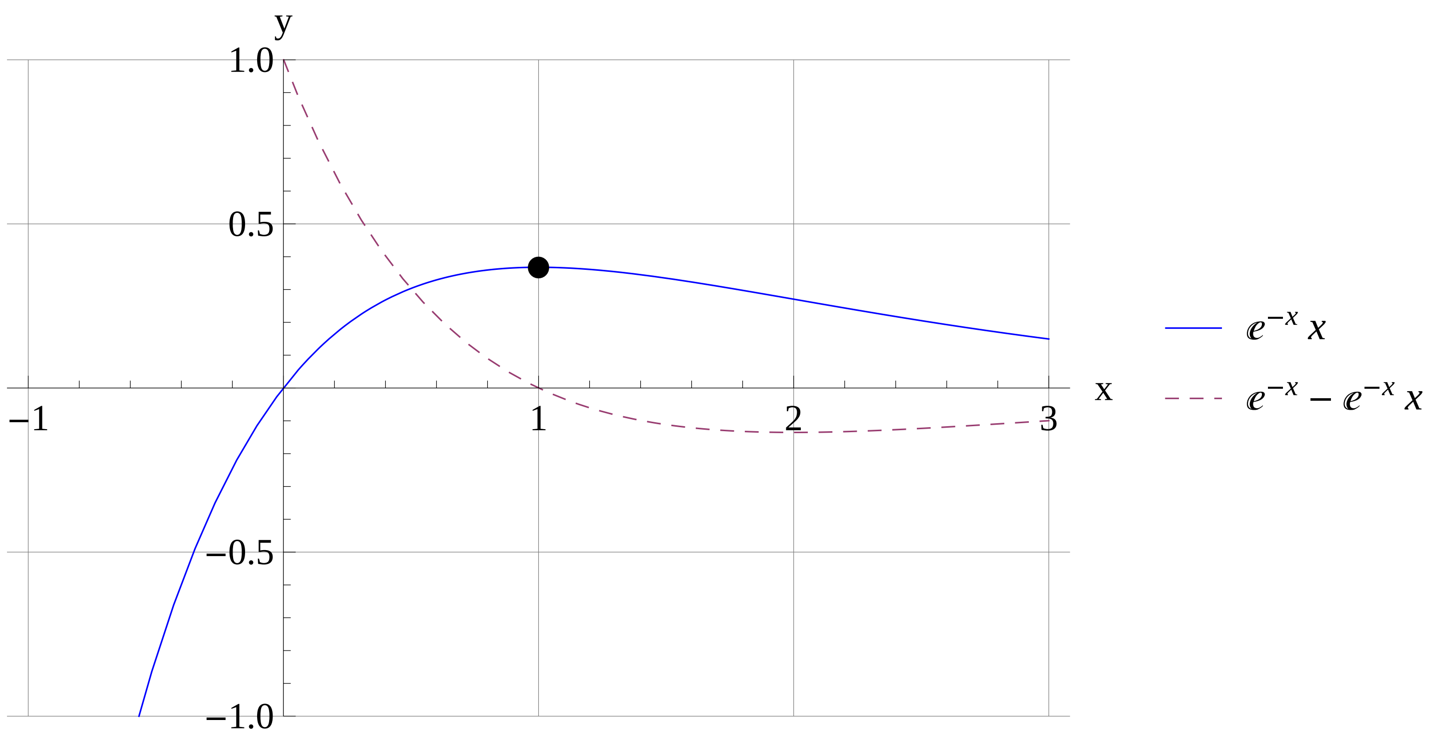
E.g.: critical point - no extrema:

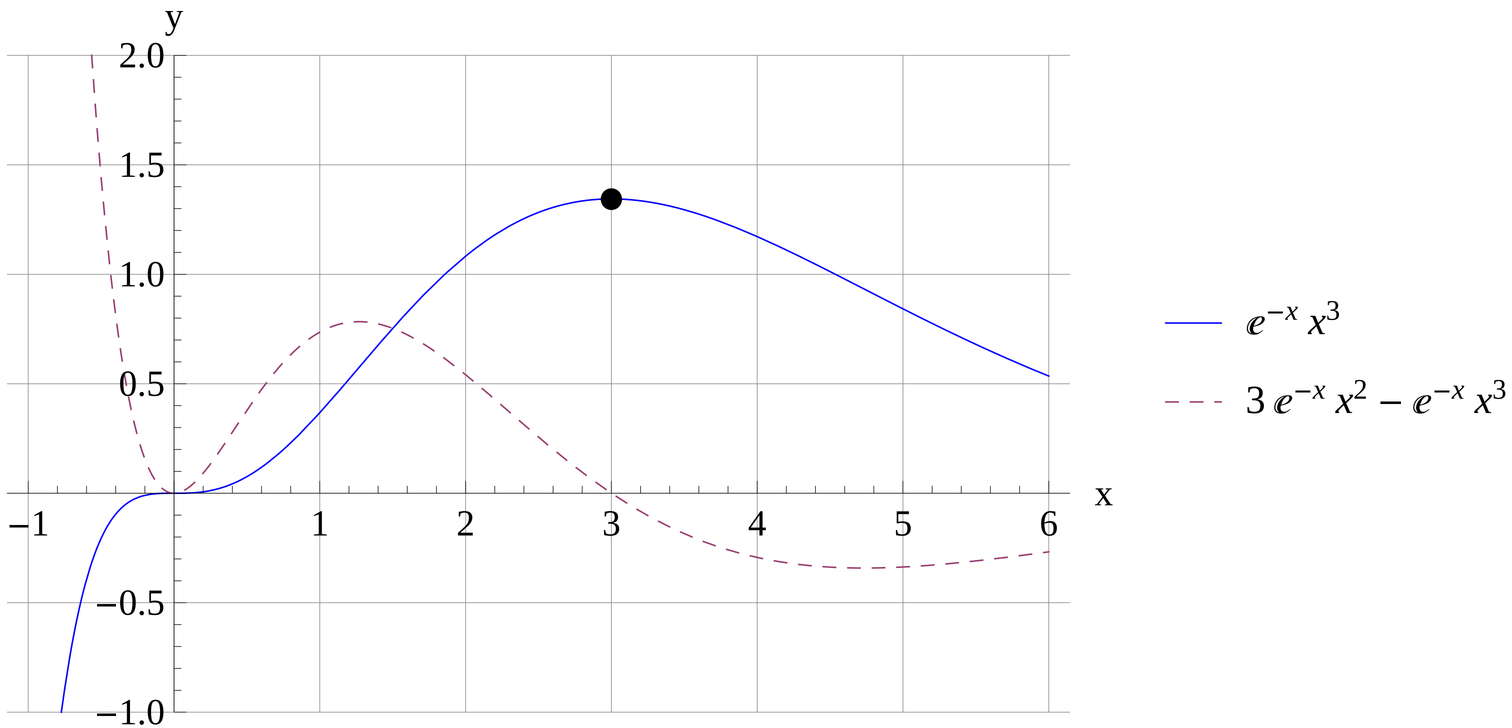


E.g.: critical point - no extrema:

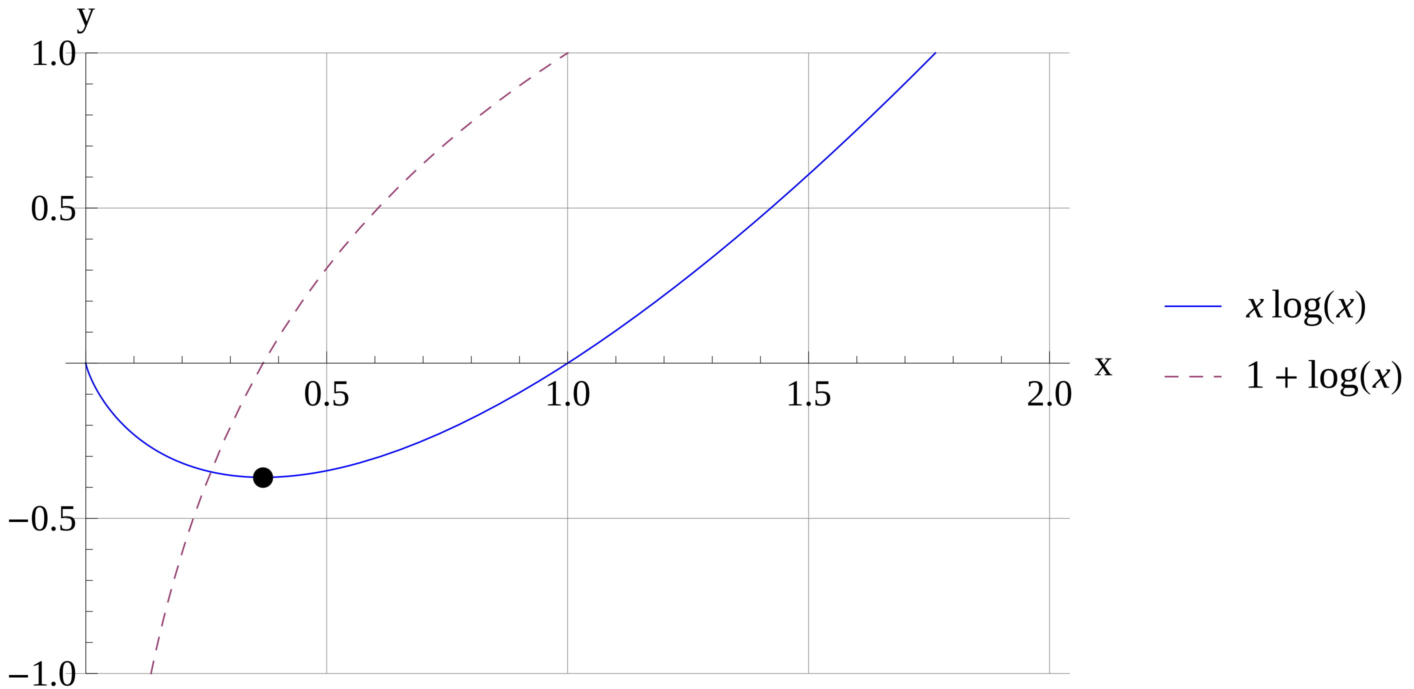


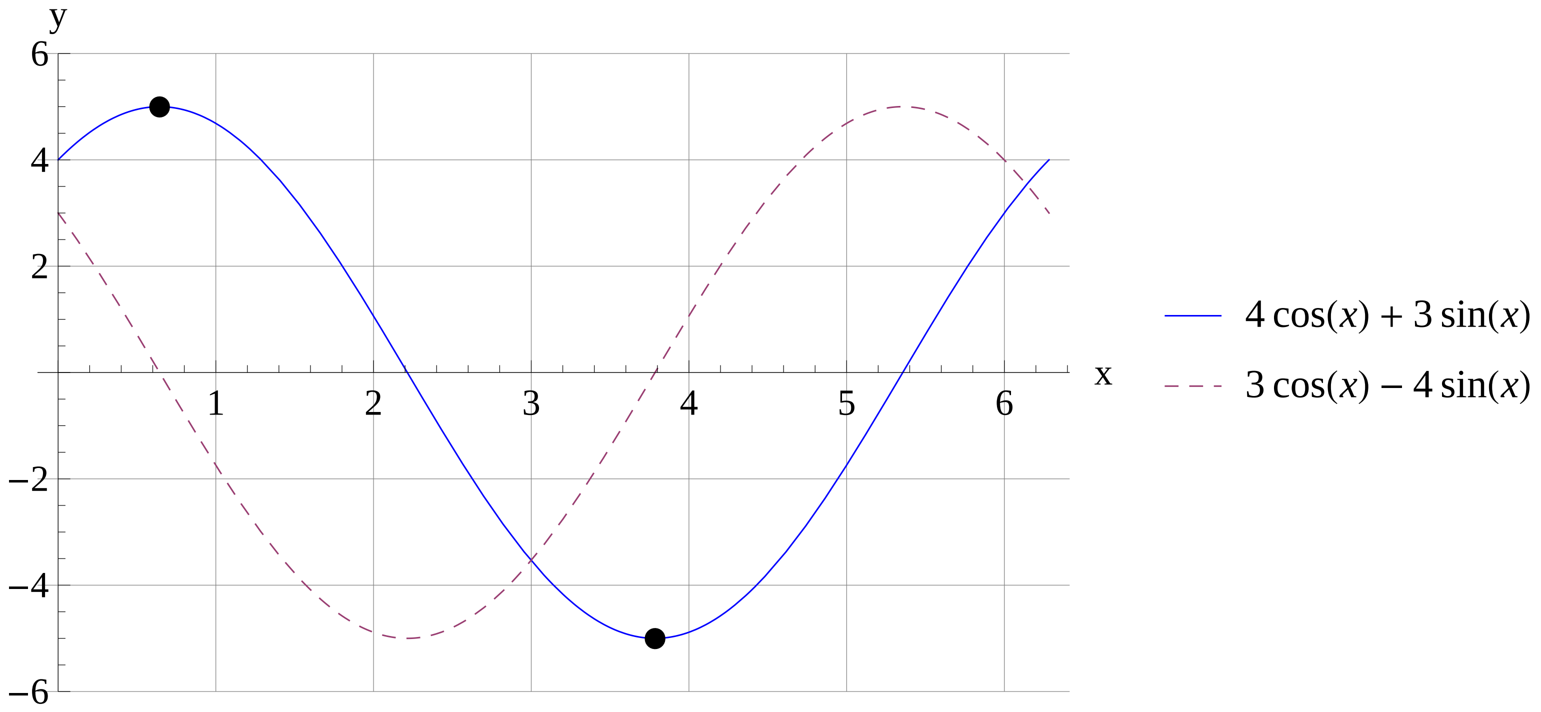

y
4
3
2
1
−1
x
−1
1
2
3
$1 + 2x - \frac{3x^2}{2} + \frac{x^3}{3}$
$2 - 3x + x^2$


y
1.0
0.5
−0.5
−1.0
x
−1
1
2
3
$e^{-x} x$
$e^{-x} - e^{-x} x$



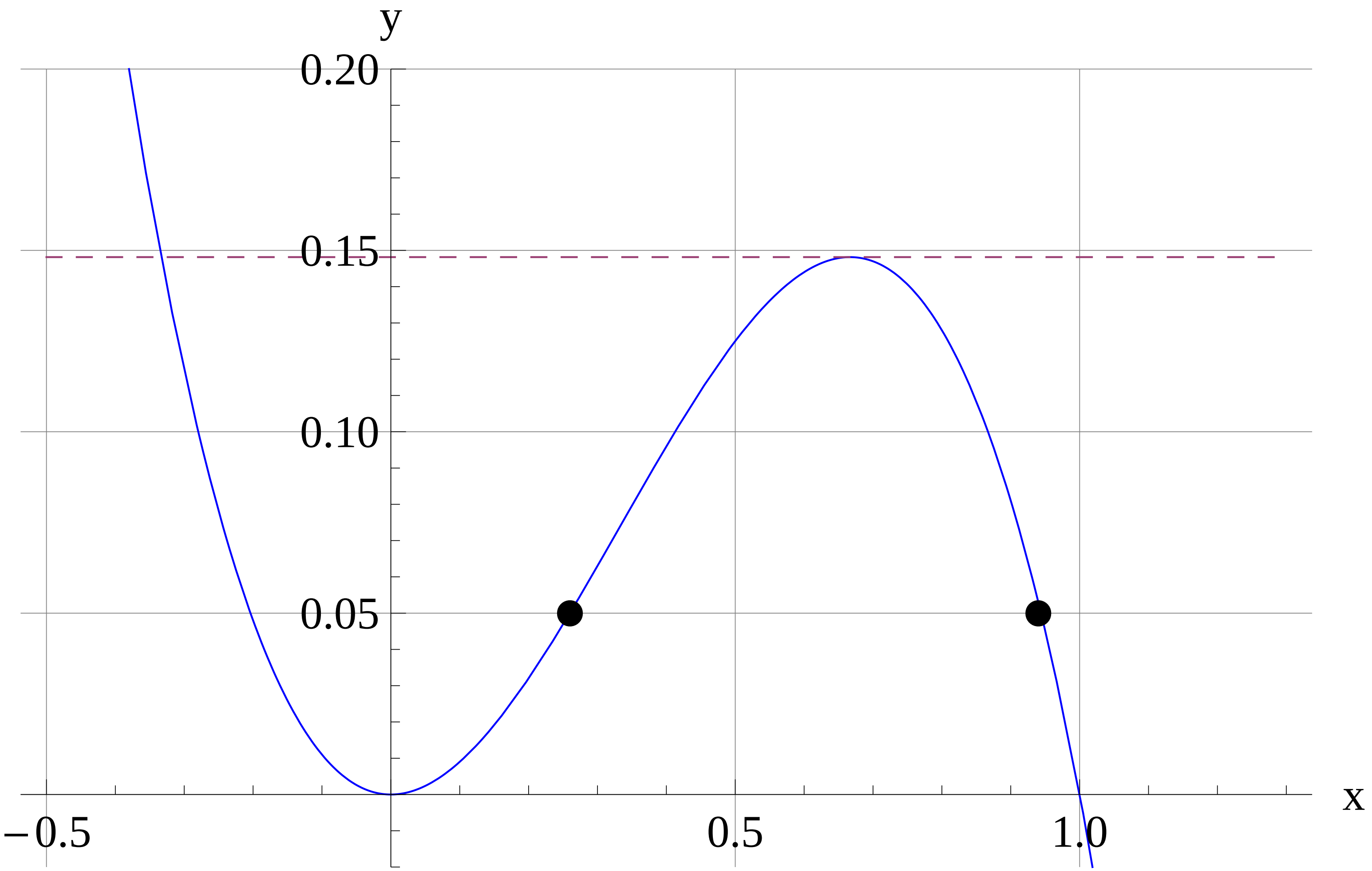
y
2.0
1.5
1.0
0.5
−0.5
−1.0
x
−1
1
2
3
4
5
6
$e^{-x} x^3$
$3 e^{-x} x^2 - e^{-x} x^3$


y
1.0
0.5
−0.5
−1.0
x
0.5
1.0
1.5
2.0
x log(x)
1 + log(x)

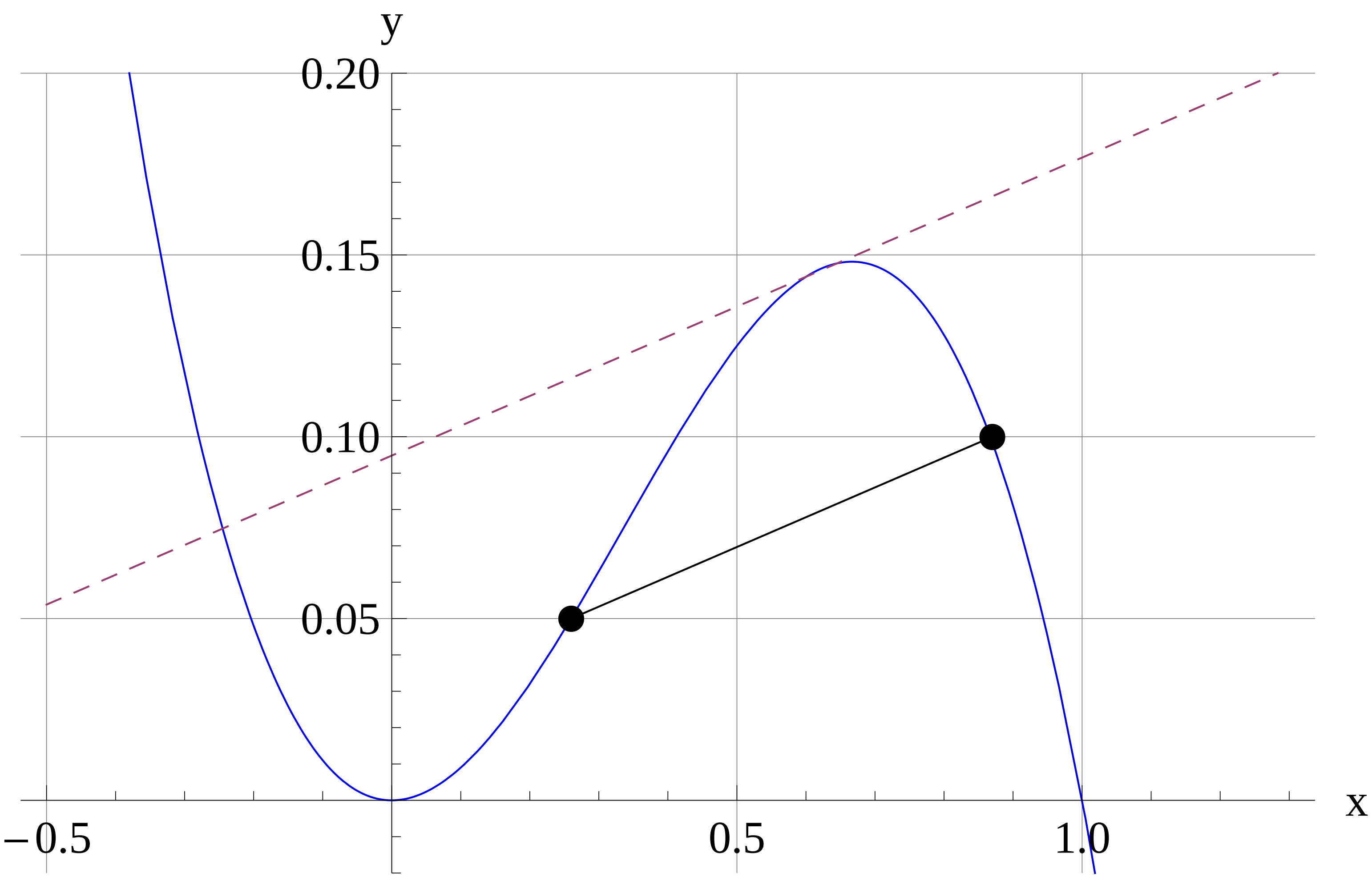


y
6
4
2
−2
−4
−6
1
2
3
4
5
6
x
$4\cos(x) + 3\sin(x)$
$3\cos(x) - 4\sin(x)$

Rolle's Theorem:



Mean Value Theorem:



Corollary: If

$$(f)' \equiv 0\,,\ f \equiv const$$

Corollary: If

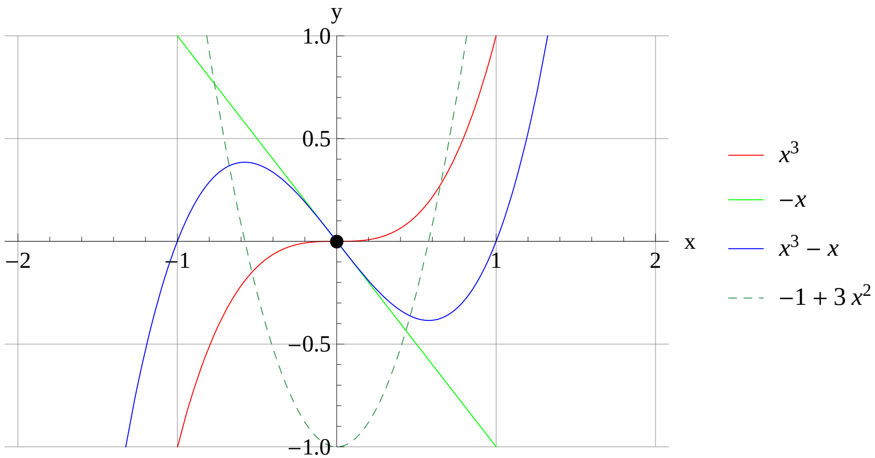
$$(f)' \equiv (g)'\,,\ f \equiv g + C$$

E.g.:

$$a = const\,,\ v = at + V_0\,,\ x = \frac{1}{2}at^2 + V_0 t + X_0$$

MONOTONICITY

Corollary:
If $(f)'$ keeps sign on (a,b), f is monotonic.



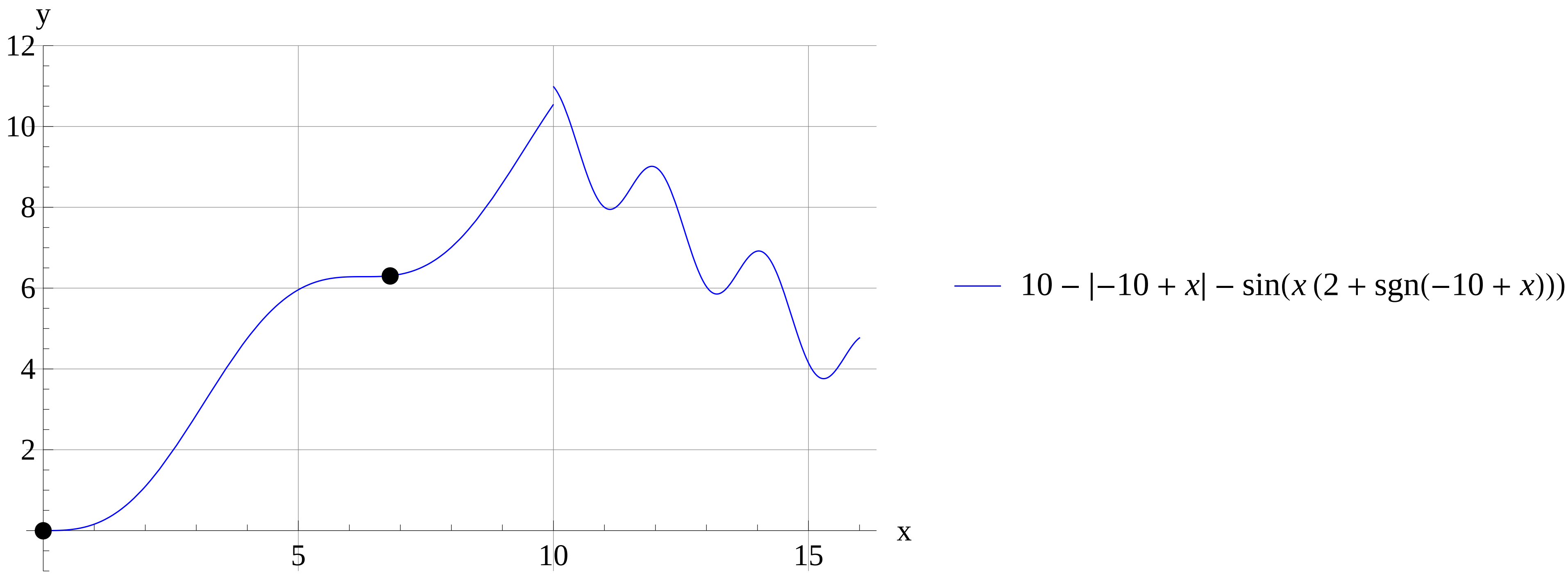
1st Derivative Test

- $(f)'$ changes from "$+$" to "-" - local max.

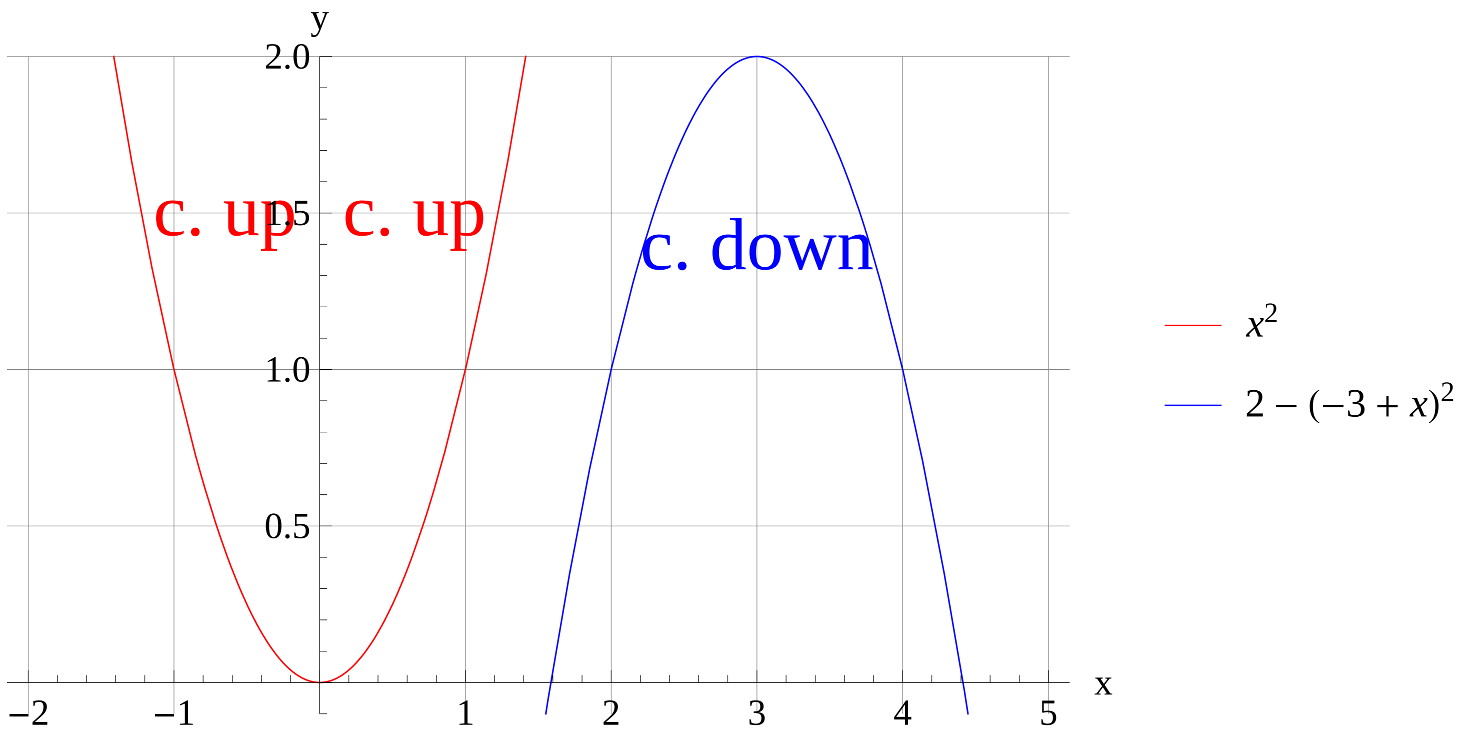
- $(f)'$ changes from "-" to "$+$" - local min.

- if $(f)'$ does not change sign - no local max or min

Note: Often can use 2nd derivative, but the 1st test does not require the existence of derivatives at $x = c$, e.g $f = |x|$.


y
12
10
8
6
4
2
5
10
15
x
10 − |−10 + x| − sin(x (2 + sgn(−10 + x)))

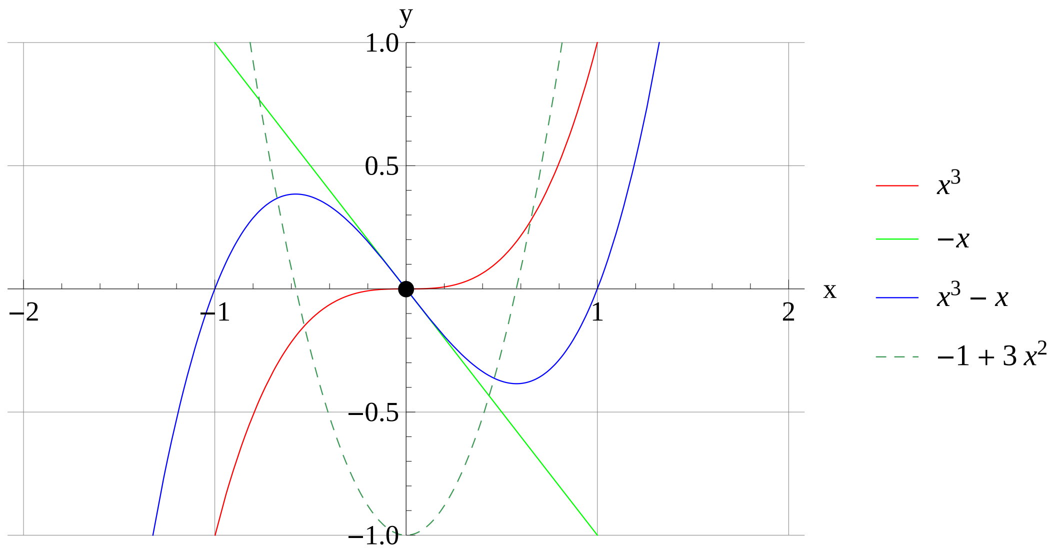
Concavity:



Concave up: $(f)'$ increasing

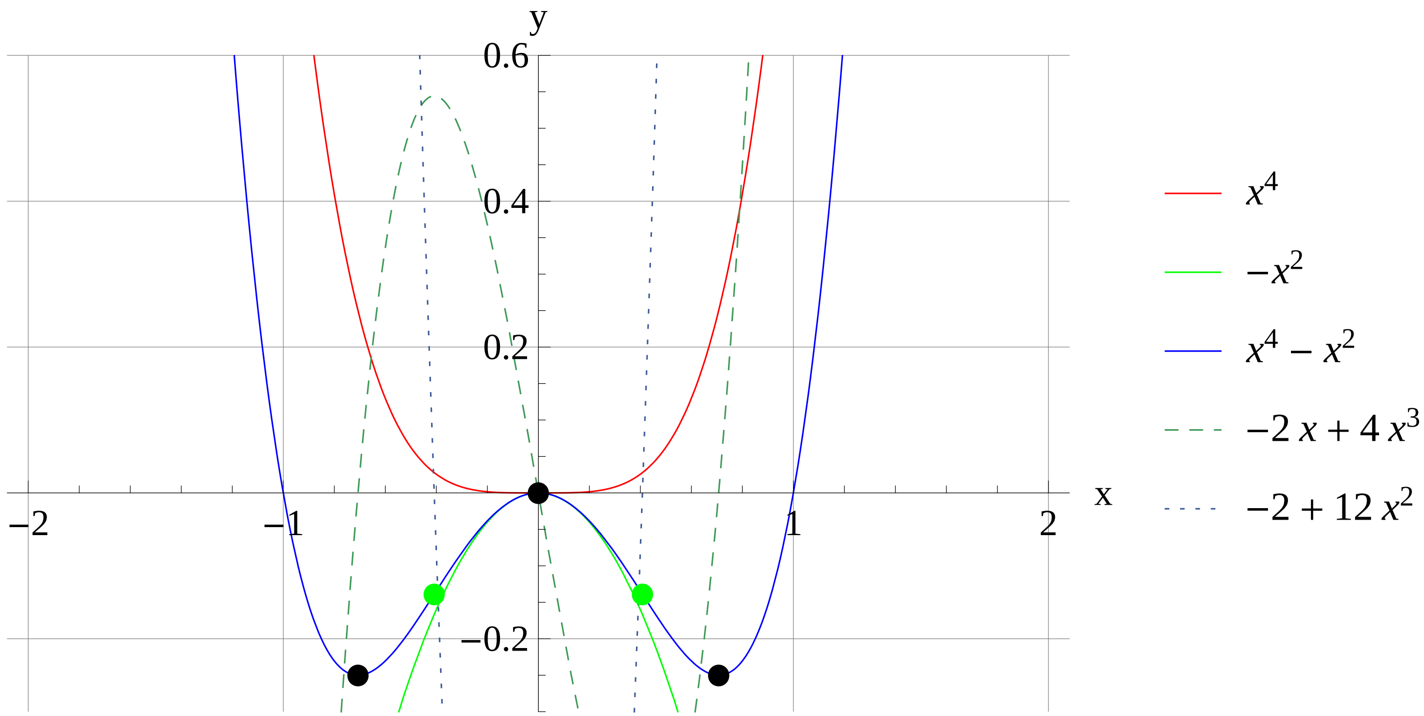
Concave down: $(f)'$ decreasing

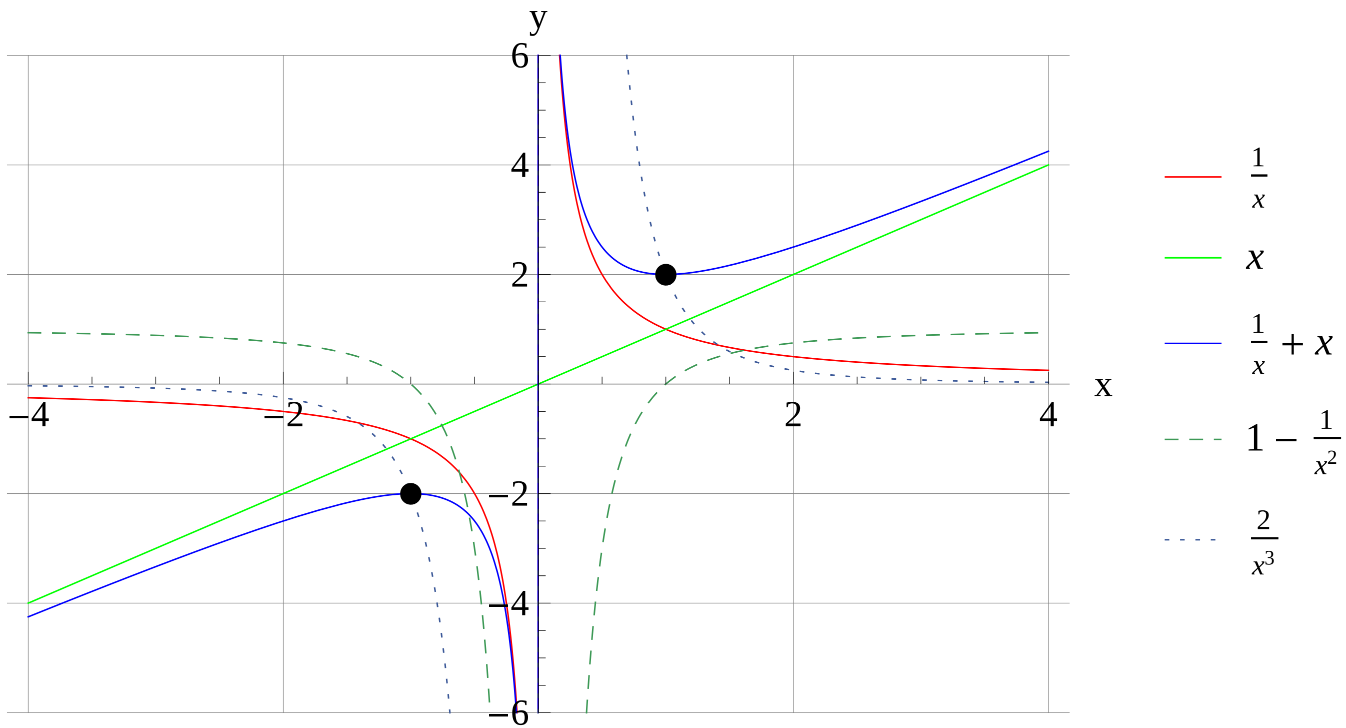
Change of concavity: inflection point

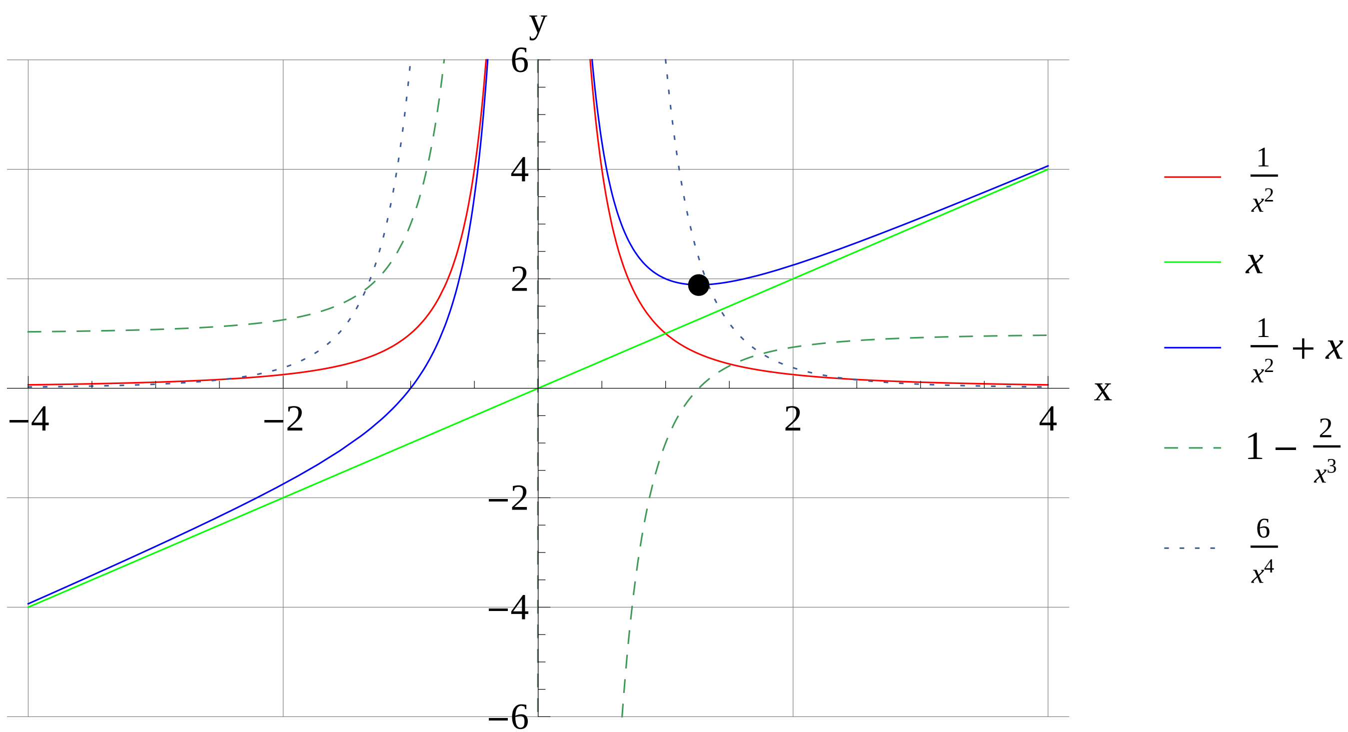


2nd Derivative Test for Concavity:
Concave up: $f'' > 0$
Concave down: $f'' < 0$
Change of concavity: inflection point $f'' = 0$ or does not exist

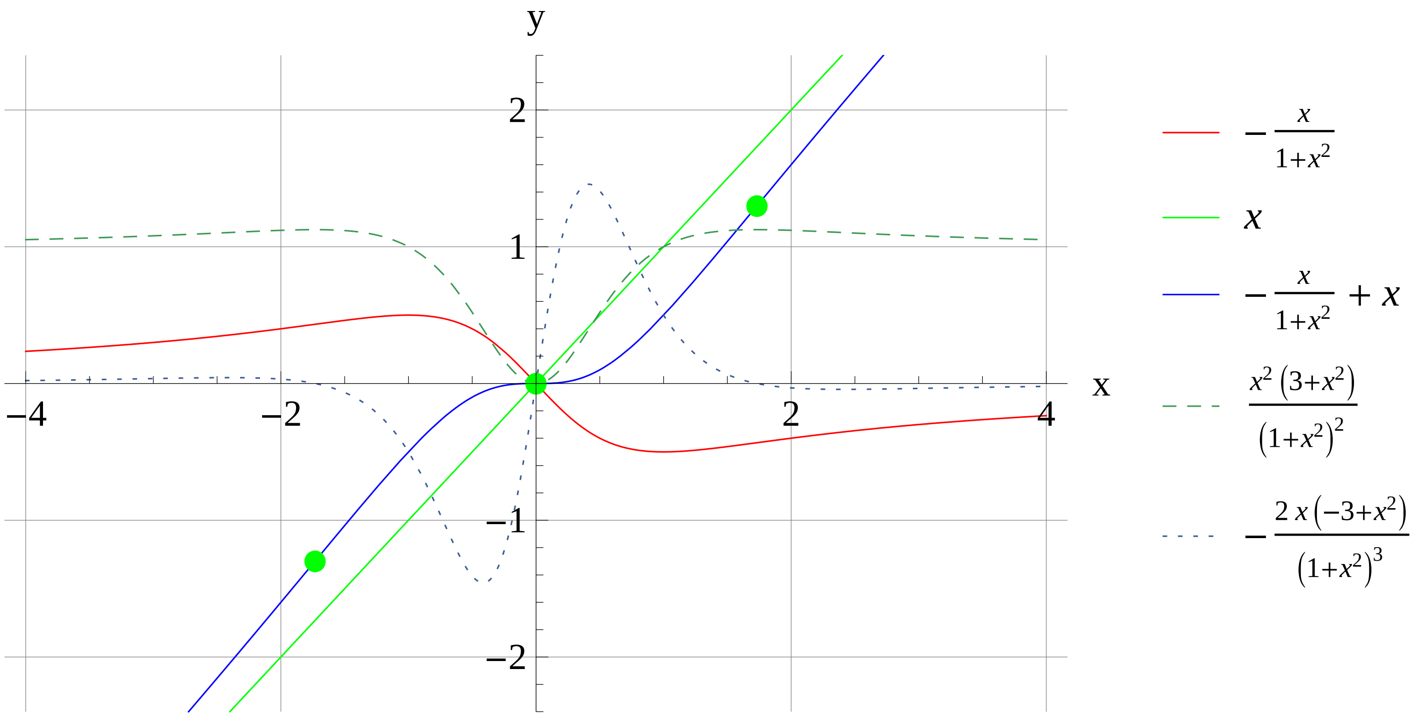
2nd Derivative Test for Local Extrema:
if $f' = 0$ and $f'' > 0$ - local min (e.g., x^2)
if $f' = 0$ and $f'' < 0$ - local max (e.g., $-x^2$)
if $f' = 0$ and $f'' = 0$ - test fails (e.g., x^3 or x^4)

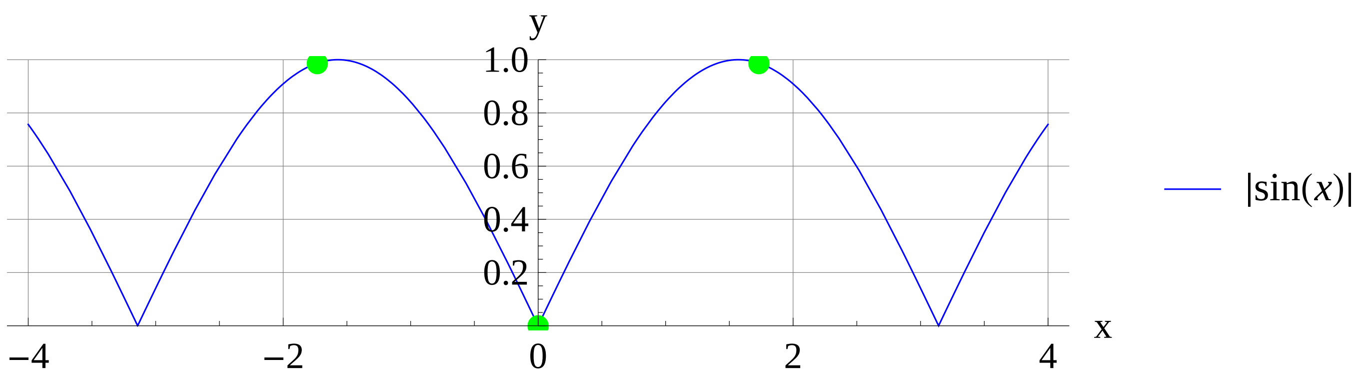

y
0.6
0.4
0.2
−0.2
x
−2
−1
1
2
x^4
$-x^2$
$x^4 - x^2$
$-2\,x + 4\,x^3$
$-2 + 12\,x^2$


y
6
4
2
−2
−4
−6
x
−4
−2
2
4
$\frac{1}{x}$
x
$\frac{1}{x} + x$
$1 - \frac{1}{x^2}$
$\frac{2}{x^3}$



y
6
4
2
−2
−4
−6
x
−4
−2
2
4
$\frac{1}{x^2}$
x
$\frac{1}{x^2} + x$
$1 - \frac{2}{x^3}$
$\frac{6}{x^4}$

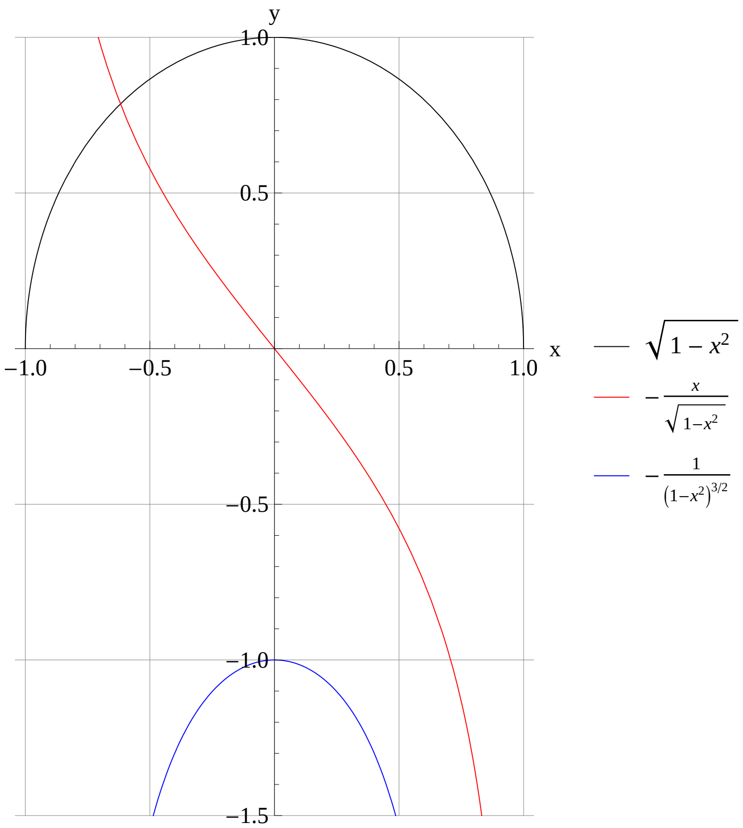

y
x
2
1
−1
−2
−4
−2
2
4
$-\frac{x}{1+x^2}$
x
$-\frac{x}{1+x^2}+x$
$\frac{x^2(3+x^2)}{(1+x^2)^2}$
$-\frac{2x(-3+x^2)}{(1+x^2)^3}$



Domain:

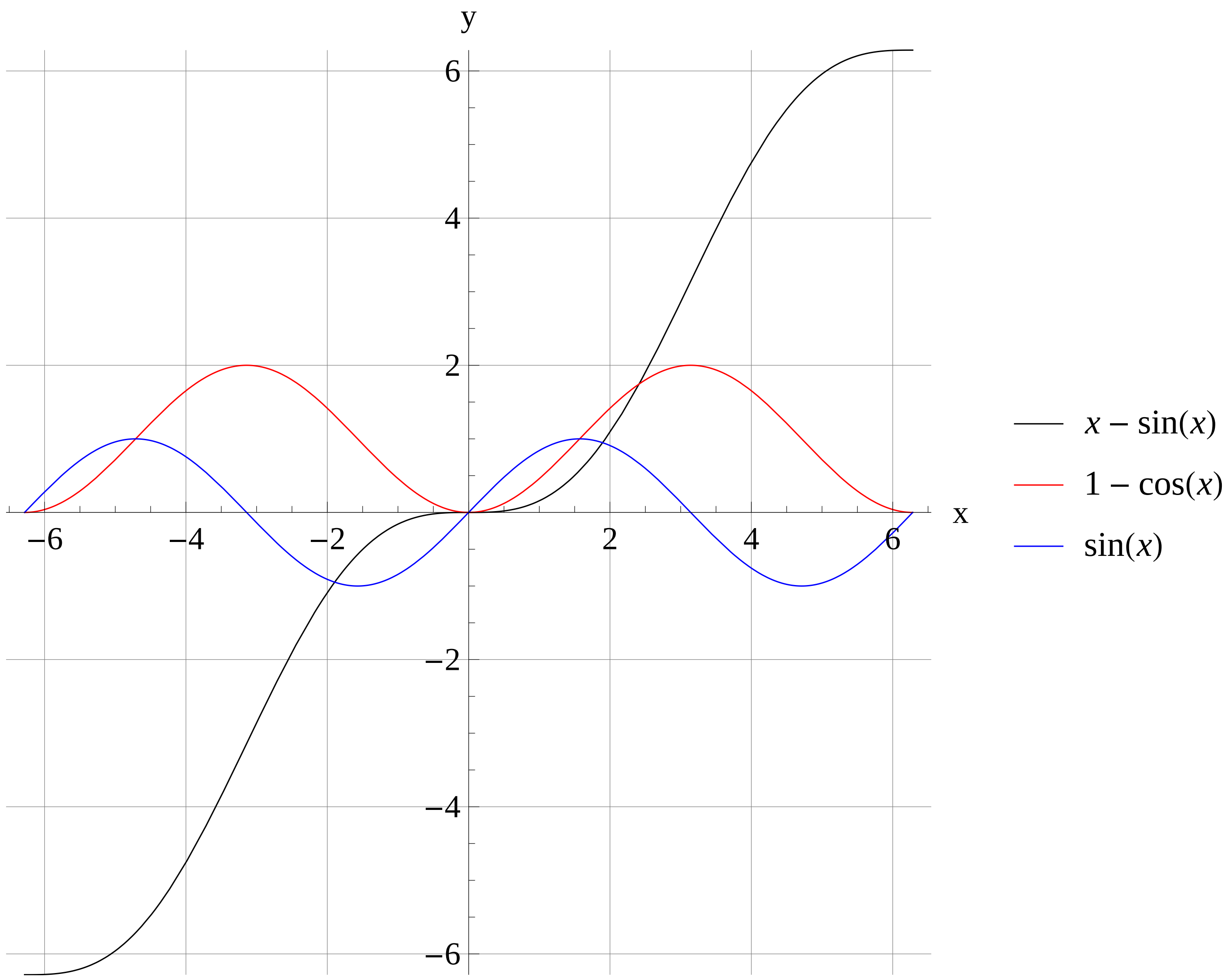
$$f(x) : (-\infty, \infty)$$

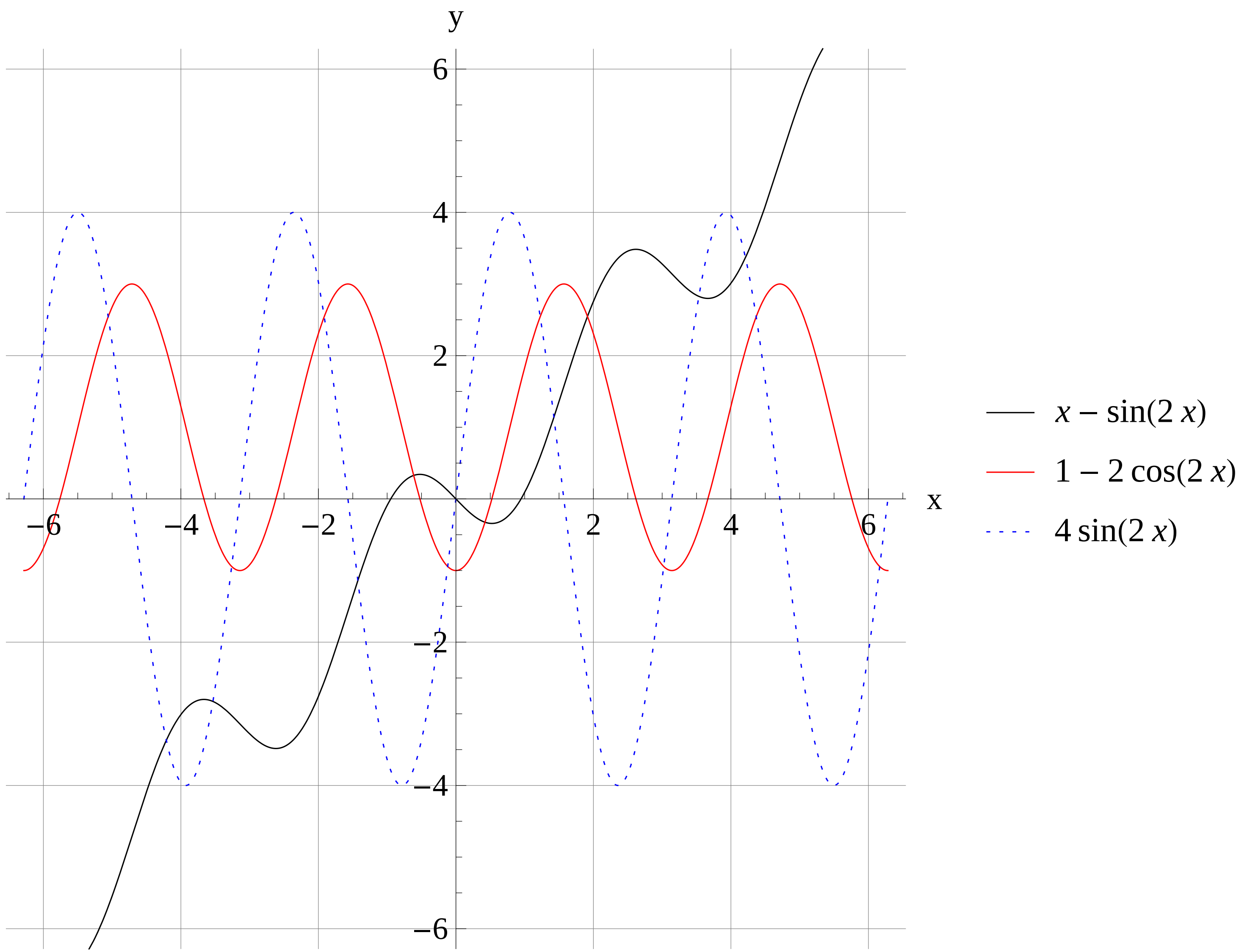
$$f'(x) : (-\infty, \infty), \ x \neq n\pi$$

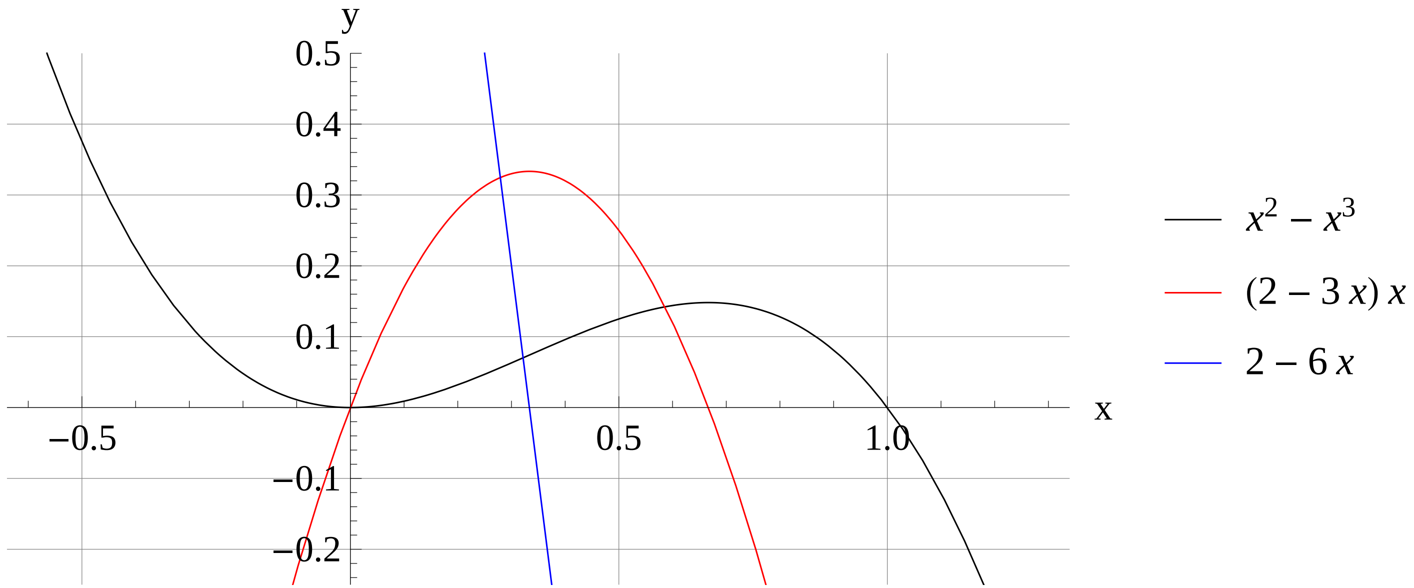


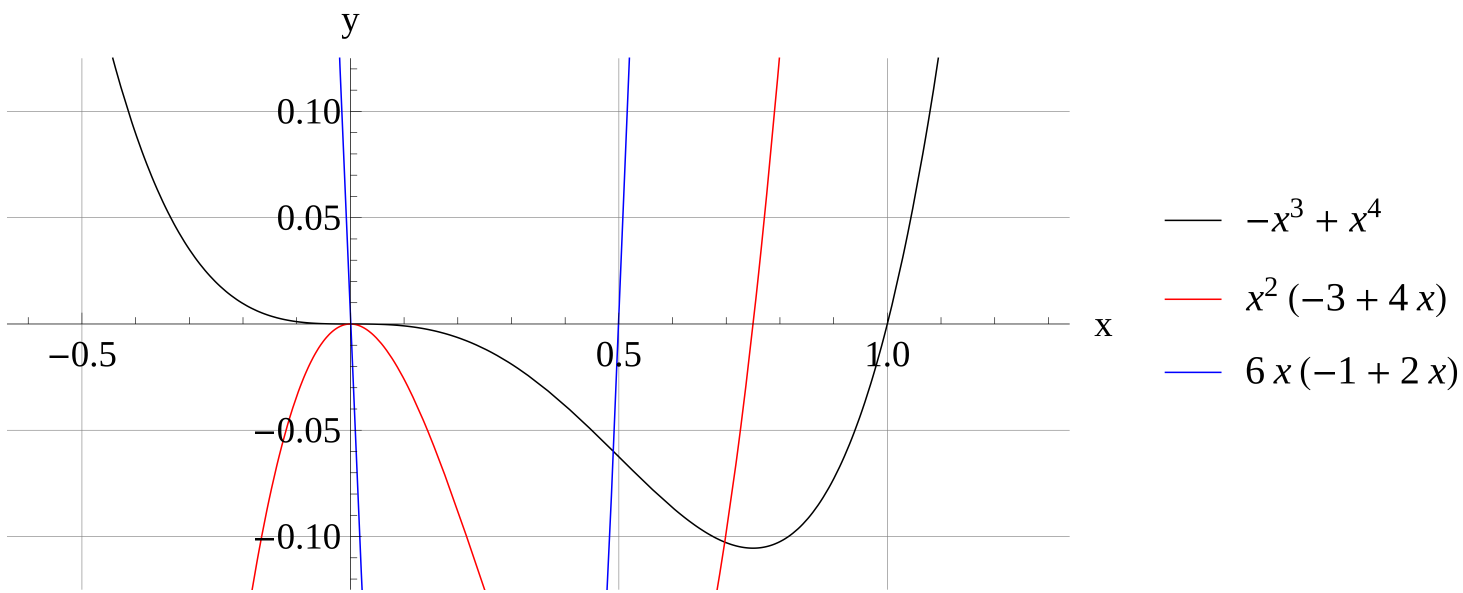
Domain:

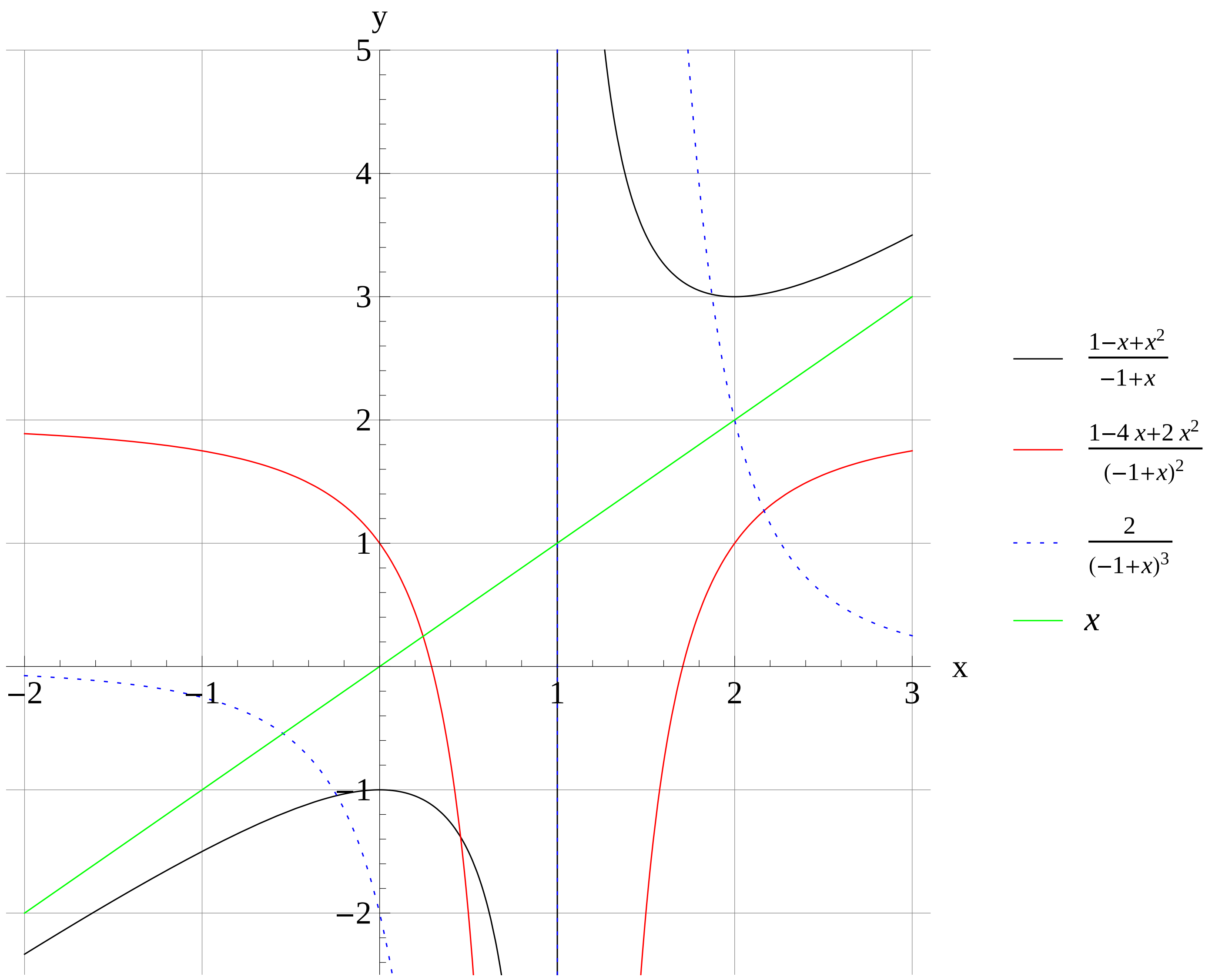
$$f(x)\ :\ [-1,\, 1]\,,\ f'(x)\ :\ (-1,\, 1)$$


y
6
4
2
−2
−4
−6
x
−6
−4
−2
2
4
6
x − sin(x)
1 − cos(x)
sin(x)


y
x
6
4
2
−2
−4
−6
−6
−4
−2
2
4
6
x − sin(2 x)
1 − 2 cos(2 x)
4 sin(2 x)


y
0.5
0.4
0.3
0.2
0.1
−0.1
−0.2
x
−0.5
0.5
1.0
$x^2 - x^3$
$(2 - 3x)x$
$2 - 6x$


y
0.10
0.05
−0.05
−0.10
x
−0.5
0.5
1.0
$-x^3 + x^4$
$x^2(-3 + 4x)$
$6x(-1 + 2x)$


y
x
5
4
3
2
1
−1
−2
−2
−1
1
2
3
$\frac{1-x+x^2}{-1+x}$
$\frac{1-4\,x+2\,x^2}{(-1+x)^2}$
$\frac{2}{(-1+x)^3}$
x

l'Hopital Rule

If $f(c) = 0$ AND $g(c) = 0$ with $g'(c) \neq 0$, then:

$$\lim_{x \to c} \frac{f}{g} = \frac{f'(c)}{g'(c)} \tag{52}$$

Demonstration (not proof):

$$f(c + dx) \simeq 0 + f'(c) \cdot dx$$

$$g(c + dx) \simeq 0 + g'(c) \cdot dx$$

If $f'(c) = g'(c) = 0$, repeat to get $f''(c)/\, g''(c)$, etc.

Examples "0/0":

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{2x}{1}\Big|_{x=1}=2$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\frac{\cos x}{1}\Big|_{x=0}=1$$

$$\lim_{x\to 0}\frac{\tan x}{x}=\frac{1/\cos^2 x}{1}\Big|_{x=0}=1$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1/(2\sqrt{1+x})}{1}\Big|_{x=0}=\frac{1}{2}$$

Caution: $f(c)=g(c)=0$ is crucial! E.g.

$$\lim_{x\to 0}\frac{\sin x-1}{x}=-\infty\,,\ not\ \ \frac{\cos x-0}{1}\Big|_{x=0}=1$$

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \left.\frac{1-\cos x}{3x^2}\right|_{x\to 0} = \left.\frac{\sin x}{6x}\right|_{x\to 0} = \frac{1}{6}$$

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \left.\frac{1/\cos^2 x - 1}{3x^2}\right|_{x\to 0} = \left.\frac{2\sin x/\cos^3 x}{6x}\right|_{x\to 0} = \frac{1}{3}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2} = \left.\frac{1/(2\sqrt{1+x})-1/2}{2x}\right|_{x\to 0} =$$

$$= \left.\frac{-1/4\cdot(\sqrt{1+x})^{-3}}{2}\right|_{x\to 0} = -\frac{1}{8}$$

l'Hopital Rule for "∞/∞":

If $f(x) \to \infty$ AND $g(x) \to \infty$ as $x \to c$ with $g'(x) \to const \neq 0$, then:

$$\lim_{x \to c} \frac{f}{g} = \left. \frac{f'(x)}{g'(x)} \right|_{x \to c} \tag{53}$$

$$\lim_{x\to 0^+} x^n \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-n}} = \lim_{x\to 0^+} \frac{1/x}{-nx^{-n-1}} =$$

$$= -\frac{1}{n}x^n\Big|_{x\to 0^+} = 0\,,\ n > 0$$

$$\lim_{x\to 0^+} x\ln x = 0 \qquad (54)$$

$$\lim_{x\to\infty} x^{-n}e^x = \lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \ldots$$

$$= \lim_{x\to\infty} \frac{e^x}{n!} \to \infty$$

$$"\infty \cdot 0 \;\to\; 0/0":$$

$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{-(1/x^2)\cos(1/x)}{-1/x^2} = 1$$

$$"\infty - \infty \;\to\; 0/0":$$

$$\frac{1}{\sin x} - \frac{1}{x}\bigg|_{x\to 0} = \frac{x-\sin x}{x\sin x}\bigg|_{x\to 0} = \frac{1-\cos x}{x\cos x + \sin x}\bigg|_{x\to 0} =$$

$$= \frac{\sin x}{-x\sin x + \cos x + \cos x}\bigg|_{x\to 0} = 0$$

"1^∞":

$$\lim_{x \to 0^+} (1 + zx)^{1/x} = e^z$$

"0^0":

$$\lim_{x \to 0^+} x^x = 1$$

(both via logarithm)

Applied Optimization

"Best rectangle": $P = 2(x+y)$ -fixed; $S = xy = \max$.

$$S = x(P/2 - x)\,,\ dS/dx = P/2 - 2x = 0$$

$$x = P/4 = y$$

"Best can": $V = \pi r^2 h$ - fixed; $S = 2\pi r h + 2\pi r^2 = \min$.
$h = V/\pi r^2$

$$S = 2\pi\left(\frac{V}{\pi r} + r^2\right)\,,\ \frac{1}{2\pi} dS/dx = -\frac{V}{\pi r^2} + 2r = 0$$

$$r = \left(\frac{V}{2\pi}\right)^{1/3}, h = 2r$$

Rectangle in a semicircle: $S = 2x\sqrt{R^2 - x^2} = \max$.
Ans.: $x = R/\sqrt{2}$.

Refraction and reflection from Fermat principle (in class).

Distance from a parabola (in class).

Using linearization for approximate calculations:
Let $F(c)$ be easy to evaluate, then for small dx

$$F(c+dx) \simeq F(c) + F'(c) \cdot dx$$

E.g. $F(x) = x^{1/3}$, $F'(x) = \frac{1}{3}x^{-2/3}$. Find $29^{1/3}$.
Solution: Use $c = 27$:

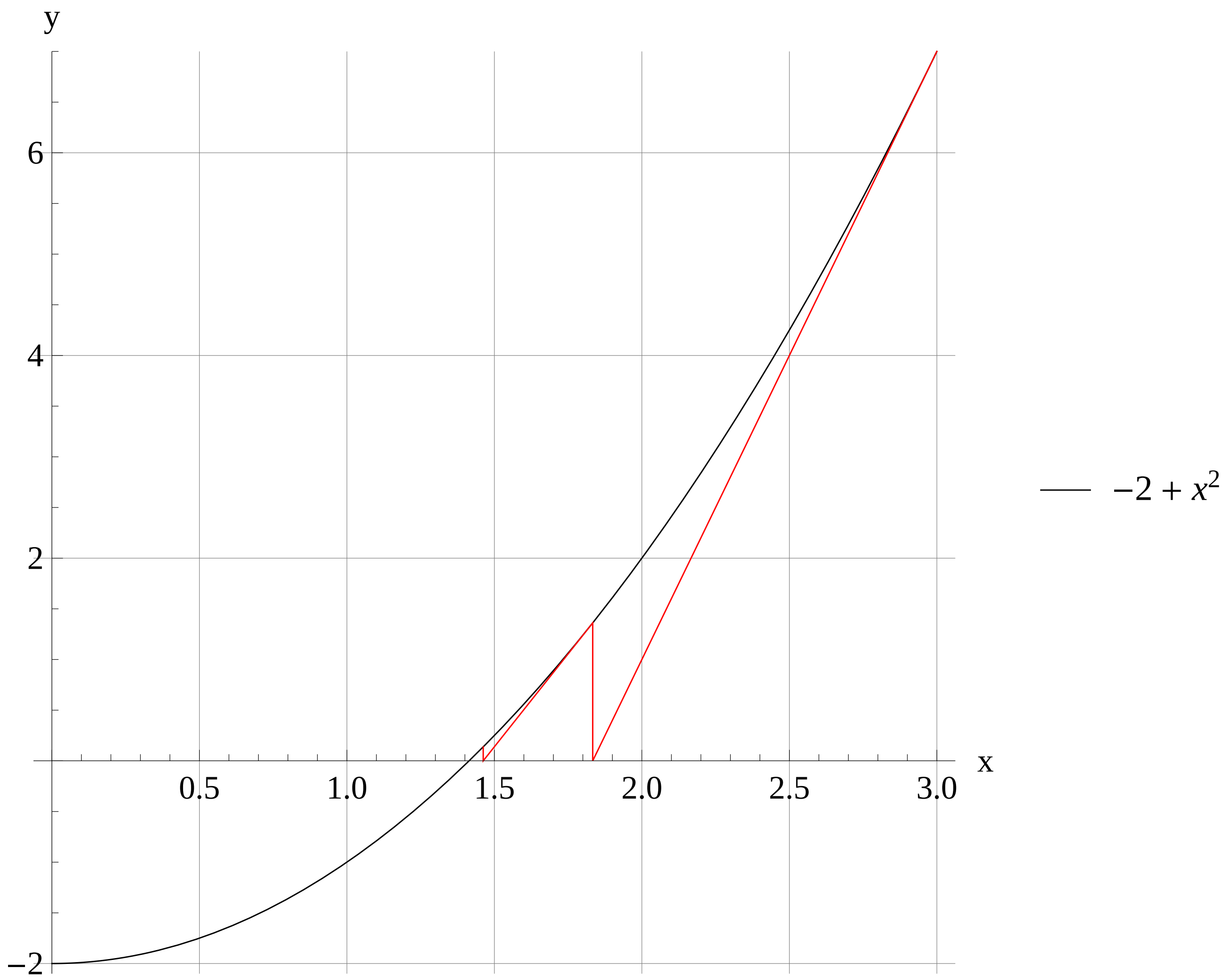
$$(27+2)^{1/3} \simeq 3 + \frac{1}{3}27^{-2/3} \cdot 2 = 3\frac{2}{27} = \frac{83}{27}$$

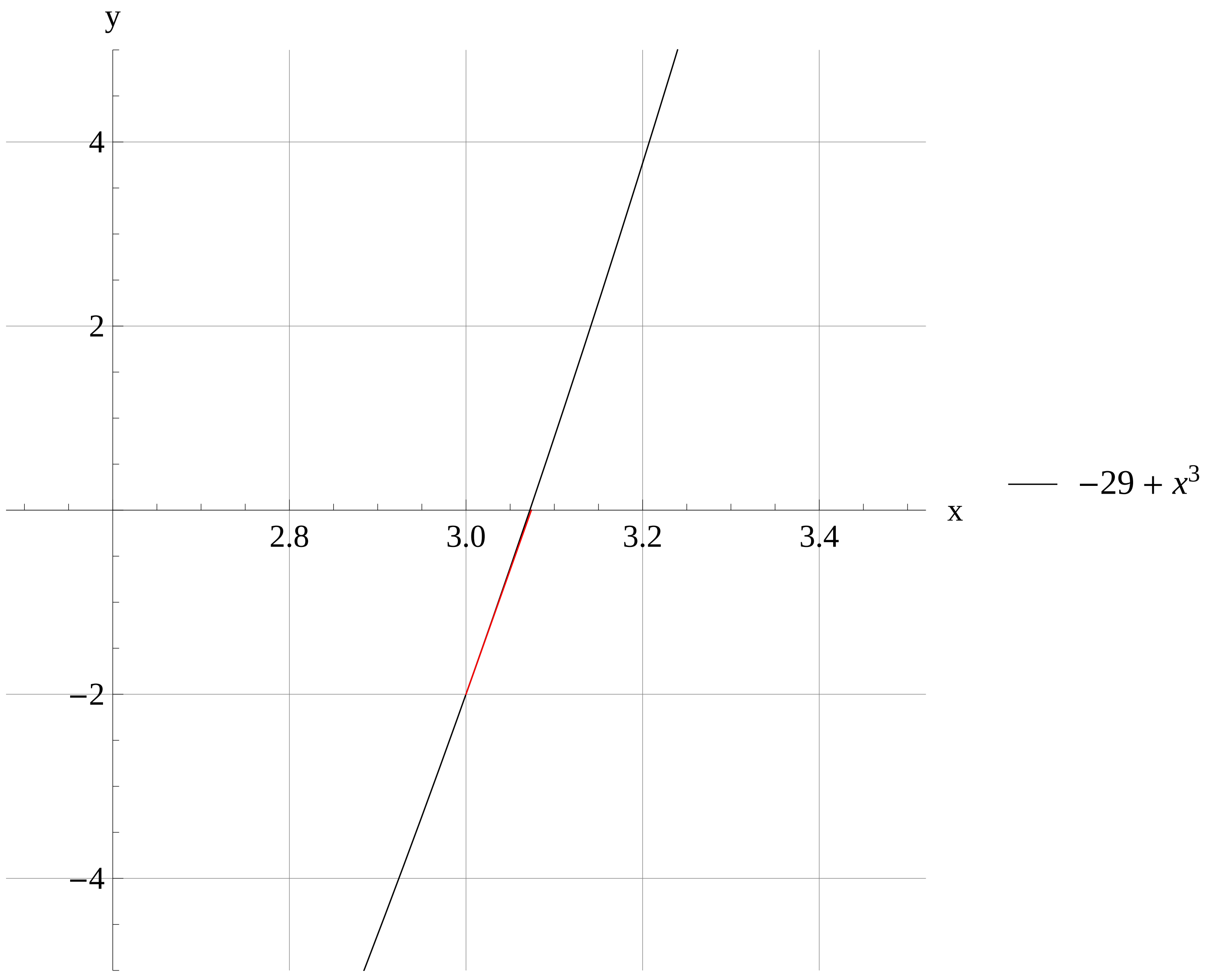
Alternatively (1st step of Newtons method): $f(x) = x^3 - 29$. look for a root in linearized approximation; starting guess $x_0 = 3$

$$0 = f(x) \simeq f(x_0) + (x - x_0) \cdot f'(x_0)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - (-2)/(3 \cdot 9) = 83/27$$

same thing.


y
6
4
2
−2
0.5
1.0
1.5
2.0
2.5
3.0
x
$-2 + x^2$


y
4
2
−2
−4
2.8
3.0
3.2
3.4
x
$-29 + x^3$

General Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{55}$$

E.g., $f = x^2 - 2$, starting guess $x_0 = 3/2$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\,,$$

$$\left\{\frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}\right\}$$

"Errors":

$$\{0.25,\ 0.007,\ 10^{-6},\ 10^{-12}\}$$

$f = x^3 - 29$, starting guess $x_0 = 3$:

$$x_{n+1} = \frac{29}{3x_n^2} + \frac{2x_n}{3}$$

$$\left\{3, \frac{83}{27}, \frac{1714381}{558009}, \frac{1511621803254658 3823}{4920136460591269347}\right\}$$

"Errors":

$$(-2.\,,\ 0.05\,,\ 10^{-5}\,,\ 10^{-12})$$

Approximating $\ln 3 = 1.09861\ldots$ with Newton

$$f(x) = e^x - 3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + 3e^{-x_n} - 1$$

If start with $x_0 = 1$,

$$x_1 = 1 + \frac{3}{e} - 1 = \frac{3}{e} \simeq 1.10364$$

with error $5 \cdot 10^{-3}$. Next, $x_2 \simeq 1.09862$ with error 10^{-5}

Antiderivatives and indefinite integrals:
$F(x)$ is A.D. of $f(x)$ if

$$(F)' = f$$

Note: $F(x)+C$ - also an A.D. All together - *indefinite integral*, $\int f(x)\,dx$

E.g.

$$\int dx = x + C \tag{56}$$

$$\int x\,dx = \frac{1}{2}x^2 + C \tag{57}$$

$$\int \frac{dx}{x^2} = -\frac{1}{x} + C \tag{58}$$

Tables (remember constant!):

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}\,,\ n \neq -1 \tag{59}$$

$$\int dx/x = \ln|x| \tag{60}$$

$$\int \sin x\,dx = -\cos x \tag{61}$$

$$\int \cos x\,dx = \sin x \tag{62}$$

$$\int dx/\cos^2 x = \tan x \tag{63}$$

$$\int dx/\sqrt{1-x^2} = \arcsin x \tag{64}$$

$$\int dx/(x^2+1) = \arctan x \tag{65}$$

$$\int e^x\,dx = e^x \tag{66}$$

Expanding the Tables:

- $$\int (af(x) + bg(x))\, dx = a \int f\, dx + b \int g\, dx + C$$

- If $F(x) = \int f(x)\, dx$, then
$$\int f(kx)\, dx = \frac{1}{k} F(kx) + C$$

E.g.

$$\int \sin(\omega t)\, dt = -\frac{1}{\omega} \cos(\omega t) + C\ ,\ \ \int e^{-2x}\, dx = -\frac{1}{2} e^{-2x} + C\ ,\ \ \ldots$$

Finding C.

Let $F(x)$ - some A.D. of $f(x)$. We want to construct a specific A.D. which satisfies $F(x_0) = A$. The general form is $F(x) + C$ AND

$$F(x_0) + C = A\,,\ i.e.\, C = A - F(x_0)$$

Thus, we solved a *differential equation* $(F)' = f$ with additional "initial" (or "boundary") condition $F(x_0) = A$.

E.g., $dx/dt = 4t^3 + 4t\,,\ x(0) = 7$

$$x(t) = \int (4t^3 + 4t)\,dt = t^4 + 2t^2 + C\,,\ C = 7$$

E.g., free fall with $dv/dt = -g = const\,,\ v(0) = V_0$, $dy/dt = v\,,\ y(0) = Y_0$. (2 equations)

$$v(t) = \int (-g)\,dt = -gt + C\,,\ C = V_0$$

$$y(t) = \int v(t)\,dt = -\frac{1}{2}gt^2 + V_0 t + C_1\,,\ C_1 = Y_0$$

E.g., 1-d motion with $a = 4e^{2t}$, $v(0) = 7$, $x(0) = 3$.

$$v(t) = 2e^{2t} + C\,,\ C = 7 - 2e^0 = 5$$

$$x(t) = e^{2t} + 5t + C_1\,,\ C_1 = 3 - e^0 - 5 \cdot 0 = 2$$

Small x approximations (from l'Hopital)

$$(1+x)^k - 1 \simeq kx \tag{67}$$

$$e^x - 1 \simeq x\,,\ a^x - 1 \simeq x \ln a \tag{68}$$

$$\ln(1+x) \simeq x \tag{69}$$

$$\sin x \simeq \tan x \simeq x \tag{70}$$

$$x - \sin x \simeq x^3/6 \tag{71}$$

$$\tan x - x \simeq x^3/3 \tag{72}$$

$$1 - \cos x \simeq x^2/2 \tag{73}$$

More examples on limits

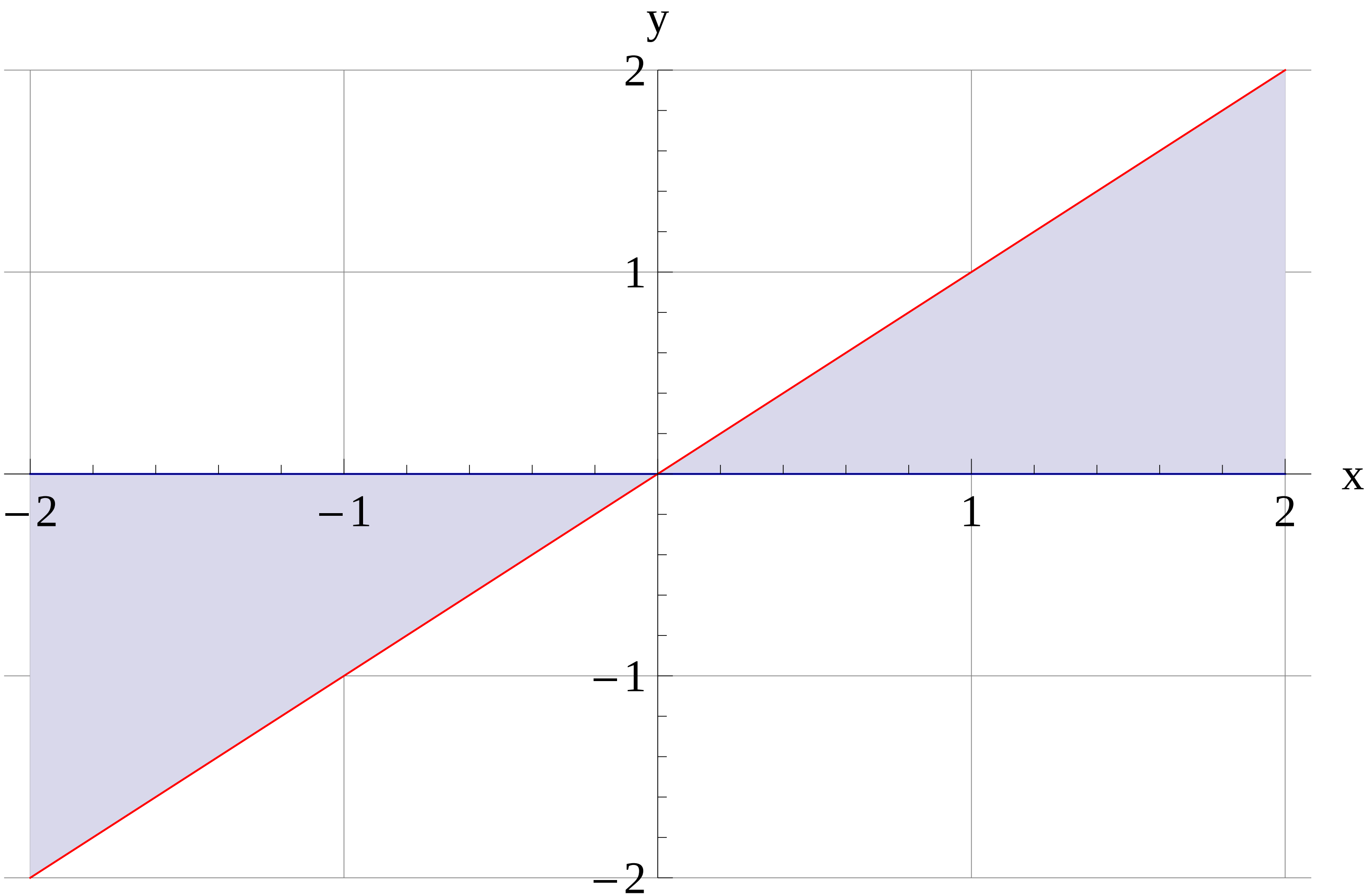
$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x-\pi} = -2$$

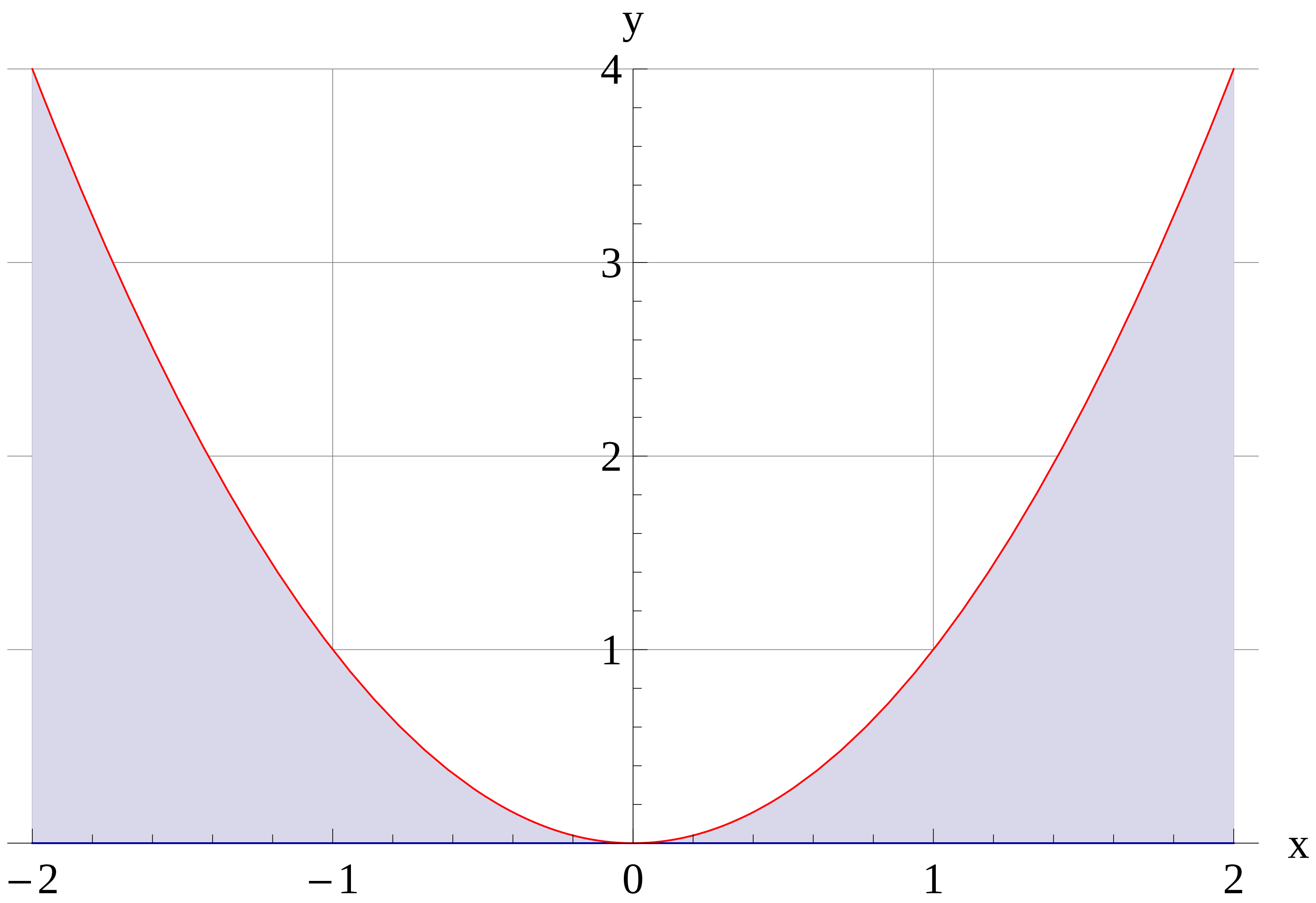
$$\lim_{x\to 0} \frac{2^{\sin x}-1}{x} = \ln 2$$

$$\lim_{x\to 0} (1+2x)^{5/x} = e^{10}$$

$$\lim_{x\to 0} \cos\frac{\pi x}{\sin x} = -1$$

$$\lim_{x\to 0} \left(e^{2x}+e^{x}-1\right)^{2/x} = e^{6}$$


y
2
1
−1
−2
x
−2
−1
1
2


y
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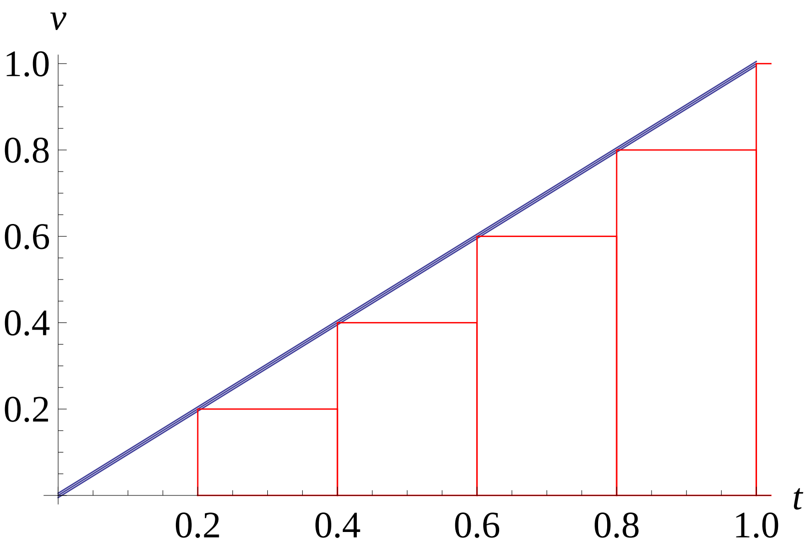
3
2
1
−2
−1
0
1
2
x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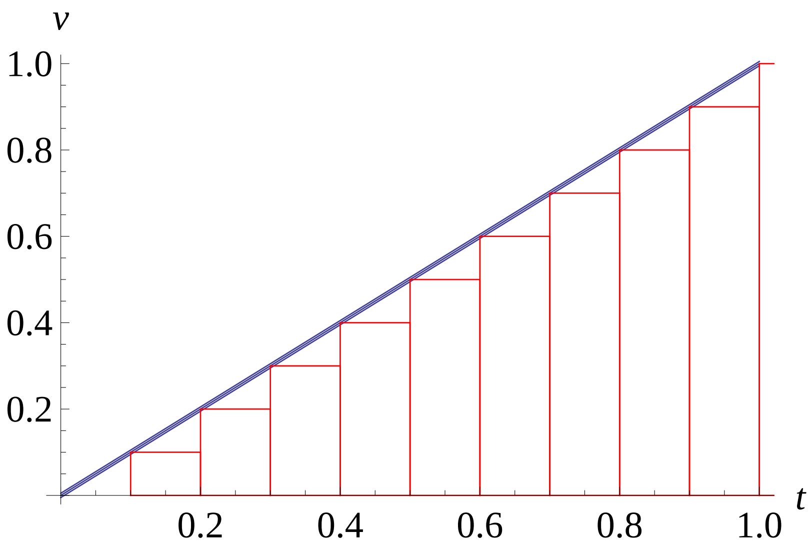
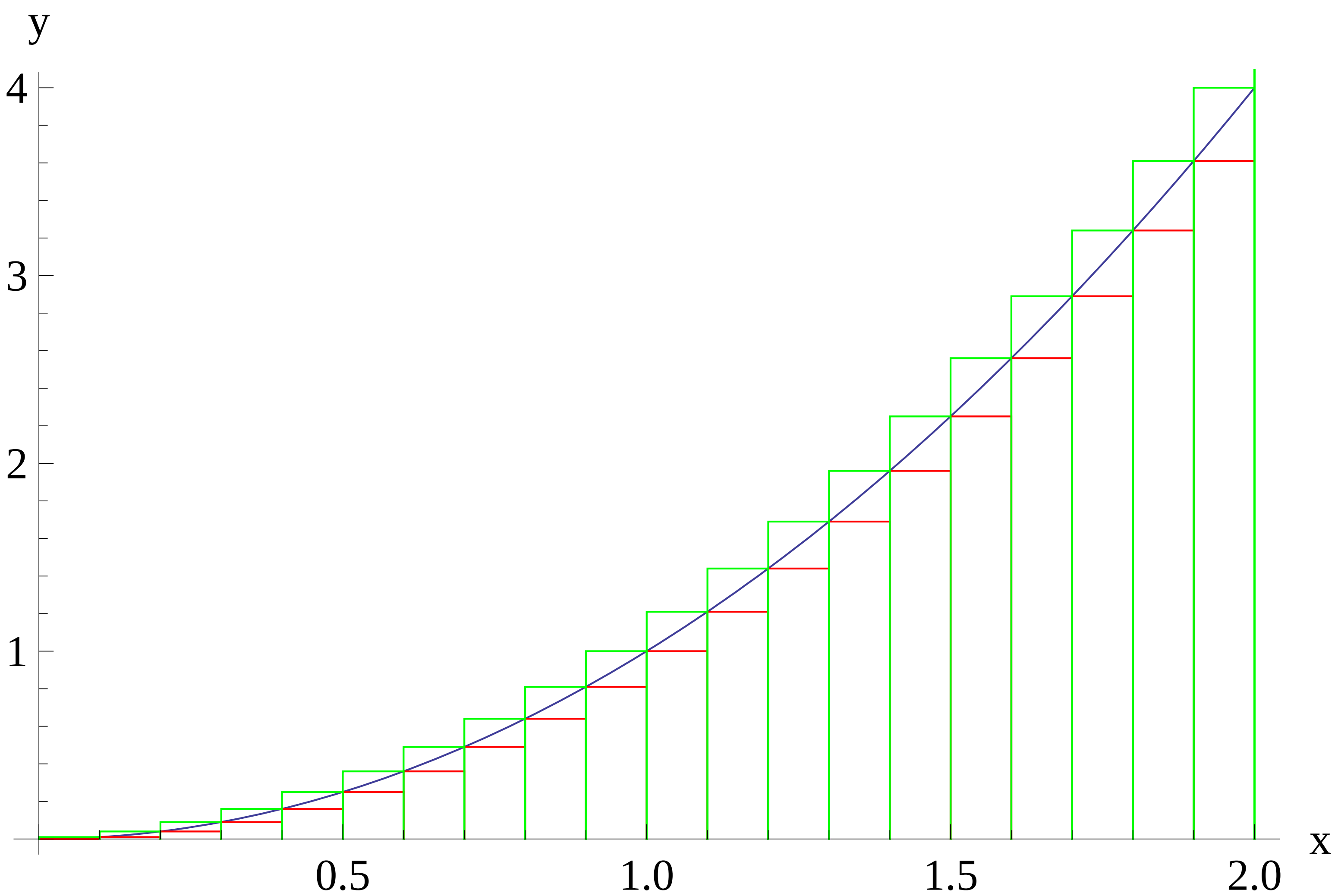
0.6
0.4
0.2
t
0.2
0.4
0.6
0.8
1.0


v
1.0
0.8
0.6
0.4
0.2
t
0.2
0.4
0.6
0.8
1.0


y
4
3
2
1
x
0.5
1.0
1.5
2.0

Σ-notations

$$\sum_{k=1}^{N} a_k = a_1 + a_2 + \ldots + a_N$$

k - "dummy" variable:

$$\sum_{k=1}^{N} a_k = \sum_{i=1}^{N} a_i = \sum_{m=0}^{N-1} a_{m+1}$$

$$\sum_{k=1}^{N} f(a_k) = f(a_1) + f(a_2) + \ldots f(a_N)$$

$$\sum_{k=1}^{N} k = 1 + 2 + \ldots + N = \frac{N(N+1)}{2} \tag{74}$$

Indeed, consider and add

$$1 + 2 + \ldots + (N-1) + N$$

$$N + (N-1) + \ldots + 2 + 1$$

Also,

$$\sum_{k=1}^{N} k^2 = 1^2 + 2^2 + \ldots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

Area under a straight line

Consider $v(t) = t$ (acceleration $= 1$ and $v(0) = 0$) and look for displacement from $t = 0$ to arbitrary t (in the picture, $t = 1$). Use discrete

$$t_k = k\Delta t = v_k\,,\ \Delta t = \frac{t}{N}$$

$$\sum_{k=1}^{N-1} v_k \Delta t = (\Delta t)^2 \frac{N(N-1)}{2} = \frac{1}{2} t^2 (1 - \frac{1}{N})$$

Area under a parabola

Consider $y = x^2$ and look for area between $a = 0$ and arbitrary $b > 0$. Use discrete

$$x_k = k\Delta x\,,\; y_k = (k\Delta x)^2\,,\; \Delta x = \frac{b-a}{N}$$

$$\sum_{k=1}^{N-1} y_k \Delta x = (\Delta x)^3 \frac{N(N-1)(2N-1)}{6} = \frac{1}{3} b^3 (1 - \frac{1}{N})(1 - \frac{1}{2N})$$

Riemann Sums

Partition

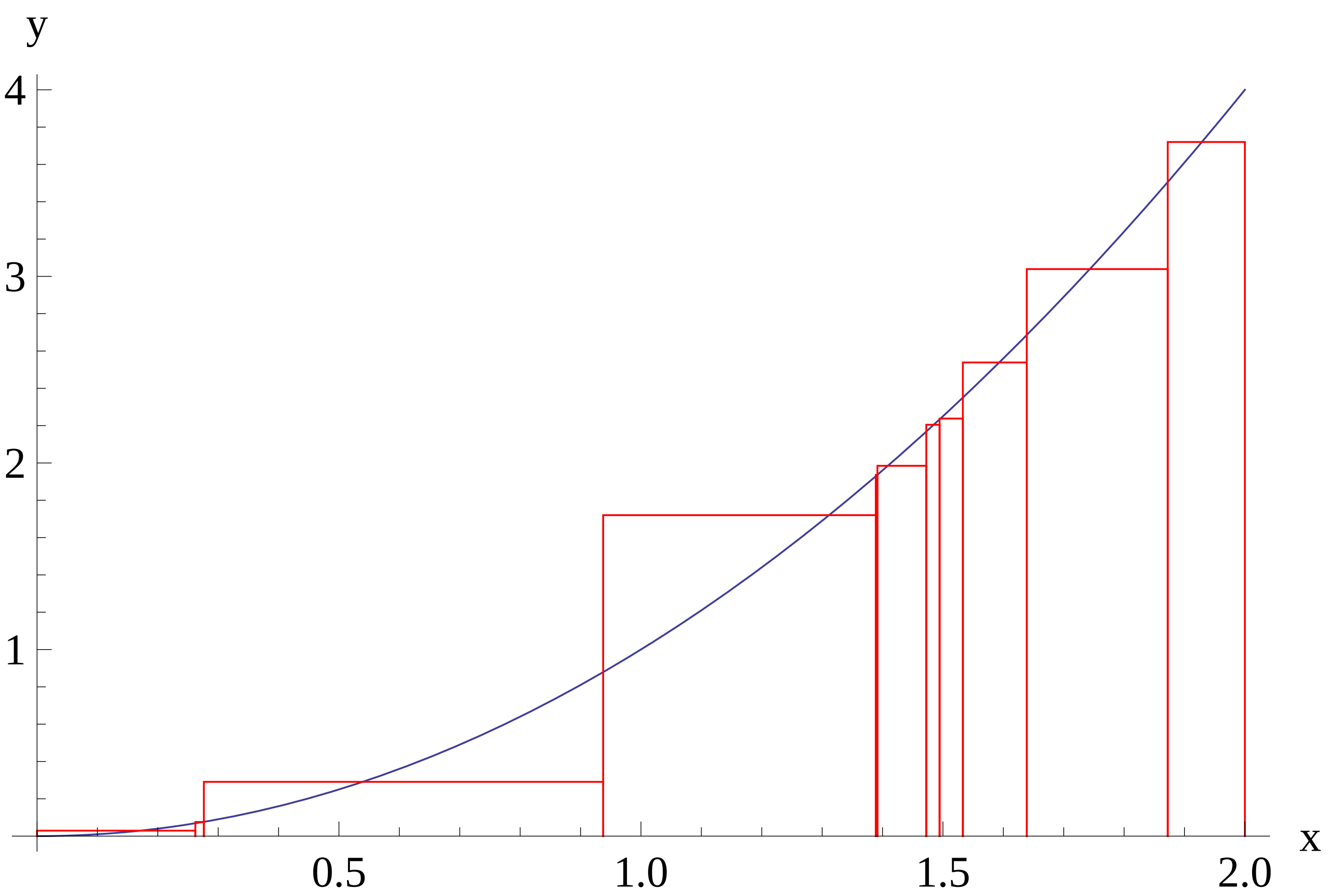
$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b\}$$

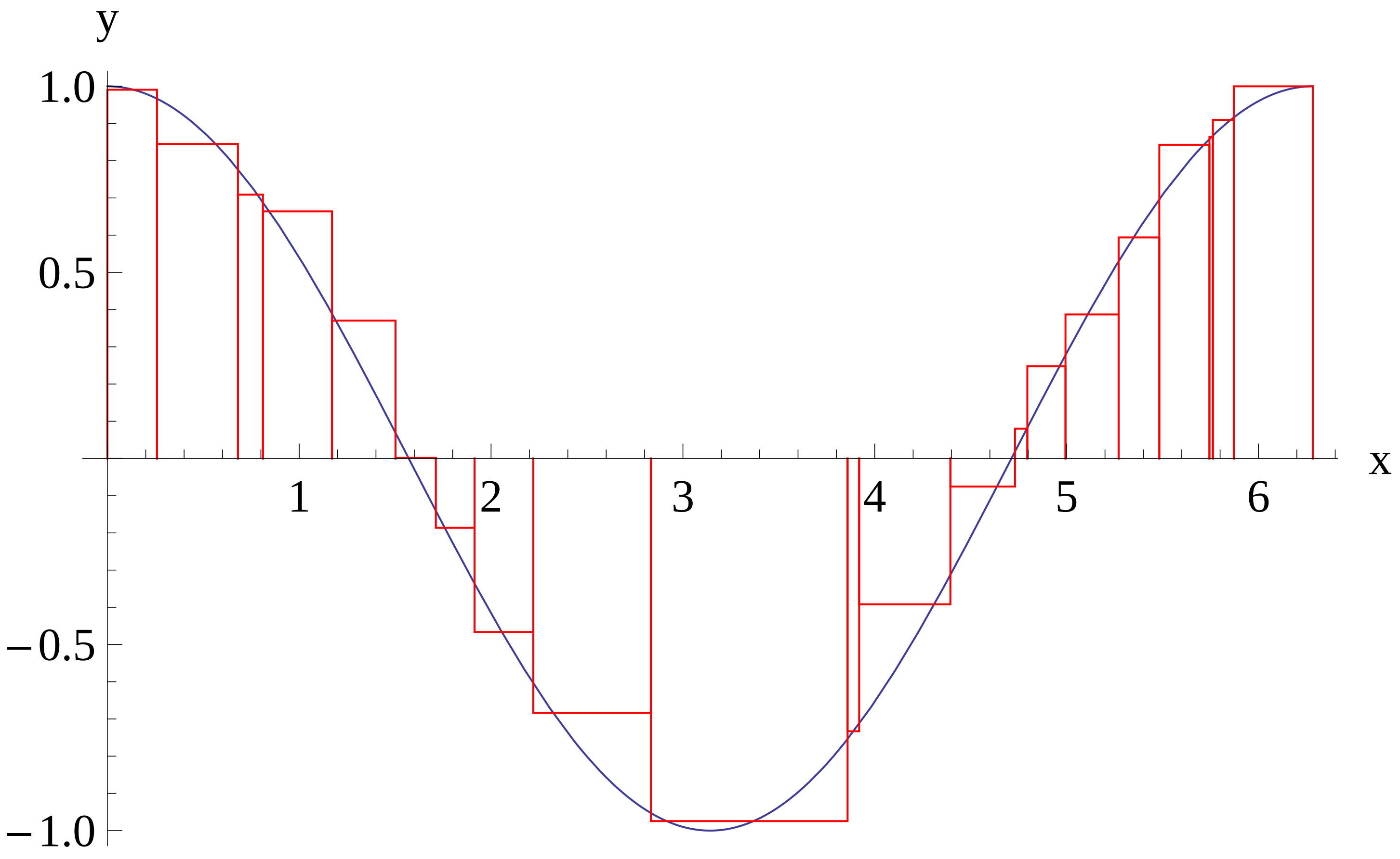
$$\Delta x_k = x_k - x_{k-1}\,,\ ||P|| = \max\,[\Delta x_k]$$

$$x_{k-1} \le c_k \le x_k$$

$$R.S.:\ \sum_{k=1}^{N} f(c_k)\Delta x_k \qquad (75)$$

$f(c_k) = \max f(x)\,,\ x_{k-1} \le x \le x_k$ - upper sum.
$f(c_k) = \min f(x)\,,\ x_{k-1} \le x \le x_k$ - lower sum.


y
4
3
2
1
0.5
1.0
1.5
2.0
x


y
1.0
0.5
−0.5
−1.0
x
1
2
3
4
5
6

Definite integral:

$$\int_a^b f(x)\,dx = \lim_{||P||\to 0} \sum_{k=1}^{N} f(c_k)\Delta x_k = \tag{76}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} f(c_k)\Delta x\,,\ \Delta x = \frac{b-a}{N} \tag{77}$$

$$area = \int_a^b f(x)\,dx\,,\ f(x) \geq 0\,,\ b > a \tag{78}$$

Mean Value Theorem:

If $f(x)$ is continuous on $[a, b]$ then at some $a \leq c \leq b$

$$f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f \, dx$$

Proof: From min-max inequality $\min f \leq \bar{f} \leq \max f$.
Then use I.V.T.

E.g., if $\int_a^b f \, dx = 0$, there is a root of f

Fundamental Theorem of Calculus. Pt.I

If $f(x)$ is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous and diff. on (a,b) with

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \tag{79}$$

Part II ("Evaluation Theorem"):

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b \tag{80}$$

where F is *any* A.D. of f.

E.g. (Pt. I):

$$\frac{d}{dx}\int_a^x t\sin t\,dt = x\sin x$$

(a does not matter!)

$$\frac{d}{dx}\int_x^a t\sin t\,dt = -x\sin x$$

$$\frac{d}{dx}\int_a^{x^2} t\sin t\,dt = (x^2\sin x^2)\frac{dx^2}{dx} = \ldots$$

E.g. (Pt.II):

$$\int_0^{\pi} \sin x \, dx = -\cos x|_0^{\pi} = 2$$

$$\int_0^{\pi/4} \frac{dx}{\cos^2 x} = \tan x|_0^{\pi/4} = 1$$

$$\int_0^{1} \frac{dx}{x^2+1} = \arctan x|_0^{1} = \pi/4$$

$$\int_1^{2} dx/x = \ln x|_1^{2} = \ln 2$$

$$\int_0^{b} e^{\alpha x} \, dx = \frac{1}{\alpha} e^{\alpha x}|_0^{b} = \frac{e^{\alpha b} - 1}{\alpha}$$

The "net change theorem"

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt$$

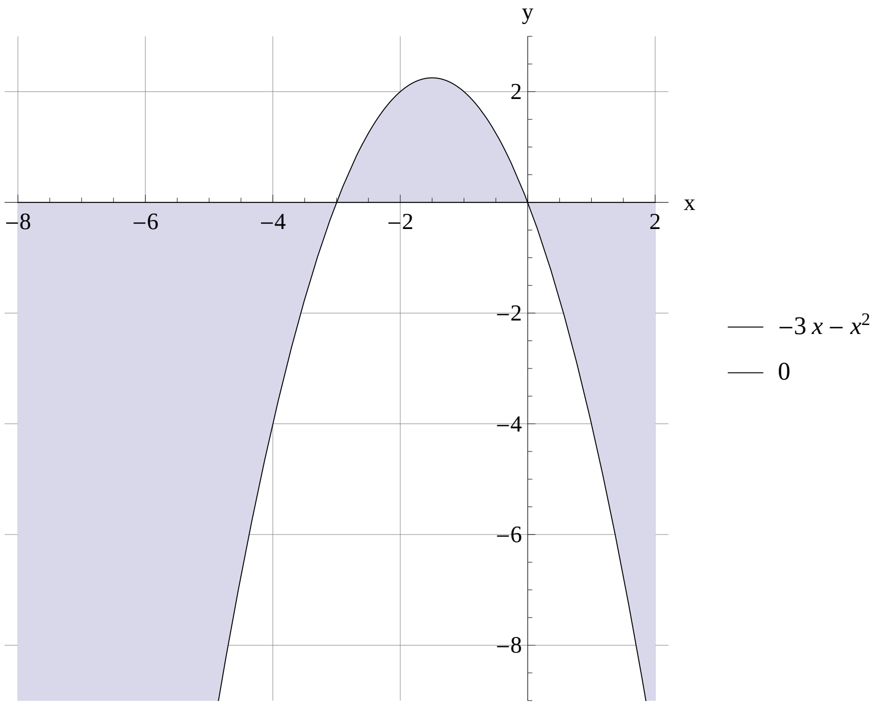
$$area = \int_a^b |f(x)|\, dx$$

not convenient since don't know A.D. for $|f|$.

$$\int_0^{2\pi} |\sin x|\, dx = -\cos x|_0^{\pi} + \cos x|_{\pi}^{2\pi} = 4$$

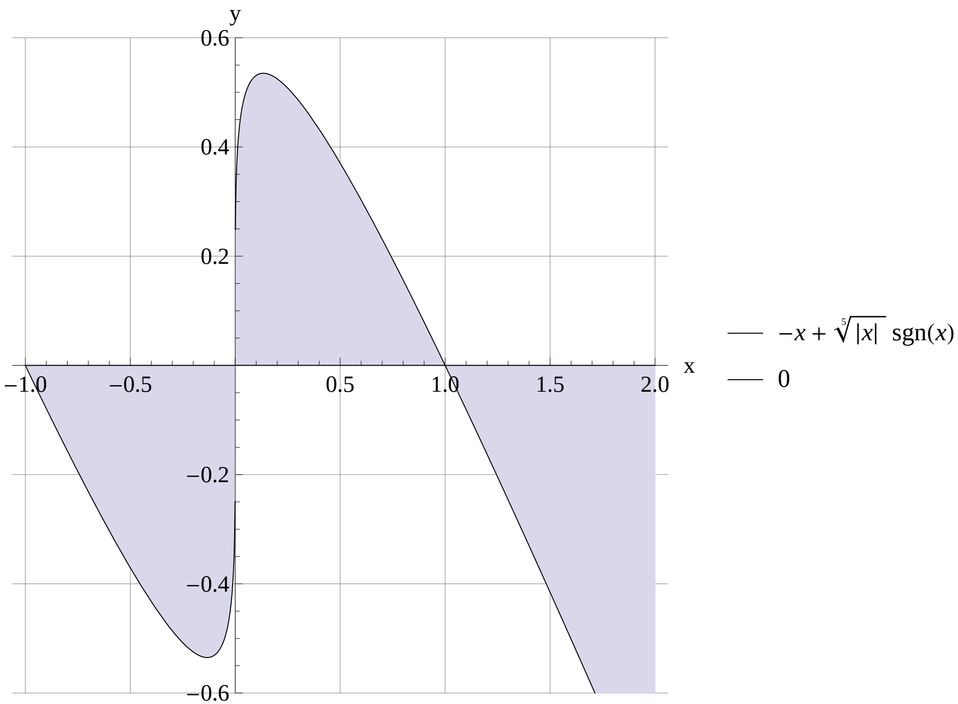
$$f = x(x+1)(x-2)\,,\ F(x) = x^4/4 - x^3/3 - x^2$$

$$area = \int_{-1}^{0} f\, dx - \int_0^2 f\, dx = F(0) - F(-1) - [F(2) - F(0)] = \dots$$



$$F(x) = -x^3/3 - 3x^2/2\,,$$

$$A = -[F(-3) - F(-8)] + F(0) - F(-3) - [F(2) - F(0)]$$



$$F(x) = \frac{5}{6}x^{6/5} - x^2/2$$

$$area = 2[F(1) - F(0)] - [F(2) - F(1)] = \ldots$$

Substitution method

$$\int f(u)u'(x)\,dx = \int f(u)\,du \tag{81}$$

$$dx = \frac{1}{k}\,d(kx+b) \tag{82}$$

$$x\,dx = d(x^2/2) \tag{83}$$

$$dx/x^2 = -d(1/x) \tag{84}$$

$$dx/x = d(\ln|x|) \tag{85}$$

$$\sin x\,dx = -d(\cos x) \tag{86}$$

$$\cos x\,dx = d(\sin x) \tag{87}$$

$$dx/\cos^2 x = d(\tan x) \tag{88}$$

$$e^{ax}\,dx = \frac{1}{a}d(e^{ax}) \tag{89}$$

$$\int f(kx+b)\,dx = \frac{1}{k}\int f(kx+b)\,d(kx+b) = \frac{1}{k}F(kx+b)+C$$

$$\int \sqrt{kx+b}\,dx = \frac{1}{k}\frac{2}{3}(kx+b)^{3/2}+C$$

$$\int \sec^2(kx+b)\,dx = \frac{1}{k}\tan(kx+b)+C$$

$$\int xe^{x^2}\,dx = \int e^{x^2}\,d(x^2/2) = \frac{1}{2}e^{x^2}+C$$

$$\int x^2e^{x^3}\,dx = \int e^{x^3}\,d(x^3/3) = \frac{1}{3}e^{x^3}+C$$

$$I = \int \sec x \, dx = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{du}{1-u^2} \,, \quad u = \sin x$$

$$I = \int \frac{1}{2} \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + C$$

$$I = \int x\sqrt{2x+1} \, dx = \int \frac{u-1}{2} \sqrt{u} \, d\frac{u}{2} \,, \; u = 2x{+}1 \,, \; x = (u{-}1)/2$$

$$I = \frac{1}{4}(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}) + C = \ldots$$

$u = \cos x$:

$$\int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{du}{u} = -\ln|u|{+}C = \ln|\sec x|{+}C$$

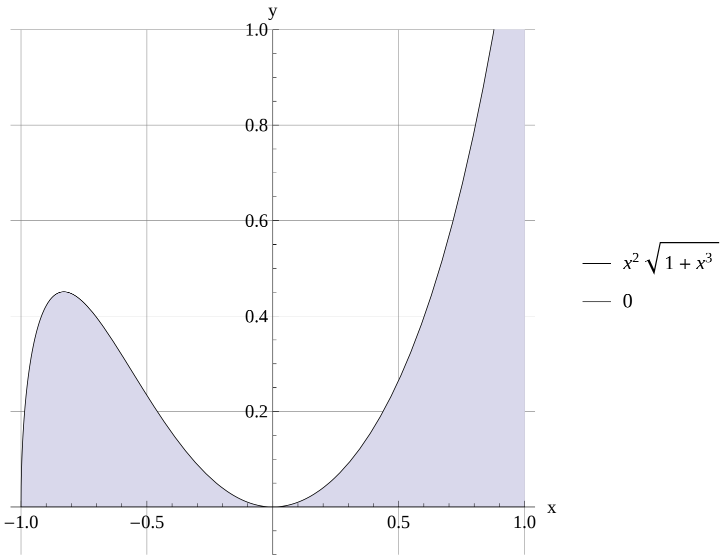
Substitution method for definite integrals

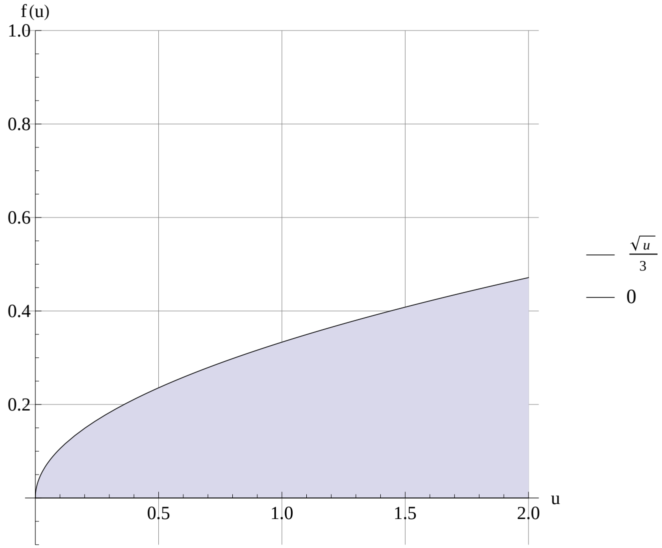
$$\int_a^b f(u)u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du \tag{90}$$

$$\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx = \int_{-1}^{1}\sqrt{x^3+1}\,d\frac{(x^3+1)}{3} = \frac{1}{3}\cdot\frac{2}{3}(x^3+1)^{3/2}\Big|_{-1}^{1} = \frac{2}{9}2^{3/2}$$

or $u = x^3+1\,,\ u(-1)=0\,,\ u(1)=2$

$$\frac{1}{3}\int_0^2 \sqrt{u}\,du = \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{9}2^{3/2}$$


y
1.0
0.8
0.6
0.4
0.2
−1.0
−0.5
0.5
1.0
x
$x^2\sqrt{1+x^3}$
0

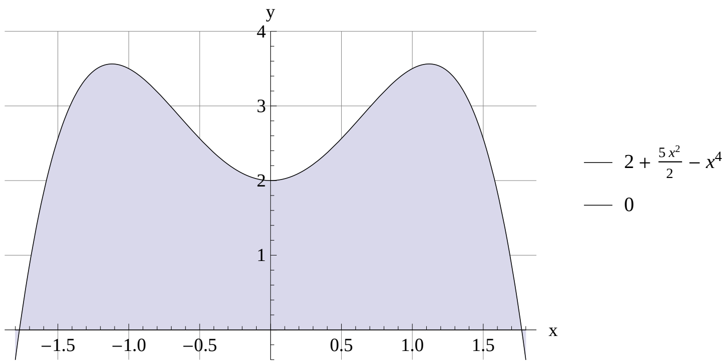

f(u)
1.0
0.8
0.6
0.4
0.2
0.5
1.0
1.5
2.0
u
$\frac{\sqrt{u}}{3}$
0

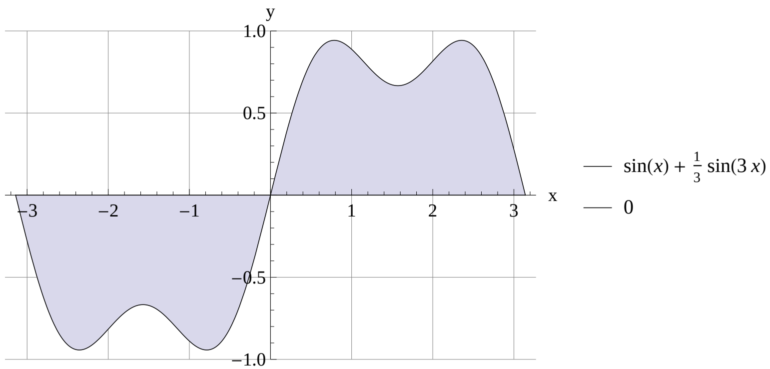
$u = \cot x$:

$$\int_{\pi/4}^{\pi/2} \cot x \csc^2 x \, dx = -\int_{\cot\frac{\pi}{4}}^{\cot\frac{\pi}{2}} u \, du = -\frac{1}{2}u^2\Big|_1^0 = \frac{1}{2}$$

$u = \cos x$:

$$\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x \, dx}{\cos x} = -\int_{\cos(-\pi/4)}^{\cos(\pi/4)} \frac{du}{u} = -\int_{1/\sqrt{2}}^{1/\sqrt{2}} \ldots = 0$$





$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f\,dx\,,\ (f\ -\ even)$$

$$\int_{-a}^{a} f(x)\,dx = 0\,,\ (f\ -\ odd)$$

Proof

$$I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = I_1 + I_2$$

$$I_1 = \int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-u)\,du = \pm \int_{0}^{a} f(u)\,du =$$

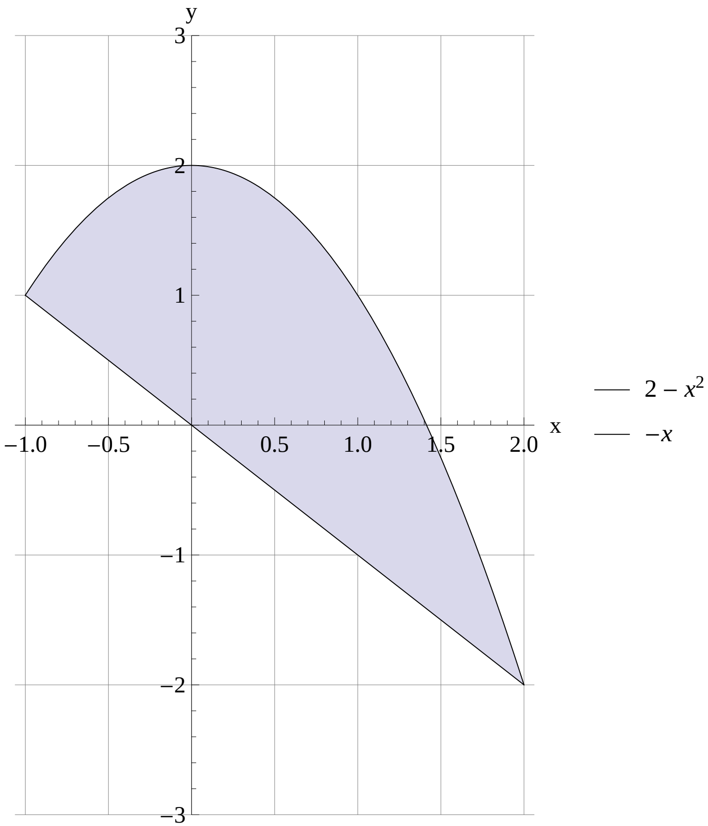
$$= \pm I_2\,,\ ('' + ''\ for\ even\,,\ '' - ''\ for\ odd)$$

E.g.

$$\int_{-1}^{1} (x^4 - 3x^3 + 4x^2 + x - 2)\,dx = 2\int_{0}^{1} (x^4 + 4x^2 - 2)\,dx = \ldots$$

Area between two curves

$$A = \int_a^b (f - g)\, dx\,,\ f(x) \geq g(x)$$



$$f = 2 - x^2\,,\ g = -x$$

Intersection:

$$f(x) = g(x)\,,\ x = -1\ and\ x = 2$$

$$A = \int_{-1}^{2} (2 - x^2 + x)\,dx = (2x - x^3/3 + x^2/2)\Big|_{-1}^{2} =$$

$$= 2(2+1) - \frac{1}{3}(2^3 + 1) + \frac{1}{2}(2^2 - 1) = \ldots > 0$$