Differentiation rules

$$(af(x) + bg(x))' = af' + bg' \tag{17}$$

Proof. Consider

$$\frac{af(z) + bg(z) - af(x) - bg(x)}{z - x} \equiv \frac{a[f(z) - f(x)] + b[g(z) - g(x)]}{z - x}$$

and take the limit $z \to x$.

$$[f(x) \cdot g(x)]' = f'g + fg' \tag{18}$$

Proof. Consider

$$\frac{f(z)g(z) - f(x)g(x)}{z - x} \equiv \frac{[f(z) - f(x)]g(z) + f(x)[g(z) - g(x)]}{z - x}$$

and take the limit $z \to x$.

Example:

$$(x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'}{f^2} \tag{19}$$

Proof. Consider

$$\frac{1/f(z) - 1/f(x)}{z - x} \equiv \frac{f(x) - f(z)}{f(z)f(x)(z - x)}$$

and take the limit $z \to x$.

Example: $(1/x)' = -1/x^2$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2} \tag{20}$$

Derivatives of elementary functions

$$(x^n)' = nx^{n-1}\,,\ \text{any } n \tag{21}$$

$$(\sin x)' = \cos x\,,\ (\cos x)' = -\sin x \tag{22}$$

$$(e^x)' = e^x\,,\ (a^x)' = a^x \cdot \ln a \tag{23}$$