ELECTRICITY & MAGNETISM

Lecture notes for Phys 121. Part II.

Dr. Vitaly A. Shneidman

Department of Physics, New Jersey Institute of Technology, Newark, NJ 07102

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Abstract

This part of Lecture Notes is mostly on magnetism. It continues Part I on electrostatics and current. Please refer to Part I for the general information (Abstract) and for the Introduction on vectors and vector fields.

Contents

I. MAGNETIC FIELD AND MAGNETIC FORCE.

We now start studying a new vector field, the magnetic field $\vec{B}$:

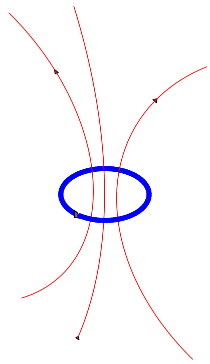
$$\text{Notations: } \vec{B}\,, \;\; \text{Units: } [B] \;=\; T(\text{tesla})$$

Earth $B < 10^{-4}\,T$ (depends on location, magnetic "South pole" is close to geographic "North")

a refrigerator magnet: $B \sim 10^{-2}\,T$, MRI: $B \sim 1.5\,T$, Lab magnet: $B > 20\,T, \ldots$

Safety: time-independent magnetic field of several teslas or less is mostly safe, but if field is time-dependent or if you move in or out of magnetic field, it can become dangerous even for modest values of the field (will discuss later in connection with Faraday's effect).

How does the magnetic field originate (in other words, What are the *sources* of this field)? Unlike the electrostatic field E which is due to electric charges, **there are no magnetic charges** (or, more cautiously, such charges have not been discovered despite intense search during the last 100 years - see the cover of *Physics Today* from a few years ago). In the absence of "elementary" sources of magnetic field, one can identify currents, including the microscopic current as electrons in orbital motion around a nucleus, and permanent magnets. Many elementary particles, including electrons, have their own magnetic moments. Competition of the inherent magnetic moment of the electrons with magnetic effects due to orbital motion determines magnetic properties of a material; usually, orbital motion prevails but on occasions (iron, nickel) the inherent magnetic moment takes the upper hand.



"Elementary" sources of magnetic field. Left -there are NO known "magnetic charges" (particles with only North or only South poles). Right - a microscopic ring current, as electrons around a nucleus (note that direction of motion is opposite for electrons, due to a negative charge). The North pole is on top, and the South pole is on the bottom. Many elementary particles have their own magnetic dipole moments, with a similar structure of field.

Conventions for pictures

Problems with magnetism are always 3D. To simplify graphics, will use a 2D convention, as in Fig. 1: if field goes into the page you see "feathers-of-an-arrow" which flies away fro you; if field goes out of the page, you see "head-of-an-arrow" flying towards you. (A similar convention will be used later when drawing currents which flow in or out of the page).

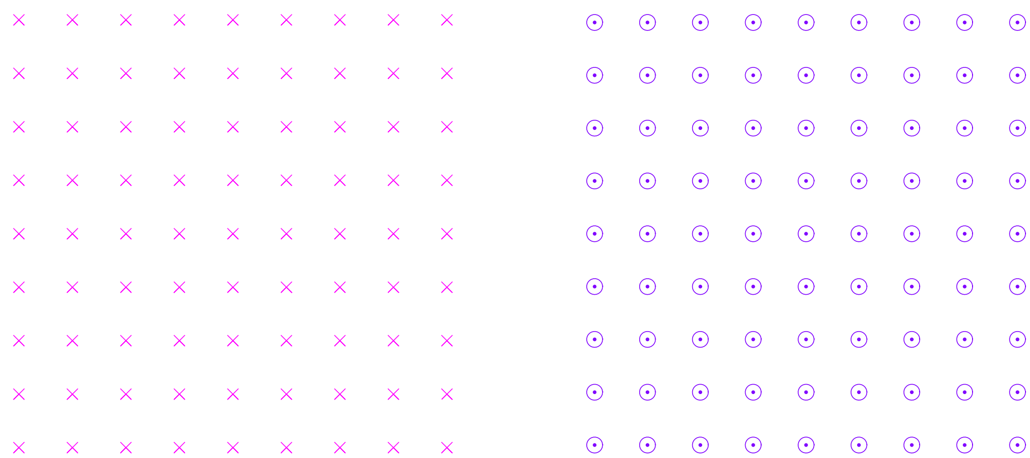


FIG. 1: Convention for 2D pictures of magnetic field. Left - field goes into the page; right - field goes out of the page.

1. Properties of magnetic field lines

Similar to electric field:

- tangent - direction of $\vec{B}$
- density - proportional to $\left|\vec{B}\right|$
- do not cross

Different from electric field:

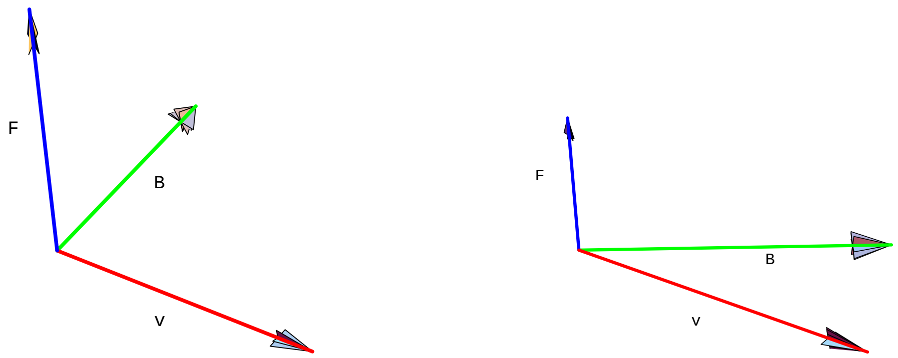
- magnetic lines never start or end (i.e. there are NO magnetic charges)
- magnetic lines can loop (they loop around currents). This does not contradict conservation of energy since the static magnetic field does not do work.

II. MAGNETIC FORCE

The force acting on a charge moving with velocity $\vec{v}$ in field $\vec{B}$ is given by

$$\boxed{\vec{F} = q\vec{v} \times \vec{B}} \tag{1}$$

Similarly to electric force, the magnetic force is proportional to charge and field magnitudes. But, this force acts only on *moving* charges and is perpendicular (not parallel) to direction of the field $\vec{B}$.



Magnetic force on a positive particle, $\vec{F} = q\vec{v} \times \vec{B}$. Note that $\vec{F}$ (blue) is perpendicular to both $\vec{v}$ (red) and $\vec{B}$ (green) , and reaches maximum when $\vec{v}$ and $\vec{B}$ are perpendicular to each other. For parallel (or antiparallel) $\vec{v}$ and $\vec{B}$ the force would be zero.

$$\text{Power: } \boxed{P = \vec{F} \cdot \vec{v} = 0} \;\Rightarrow\; \text{no work by static magnetic field} \tag{2}$$

Example. Find acceleration of an electron if $\vec{v} = V(\vec{i}+\vec{j})$, $\vec{B} = b(\vec{j}+\vec{k})$ (with $V = 10^6\, m/s$ and $b = 1\,T$ some positive numbers). Solution. Use the "ring diagram" (or, look up the components of the cross product):

$$\vec{a} = \frac{1}{m_e}\vec{F} = \frac{1}{m_e}(-|e|)\vec{v} \times \vec{B} = \frac{1}{m_e}(-|e|)Vb(\vec{i} + \vec{j}) \times (\vec{j} + \vec{k}) =$$

$$= \frac{1}{m_e}(-|e|)Vb(\vec{i}\times\vec{j}+\vec{i}\times\vec{k}+\vec{j}\times\vec{k}) = \frac{1}{m_e}(-|e|)Vb(\vec{k}-\vec{j}+\vec{i}) = \frac{1}{m_e}|e|Vb(-\vec{k}+\vec{j}-\vec{i}) \sim 10^{17}\frac{m}{s^2}$$

Example: separation of particles by charges and masses - see Fig. 2.

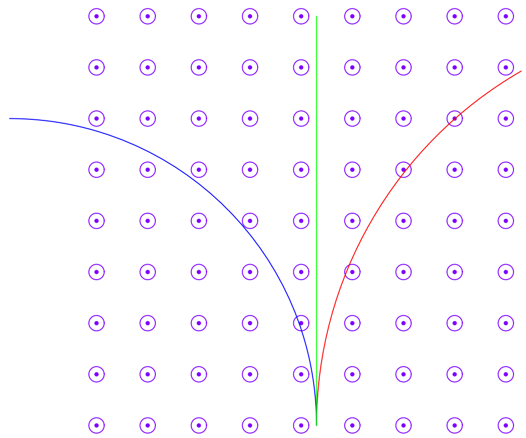
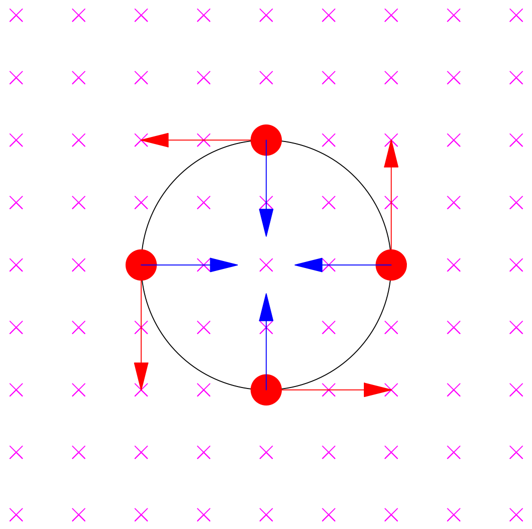


FIG. 2: Separation of particles of different signs in the magnetic field, as observed in radioactive decay. The positive (red) particles deviate to the right (they were called "alpha-particles" and turned out to be nuclei of Helium). The negative (blue) particles deviate left. (they were called "beta-particles" and turned out to be electrons). Neutral particles (green) do not deviate. (they were called "gamma-particles", and turned out to be quanta of electromagnetic radiation). Note that the path of negative particles is more curved, due to smaller mass.

A. Circular motion of a particle

Since the magnetic force $\vec{F}$ is perpendicular to the velocity, the latter changes only the direction while the magnitude ("speed") remains constant. The same is true for $\vec{F}$. Thus, the magnetic force leads to circular motion.



Circular motion of a positive particle in magnetic field. The velocity of a particle is shown in red, and the magnetic force (blue) provides a centripetal force for circular motion. The period of revolution is independent of the velocity (!) - a faster particle will make a larger circle, completing the revolution in the same time.

For the actual calculations we use the 2nd Law

$$m\frac{v^2}{r} = qvB \text{ , or}$$

$$\boxed{r = \frac{mv}{qB}} \tag{3}$$

This is the experimental way to find q/m for elementary particles, and eventually to find their masses (which otherwise are very small)- see the example on next page.

We now use the above equation to find the period of revolution $T = 2\pi r/v$:

$$\boxed{T = 2\pi\frac{m}{qB}} \tag{4}$$

Doest not depend on the velocity (energy) of a particle (!)

Example (hard). A particle is accelerated through a potential difference ΔV and the enters magnetic field B at a right angle, where it revolves in a circle of radius r. Find q/m and the period of revolution.

$$\text{from work-energy theorem } \frac{1}{2}mv^2 = q\Delta V \ \Rightarrow \ v = \sqrt{\frac{q}{m}2\Delta V}$$

we substitute this into

$$r = \frac{mv}{qB} \ \Rightarrow \ r = \frac{m}{qB}\sqrt{\frac{q}{m}2\Delta V} = \frac{1}{B}\sqrt{\frac{m}{q}2\Delta V} \ \Rightarrow \ \boxed{\frac{q}{m} = \frac{2\Delta V}{(rB)^2}}$$

$$\text{The period of revolution : } T = 2\pi\frac{m}{q}\frac{1}{B}\,, \ \text{ is independent of } \Delta V$$

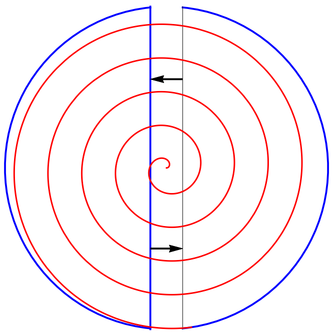
Here are a few typical q/m ratios

$$\text{proton : } \frac{q}{m} \simeq \frac{1.6\cdot 10^{-19}\,C}{1.67\cdot 10^{-27}\,kg} \sim 10^8\frac{C}{kg}$$

$$\text{electron : } \frac{q}{m} \simeq \frac{1.6\cdot 10^{-19}\,C}{9.11\cdot 10^{-31}\,kg} \sim 10^{11}\frac{C}{kg}$$

$$\text{alpha-particle : } \frac{q}{m} \simeq \frac{2\times 1.6\cdot 10^{-19}\,C}{4\times 1.67\cdot 10^{-27}\,kg} \approx 0.5\cdot 10^8\frac{C}{kg}$$

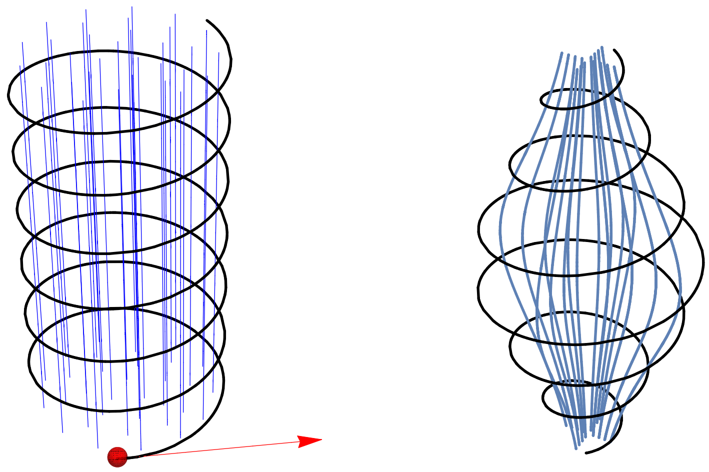
The cyclotrone (advanced topic, not on exam)



Since magnetic field alone cannot change energy, need electric field to accelerate particles. The cyclotrone is placed in strong magnetic field which is perpendicular to the plane of the picture and which makes a particle to move in a circular ark. When the particle reaches the space between the half-rings an electric field parallel to $\vec{v}$ accelerates the particle, increasing its energy and radius of the ark (but not changing the period of revolution!). When the particle, after completing a half circle, again enters the space between the half-rings, the electric field changes to the opposite direction again being parallel to velocity vector and further accelerating the particle, etc.

B. 3D motion of charged particles

In a uniform magnetic field, circular motion is observed only if $\vec{v}$ of a particle is strictly perpendicular to $\vec{B}$. More generally, the velocity can be broken into 2 components, $\vec{v}_{||}$ parallel to $\vec{B}$ and $v_{\perp}$ perpendicular to B. Since the magnetic force contains a cross product, the parallel component of the velocity will not matter and force will be only in direction perpendicular to $\vec{B}$ with magnitude determined only by $v_{\perp}$. This will determine the motion in a plane perpendicular to $\vec{B}$ while motion along $\vec{B}$ will proceed with a constant speed $v_{||}$. A combination of linear and circular motion will produce a helix - see the left picture below. The actual calculations, however, are on advanced side and can be skipped in the first reading.



Left: uniform field. Brake $\vec{v}$ into components parallel and perpendicular to $\vec{B}$

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}\,,\ \vec{F} = q\vec{v}_{\perp} \times \vec{B}$$

$$r = \frac{mv_{\perp}}{qB}\,,\ T = 2\pi\frac{m}{qB}$$

$$v_{||} = const \ \Rightarrow \ \text{step of the helix} \ = v_{||}T = 2\pi\frac{mv_{||}}{qB}$$

Right: non-uniform field. Paths of charged particles still approximately wound around the field lines. Examples: Aurora Borealis and 'magnetic bottle' - in class.

Motion in combined $\vec{B}$ and $\vec{E}$ fields

When both $\vec{B}$ and $\vec{E}$ are present, one has to consider the total force which is the sum of electric force $\vec{F}_e = q\vec{E}$ and magnetic force $\vec{F}_m = q\vec{v} \times \vec{B}$

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)} \tag{5}$$

For example, the condition that force is zero (i.e. particle moves in a straight line) results in a condition

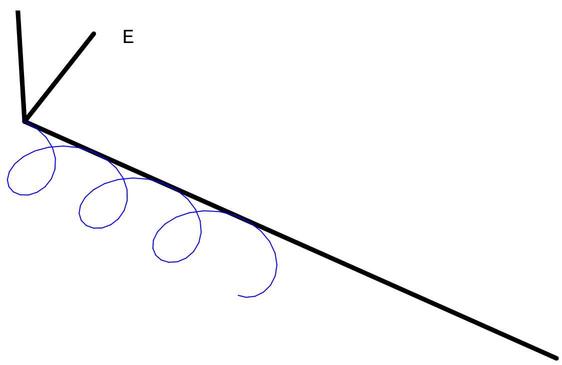
$$\vec{E} + \vec{v} \times \vec{B} = 0$$

regardless of the sign of the charge. Note that in order to have a zero force, $\vec{E}$ cannot be parallel to $\vec{B}$ and in the simplest case when it is perpendicular, $E = vB$

In a general case, eq.(5) should be combined with the Newton equation

$$m\vec{a} = \vec{F}$$

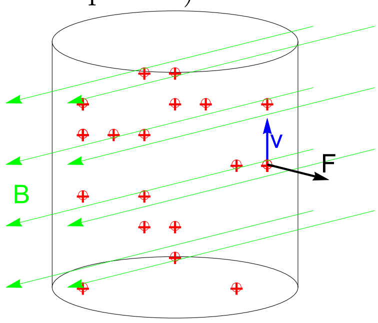
Solution is hard, and can be obtained analytically only when both fields are uniform and perpendicular to each other. Otherwise, it is the work for a computer. An example of a cycloid is given below.



Motion of a charged particle in magnetic field, $\vec{B}$ which points in the vertical z-direction; the electric field $\vec{E}$ points in the y-direction, into the page. With $v_z = 0$ the particle stays in the plane but drifts in the x-direction. [The latter is important in understanding the rather advanced Hall effect].

C. Magnetic force on a wire

Conventional current inside a wire can be represented as motion of positive charges q with drift velocity $v = v_d$. Each charge experiences a force $q\vec{v} \times \vec{B}$ (shown for a single charge in the picture).



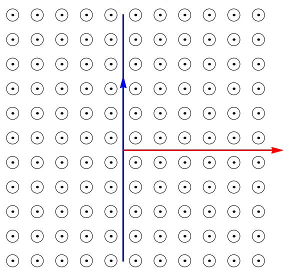
The forces add up to give the total force on the segment of wire of length L. Transition from formulas for a single charge which moves with velocity $\vec{v}$ to formulas for a small wire which carries current i and has length $\vec{L}$ (vector in direction of current), is given by

$$q\vec{v} \to i\vec{L} \tag{6}$$

Thus

$$\boxed{F = iLB}\text{, or in vector form } \vec{F} = i\vec{L} \times \vec{B} \tag{7}$$

Example: current (blue) up in the plane of the page; field out of the page. Find the direction of force (red).



Note: picture can be rotated.

Now, derivation of eq. (7). Force on a single charge q: $\vec{F_i} = q\vec{v}_d \times \vec{B}$ ($\vec{v}_d$ is the drift velocity). Force on a unit volume (with n the number of charges in unit volume and $\vec{J} = nq\vec{v}_d$ the density of current): $n \cdot \vec{F_i} = nq\vec{v}_d \times \vec{B} = \vec{J} \times \vec{B}$ Multiply by volume $V = LA$ (with L length of wire and A the cross-sectional area) and use $i = JA$ to get eq. (7).

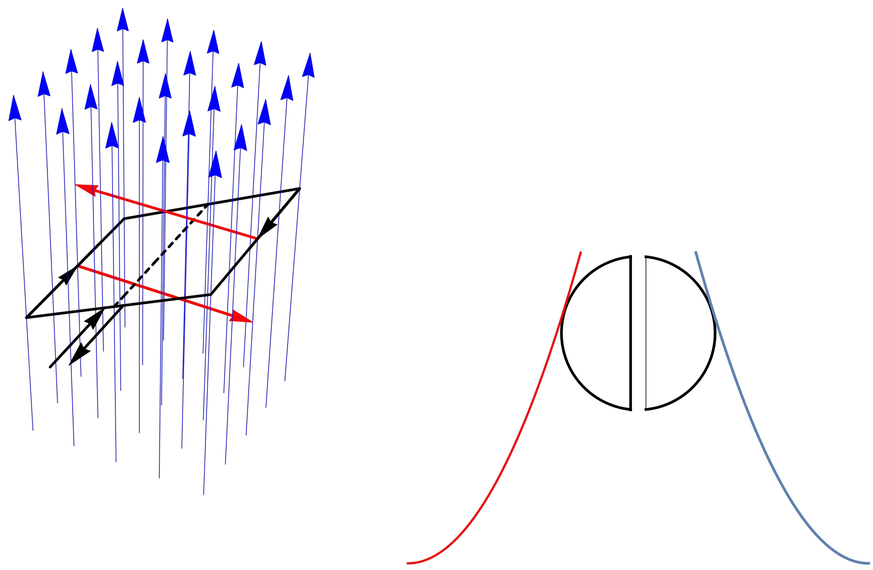


FIG. 3: Torque on a frame with current (left) and brush contacts(right) used to ensure spinning in one direction.

A loop with current will experience a torque when placed in magnetic field. Major application - electric motor. If $\vec{A}$ is the vector-area of the loop (in the direction of the normal) and θ is the angle between $\vec{A}$ and $\vec{B}$, then torque

$$\tau = AIB \sin\theta \tag{8}$$

In vector form: magnetic moment of a loop

$$\vec{\mu} = I\vec{A} \tag{9}$$

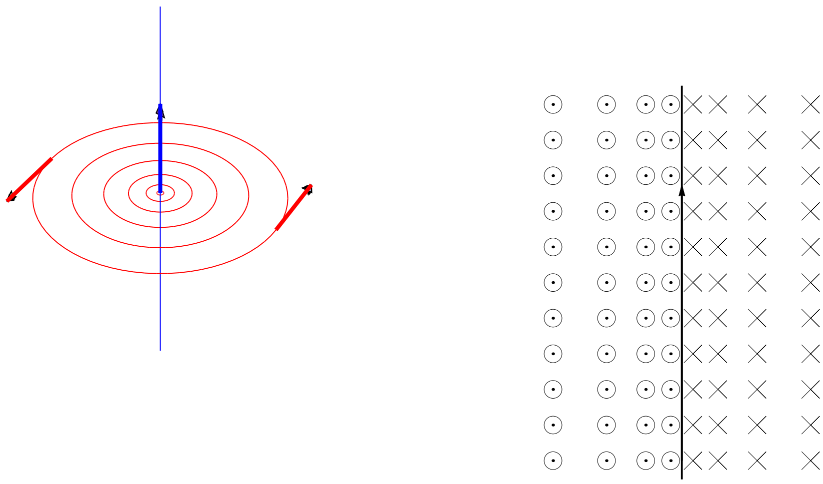
where the r.h. rule is used to find direction of μ and it should be multiplied by the number of turns N for a coil. Then,

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{10}$$

Here is the math derivation. Let the frame in fig. 3 be composed of the following vectors (starting from axis, along currents): $\vec{a}$, $\vec{b}$, $-2\vec{a}$, $-\vec{b}$, $\vec{a}$. Note, $\vec{A} = 2\vec{a} \times \vec{b}$, while forces on the "b" sides equal $\pm I\vec{b} \times \vec{B} = \pm\vec{F}_b$. Thus, $\vec{\tau} = 2\vec{a} \times \vec{F}_b = I\vec{A} \times \vec{B}$, which proves the above equations.

III. MAGNETIC FIELDS FROM CURRENTS

A. Straight wire



Magnetic field (red circular lines) due to a long straight current (blue). Note the direction of the field determined by the r.h. rule. Right: same in a 2D view.

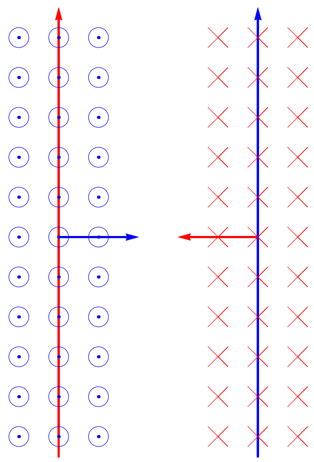
Direction: r.h. rule. Thumb of r.h. goes with the current in the wire; curled fingers show direction of $\vec{B}$. Field lines circle around the current. The formula for the magnitude $B(r)$ can be derived from various fundamental principles (whatever we agree to treat as "fundamental"). At the moment we use it as "empirical result":

$$\boxed{B = \frac{\mu_0 i}{2\pi r}}\,,\ \mu_0 = 4\pi \cdot 10^{-7}\,\frac{m \cdot T}{A} \tag{11}$$

μ_0 – magnetic permeability constant. In vector form $\vec{r}$

$$\vec{B} = \frac{\mu_0 \vec{I} \times \vec{r}}{2\pi r^2} \tag{12}$$

B. Magnetic force between two parallel wires



Consider two parallel wires, a and b with currents i_a and i_b in the same direction. Wires are separated by a distance d and have length L each. We find the force on wire b in two steps: 1) find the field due to a at the location of b and 2) find the force on b

We will do steps 1) and 2) first for the direction and then for the magnitude. From direction we get that parallel currents attract (and opposite -antiparallel- currents repel!). For magnitude, step 1) gives from eq. (11)

$$B_a = \frac{\mu_0 i_a}{2\pi d}$$

Then, from formula for force on a wire $F = iLB$:

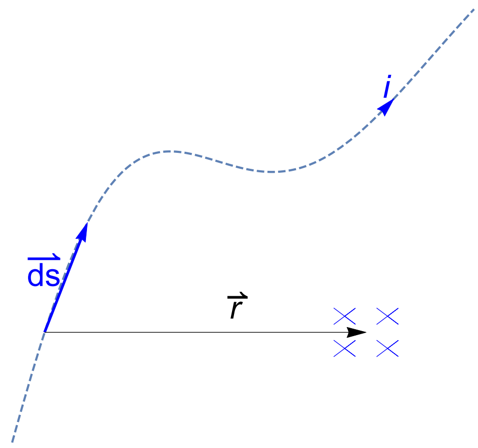
$$F_b = i_b L B_a = i_b L \frac{\mu_0 i_a}{2\pi d}$$

$$\boxed{F_b = \frac{\mu_0 i_a i_b L}{2\pi d}} \tag{13}$$

Checkpoint: find the force on wire a. Should satisfy the 3rd Law.

C. The Biot-Savarat law

How to find a field due to a wire of arbitrary complex shape?



Contribution of a small wire with current i is

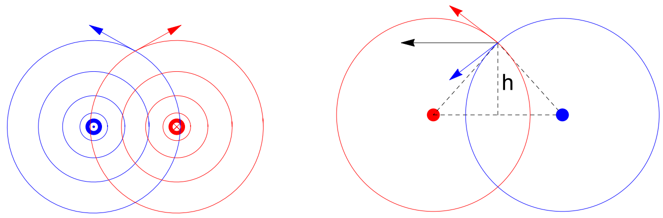
$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{s} \times \vec{r}}{r^3} \tag{14}$$

Here $d\vec{s}$ is the vector which indicates the elementary wire (in the direction of the current), while $\vec{r}$ goes from the wire to the observation point.

The superposition principle. Fields of different elementary wires add up as vectors. Can be used, in principle, to find a field from a wire of any complex shape or from several wires. Note: there will be no field in the direction of the wire.

1. Applications of Biot-Savarat

Straight wire. Can derive the above formula 11, but will get it more efficiently from the Ampere's theorem. However, will discuss an example of field due to 2 parallel wires to illustrate the superposition.

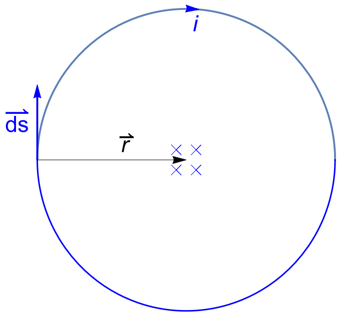


Superposition of fields due to two currents, Left: one into the page (red) and one out of the page (blue); resultant field at a point equidistant from both currents will be up. Right (for which illustrative calculations are done below): both currents out of the page.

Let $2a$ - distance between wires, r - distance from each wire to the observation point; $h = \sqrt{r^2 - a^2}$. Full field $\vec{B}$ - black (horizontal) sum of red and blue. Note triangle formed by red and blue fields is similar to the one formed by the two currents and observation point. Thus,

$$\frac{B/2}{B_1} = \frac{h}{r}, \; B = 2B_1\frac{h}{r} = \frac{\mu_0 I}{\pi r}\frac{h}{r}$$

Ring (center only).



The simplification here is that $d\vec{s}$ is always perpendicular to $\vec{r}$, and that $r = const = R$ (R is the radius of the ring) for every element of the ring. Thus, $dB = \frac{\mu_0 i}{4\pi r^2} ds$. The direction is determined by the r.h. rule (it is more convenient to curl fingers with the current; then the thumb points in the direction of the field). Contributions of individual elements just add up. Thus

$$B_{ring} = \frac{\mu_0 i}{4\pi r^2} \int_0^{2\pi R} ds = \boxed{\frac{\mu_0 i}{2R}}$$

For an *arc* with angle $\phi < 2\pi$ the field is reduced proportionally

$$B_{arc} = \frac{\mu_0 i}{2R} \cdot \frac{\phi}{2\pi} \tag{15}$$

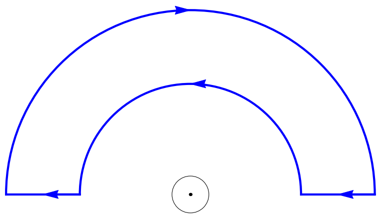
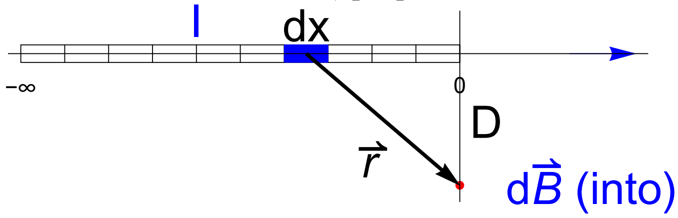


FIG. 4: Example. Superposition of field from 2 half-loops with $R > r$: $B_1 = \mu_0 I/(2r) \times 0.5$ (out of the page) and $B_2 = \mu_0 I/(2R) \times 0.5 < B_1$ (into the page).

$$B = B_1 - B_2 = \frac{\mu_0 I}{4} \left(\frac{1}{r} - \frac{1}{R} \right)$$

Example. Consider a semi-infinite wire with current I to the right. Find field B at a distance D from the end, perpendicular to the direction of current.



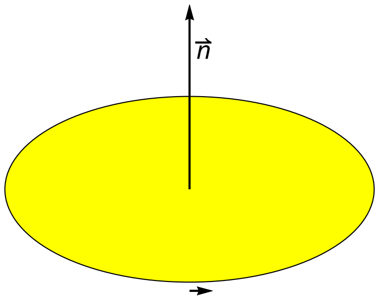
$$dB = \frac{\mu_0}{4\pi} I \frac{dx\, r \sin\alpha}{r^3}\,,\; r = \sqrt{x^2 + D^2}\,,\; \sin\alpha = \frac{D}{r}$$

$$B = \int_{-\infty}^{0} dB(x) = \frac{\mu_0}{4\pi} I D \int_{-\infty}^{0} \frac{dx}{(x^2 + D^2)^{3/2}} = \frac{\mu_0}{4\pi} I \frac{x}{D\,(x^2 + D^2)^{1/2}}\Bigg|_{-\infty}^{0}$$

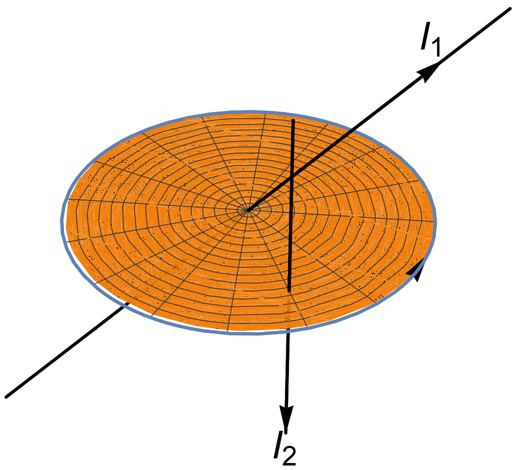
$$B = \frac{\mu_0 I}{4\pi D}$$

(which is half of the field of an infinite wire).

D. The Ampere's theorem



Preliminaries: Geometry. A closed contour (geometric line) is considered, and a positive direction along this contour is selected. A current is said to be *enclosed* if it passes inside the contour, piercing the imaginary surface stretched on this contour. The contribution of such a current is positive if it flows in the positive direction determined by the r.h. rule.



The Ampere's theorem:

$$\boxed{\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} = \mu_0 \left(I_1 - I_2 + \ldots \right)} \tag{16}$$

1. Applications of AT $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$

The AT is always valid, but to make it useful for calculations in each specific case one needs to select a good contour which is consistent with the symmetry of the problem (when it exists). Then the circulation can be evaluated even before $\vec{B}$ is known, and then used in the AT to find $\vec{B}$.

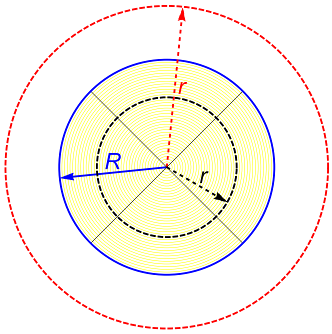


FIG. 5:

Calculating B outside and inside a long wire. Current - into the page ("X"); field lines - CW. Outside, $r > R$ - red dashed Ampere's loop; inside $r < R$ - black dashed. In both cases $\oint \vec{B} \cdot d\vec{l} = 2\pi r B$, but only current *inside* the loop contributes to AT - full current I for the outside loop, and $I(r/R)^2$ for the inside loop.

Long wire: outside Good contour - circle around the wire, in the perpendicular plane. Radius r, center at the wire. Then

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = 2\pi r B \;=\; \text{(from AT)}\; \mu_0 i$$

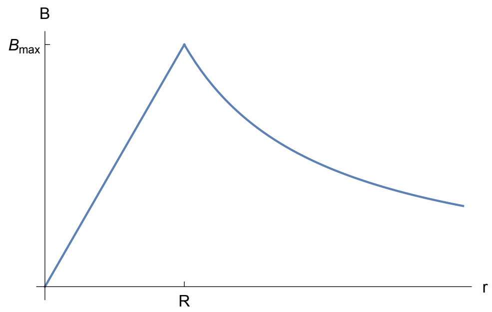
which gives eq. (11) $= \mu_0 i/(2\pi r)$ (and R of the wire does not matter as long as $r > R$).

Long wire: inside

Assume that the current is uniformly distributed (has a constant density) inside the wire with radius R. Contour - same, with the center on the axes of the cylindrical wire. Circulation - same as above, $2\pi r B$. Enclosed current is now different

$$i_{enc} = i\frac{\pi r^2}{\pi R^2} \;\Rightarrow\; B(r) = \frac{\mu_0 i}{2\pi R}\frac{r}{R}\,, \quad r \le R$$

Note that field is the same when the surface of the wire is approached either from the inside or from the outside.

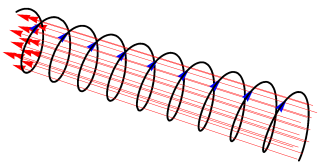


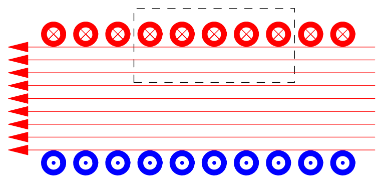
Magnetic field as a function of the distance r from the axis of the wire of radius R, both inside the wire ($r < R$) and outside the wire ($r > R$). The maximum $B_{\max} = \mu_0 I/(2\pi R)$ is achieved on the surface $r = R$.

Solenoid.

Consider a long, tightly wound coil with current i. The coil has n turns of wire per meter, and the radius turns out to be unimportant.

First, we need to understand the structure of field lines in order to get a feeling for the symmetry.One has practically straight field lines inside the solenoid, while outside they curve and are at a very large distance from each other. Direction of lines is given by the r.h. rule.





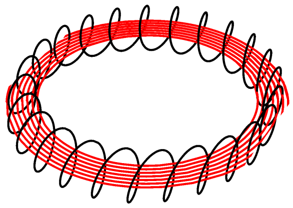
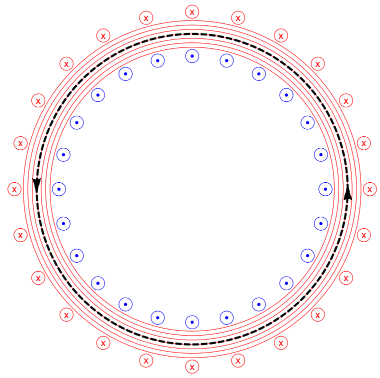
The cross section is shown on right, with the upper currents (red) going into the page and the lower currents (blue) going out of the page. The magnetic field lines are shown by horizontal red arrows.(North pole is on the left). The selected contour is shown by a dashed rectangle. Only the lower side gives a non-zero contribution to circulation, and only those currents which fall inside the rectangle contribute to the Ampere's theorem.

We now select a "good" contour. Consider a 2D cross section of the coil. For a contour select a long rectangle (with length L) which is partly inside the solenoid and which contains many turns, $N = nL \gg 1$. Two sides are parallel to the axes of the solenoid, and only one of those is inside. Note that only the latter contributes to circulation, so that one has

$$\oint \vec{B} \cdot ds = BL \; = \; (\text{from AT}) \; = \; \mu_0 I_{enc} = \mu_0 NI = \mu_0 nLI \; \Rightarrow \; BL = \mu_0 nLI$$

$$\boxed{B = \mu_0 nI} \tag{17}$$

Note that B does not depend on the distance from the axes of the solenoid, as long as one remains inside.

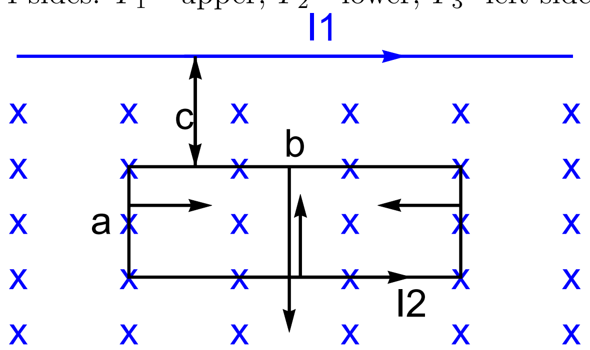
Magnetic field inside a toroidal coil with inner and outer radii R_1 and R_2 , with current I and N turns of the wire. In the cross section on right the in/out currents are shown by red/blue circles. The magnetic field lines (red) are CCW inside the toroid. The Ampere's loop (dashed) has a radius r and is also inside the toroid. Note that only the out-of-the-page (blue) currents contribute to AT.

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B \,,\ I_{enc} = NI \ \Rightarrow \ \text{from AT:}$$

$$B = \mu_0 IN/(2\pi r) \ ,\ \ R_1 < r < R_2$$

For $R_2 \approx R_1 \approx r$ one has $N/(2\pi r) = n$, the density of turns and one recovers eq.(17) for a straight solenoid.

Example. (Combination of fields and forces). A rectangular $a \times b$ frame with current I_2 is placed at a distance c from a straight wire with current I_1. Find the forces on each of the 4 sides: $\vec{F}_1$ - upper, $\vec{F}_2$ - lower, $\vec{F}_3$ -left side, $\vec{F}_4$ - right side.



Solution (2 step): First, find field (blue), then find forces - see picture.

$$F_1 = \mu_0 \frac{I_1 I_2 \, b}{2\pi c} \, , F_2 = \mu_0 \frac{I_1 I_2 \, b}{2\pi (c+a)}$$

$$dF_3 = I_2 B(x) dx \, , \; B(x) = \mu_0 I_1 / (2\pi x)$$

$$F_3 = \frac{\mu_0 I_1 I_2}{2\pi} \int_c^{c+a} \frac{dx}{x} = \frac{\mu_0 I_1 I_2}{2\pi} \ln \frac{c+a}{c} = F_4$$

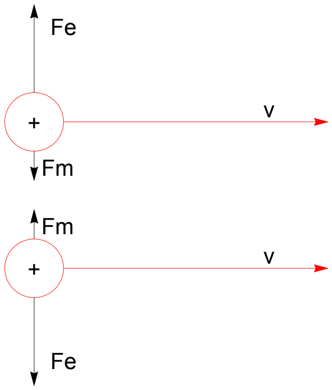
$$F_{net} = F_1 - F_2 = \frac{\mu_0 I_1 I_2}{2\pi} \frac{ab}{c(c+a)}$$

Advanced. Field and magnetic interaction of point charges.

In Biot-Savarat Law

$$Id\vec{s} \to q\vec{v}\ :\quad \vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v}\times\vec{r}}{r^3}$$

Find the ratio of F_m/F_e for 2 identical charges with velocities $\vec{v}$ separated by a distance d.



$$B = \frac{\mu_0}{4\pi} q \frac{v}{d^2} \ \Rightarrow\ F_m = qvB = \frac{\mu_0}{4\pi} q^2 \frac{v^2}{d^2}$$

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2} \ \Rightarrow\ \frac{F_m}{F_e} = \mu_0\epsilon_0 v^2$$

What is $\mu_0\epsilon_0$? Note:

$$[\mu_0\epsilon_0] = \frac{s^2}{m^2}\,,\ \frac{1}{\sqrt{\mu_0\epsilon_0}} = \left(\frac{1}{4\pi\times 10^{-7}} 4\pi\times 9\times 10^9\right)^{1/2} = \left(9\times 10^{16}\right)^{1/2} = 3\times 10^8\,\frac{m}{s} \ \Rightarrow$$

$$\boxed{\frac{F_m}{F_e} = \frac{v^2}{c^2}},\ c \text{ - speed of light}$$

IV. THE FARADAY'S LAW

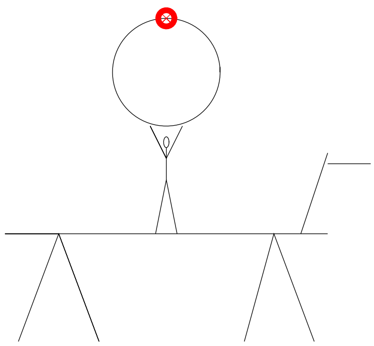


FIG. 6: An old circus show "Lady-on-a-Horse". The bulb lights up without any apparent reason??

A. Experimental

Consider a contour. If the magnetic flux (the number of lines which penetrate through the contour) changes with time, there will be an EMF induced in the contour, as in Fig. 7. If the contour is conducting, there also will be current. The faster is the change in the flux, the larger is the EMF.

B. Theory

1. Magnetic flux

Definition:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \tag{18}$$

Integration can go over ANY surface limited by a given contour.

Units: Wb (webers)

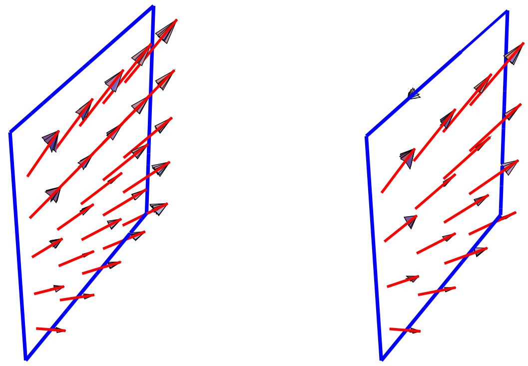


FIG. 7: Illustration of the law of induction. If the magnetic flux through the blue contour changes, there will be an induced EMF in the contour (and there will be current if the contour is conducting). The magnitude of EMF is given by the Faraday's law, and the direction is determined by the Lenz's rule. In the illustration, since the flux tends to decrease, the induced current will attempt to support the changing flux, being directed as shown.

C. The Law

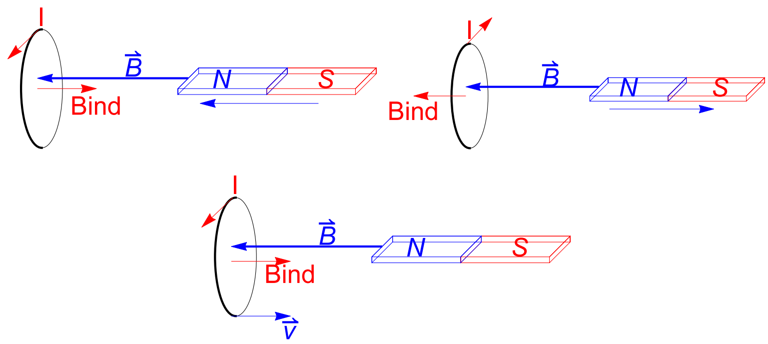
$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}} \tag{19}$$

Note that the law is valid, and has the same form, regardless of the physical reason why Φ_B is changing. Several typical situations can be identified for

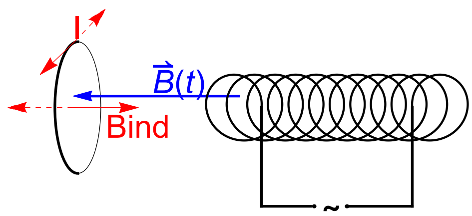
$$\Phi(t) \simeq \vec{A} \cdot \vec{B}$$

- $$B = B(t) \;\Rightarrow\; \mathcal{E} = -\vec{A} \cdot \frac{d\vec{B}}{dt}$$

  e.g. when a bar magnet approaches the contour (or the contour approaches the magnet), or the field through a given contour is generated by another coil with a time-dependent current.

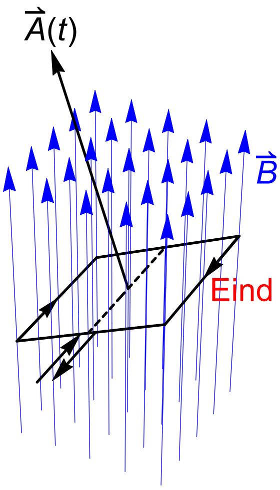


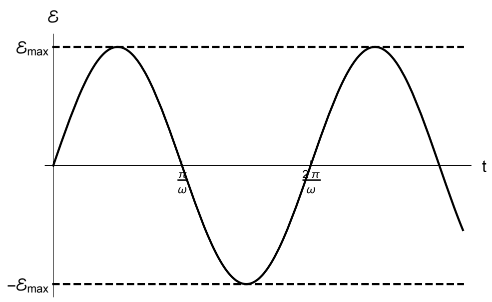
Mutual induction:



- $$\Phi_B \approx BA\cos[\phi(t)] \;\Rightarrow\; \mathcal{E} = -AB\frac{d\cos\phi}{dt}$$

as in frame revolving between the poles of a bar magnet (prototype of an electric generator).





For $\phi(t) = \omega t$ equation (19) will give

$$\mathcal{E} = \mathcal{E}_{\text{max}} \sin\omega t \;, \quad \mathcal{E}_{\text{max}} = \omega BA$$

- $$A = A(t) \;\Rightarrow\; \mathcal{E} = -B\frac{dA}{dt}$$

  E.g., for a rod which moves with velocity v - see Fig. 8:

  $$A = Lx(t) \;\Rightarrow\; \frac{dA}{dt} = Lv \;\Rightarrow\; \boxed{\mathcal{E} = vLB}$$

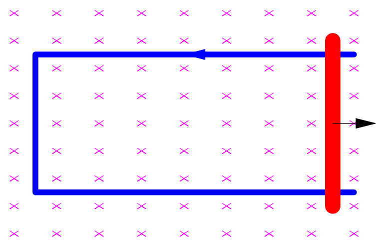


FIG. 8: Induction in a contour formed by rails and a sliding rod. The magnetic field B (which goes into the page) is constant, but the flux $\Phi_B = BA$ is changing since the area A is increasing. The magnitude of EMF is given by $\mathcal{E} = -BdA/dt = -vLB$. The direction follows from the Lenz's rule. The current will be given by $I \approx \mathcal{E}/R$ with R being the resistance of rails plus rod. For non-conduction (or, imaginary) rails there will be no current, but the EMF still will be induced

- if the contour is a conducting loop, the flux can also change due to change in current in the loop itself. This leads to *self-induction* and will be discussed separately.

D. Lenz's rule

The magnetic flux due to an induced current opposes the changes in the original magnetic flux.

This allows one to find the direction of induced current. This is consistent with conservation of energy. If current is impossible (non-conducting loop), Lenz's rule gives the direction of EMF.

Example: fig. 8. (1) Flux $\Phi = AB$ into the page and grows $\Rightarrow$ (2) induced field B_{ind} out of the page $\Rightarrow$ (3) induced current - CCW

E. Induced electric field

Consider a charge q being moved around a contour with EMF induced by a changing flux. One has

$$\mathcal{E} = \frac{W}{q} = \frac{1}{q}\oint \vec{F}\cdot d\vec{s} = \oint \vec{E}\cdot d\vec{s}$$

Thus, the Fraday's Law, eq. (19) can be written as

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{20}$$

Note that $\vec{E}$ is very different from the electrostatic field we studied before. Similarities between induced and electrostatic electric field:

- both fields act on charges with forces

$$\vec{F} = q\vec{E}$$

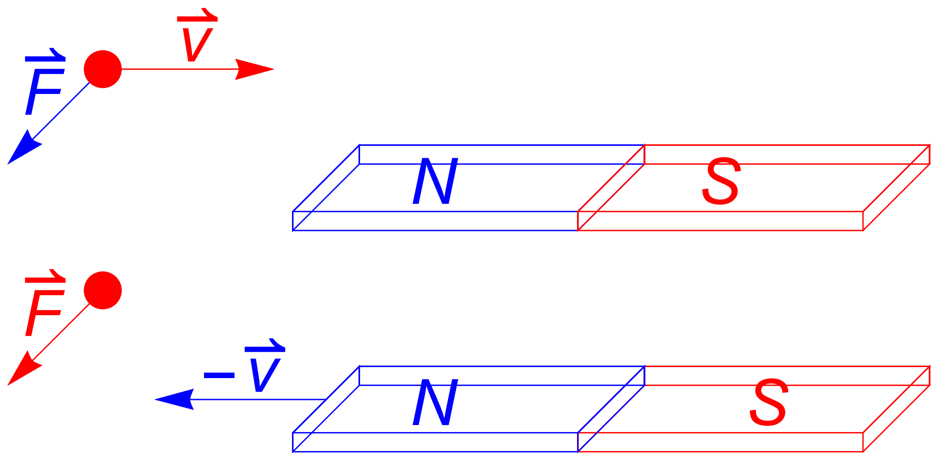
- both fields can be represented by field lines

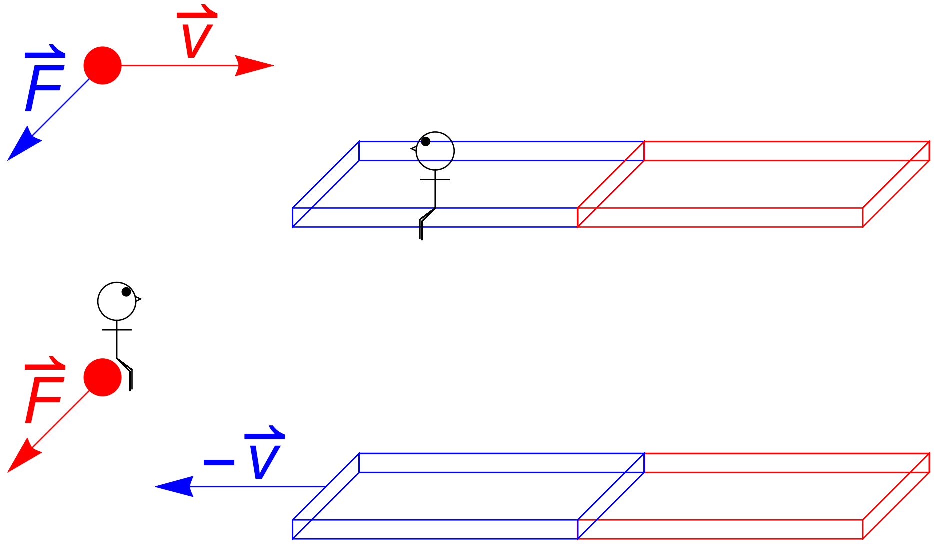
Differences:

- lines of induced electric field can loop (and this does not contradict the conservation of energy!)
- lines of induced field do not start or end (i.e. this field is produced not by charges, but by changing magnetic flux).

F. Advanced: relativity of electric and magnetic fields

will be discussed in class

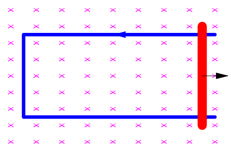




$\vec{v}$
$\vec{F}$
$\vec{F}$
$-\vec{v}$

V. SELF-INDUCTION

Slider on rails:



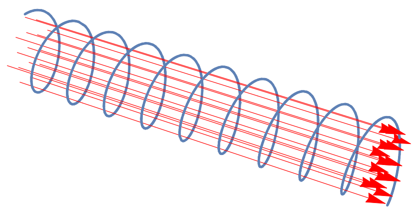
$$\mathcal{E} = lvB\,,\ \ I_{ind} = \mathcal{E}/R\,, \quad \text{What if } R = 0\ ??$$

A. Inductance - definition

Start with static current i, then $\Phi_{\text{tot}} = Li$

Units: $[L] = H$ (henry) $= T \cdot m^2/A$

Example: Solenoid, N turns, length l, area A



$$B = \mu_0 i N/l\,,\ \Phi_{\text{tot}} = NAB = NA\mu_0 iN/l \ \Rightarrow$$

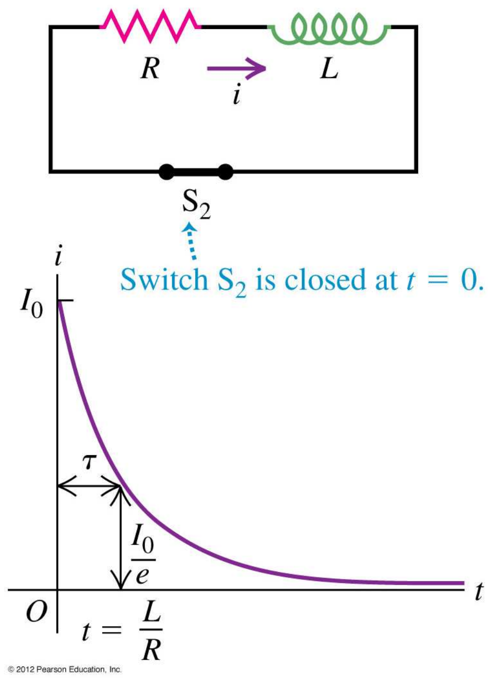
$$L/l = \mu_0 A n^2\,, \quad n = N/l \tag{21}$$

Now consider $i = i(t) \ \Rightarrow \ \Phi_{\text{tot}} = \Phi_{\text{tot}}(t)$. From Faraday's Law: $\mathcal{E}_L = -d\Phi_{\text{tot}}/dt$, thus

$$\text{Self-induced EMF: } \boxed{\mathcal{E}_L = -L\frac{di}{dt}} \tag{22}$$

B. RL circits

1. *Decay of current*

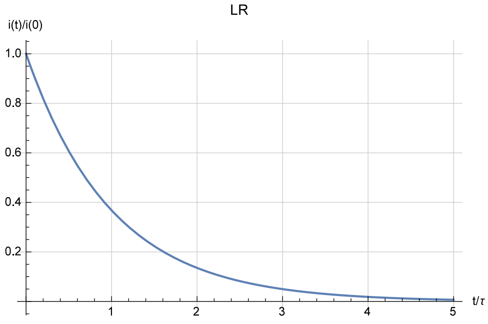


$$\text{loop rule: } \mathcal{E}_L - IR = 0\,, \;\Rightarrow\; L\frac{dI}{dt} + IR = 0 \tag{23}$$

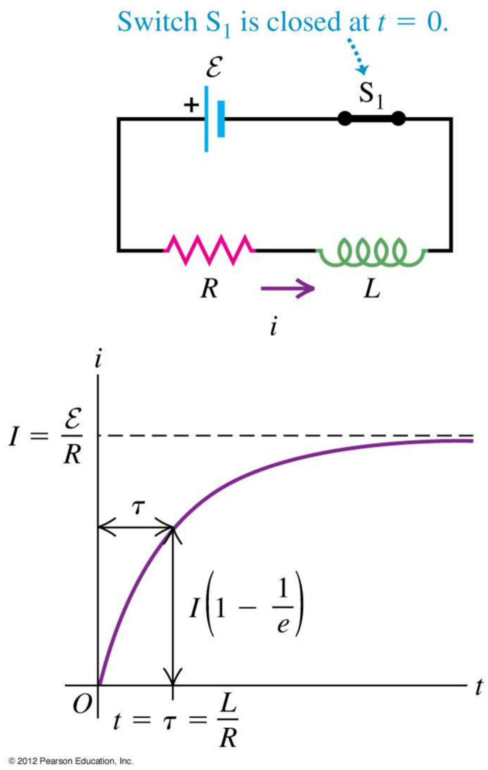
Solution:

$$I(t) = I_0 e^{-t/\tau_L}\,, \; \boxed{\tau_L = L/R} \; - \; \text{inductive time constant} \tag{24}$$

[Checkpoint: verify that $[\tau_L] = s$.] Note, if $R \to 0$ (superconductor) the current *never* decays.



2. Build-up of current



$$\text{loop rule: } \mathcal{E} + \mathcal{E}_L - iR = 0 \;\Rightarrow\; L\frac{di}{dt} + iR = \mathcal{E}$$

switch to new current $I = i - \mathcal{E}/R$, $L\frac{dI}{dt} + IR = 0$ with $I(0) = -\mathcal{E}/R$

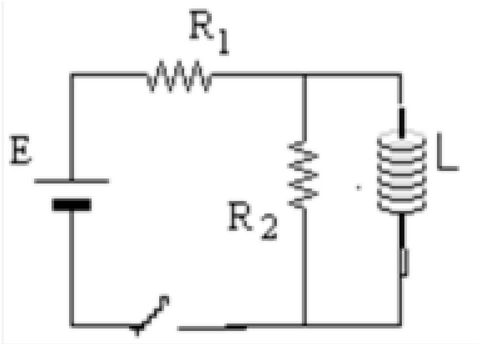
This is the same differential equation as for RL

$$I(t) = I(0)e^{-t/\tau_L} \;\Rightarrow\; i(t) = \frac{\mathcal{E}}{R} + I(0)e^{-t/\tau_L} = \frac{\mathcal{E}}{R} - \frac{\mathcal{E}}{R}e^{-t/\tau_L}$$

$$i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right) \tag{25}$$

Note: at $t = 0$ the inductor has "infinite resistance"; at $t \to \infty$ (constant current) it is just a piece of wire.

Example:



$$t \to 0: \text{ "clip off" the inductor}$$

$$I_1 = I_2 = \frac{E}{R_1 + R_2}$$

$$t \to \infty: \text{ replace inductor by a wire}$$

$$I_2 = 0\,,\; I_1 = E/R_1$$

C. Magnetic Energy of an Inductor

$$U_L = -\int_0^\infty E_L i\, dt = L\int_0^\infty i\frac{di}{dt}dt = L\int_0^I i\, di$$

or

$$\boxed{U_L = \frac{1}{2}LI^2} \tag{26}$$

Example An $L = 4.0\, mH$ inductor with some initial current I_0 is discharged through a $0.25\,\Omega$ resistor. How long does it take to lose half of the initial current? Half of the initial energy? *Solution*

$$\tau = \frac{L}{R} = 16\cdot 10^{-3}\, s\,;\;\; \frac{i(t)}{I_0} = e^{-t/\tau} = 0.5$$

$$t/\tau = -\ln 0.5 = \ln 2\,,\;\; t = \tau\ln 2 \approx 1.1\cdot 10^{-2}\, s$$

$$U(t) \sim i(t)^2 \Rightarrow \frac{U(t)}{U(0)} = e^{-2t/\tau} = 0.5$$

$$2t/\tau = -\ln 0.5 = \ln 2\,,\;\; t = \frac{1}{2}\tau\ln 2 \approx 5.5\cdot 10^{-3}\, s$$

D. Advanced: Energy stored in magnetic field

From

$$B = \mu_0 I n\,,\ \ L = \mu_0 A n^2 l$$

and volume $\mathcal{V} = lA$

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\mu_0 A n^2 l (B/\mu_0 n)^2 = \frac{B^2 \mathcal{V}}{2\mu_0}$$

$$U/\mathcal{V} = \frac{B^2}{2\mu_0} \tag{27}$$

VI. ELECTROMAGNETIC OSCILLATIONS

A. math

$$\boldsymbol{i} = \sqrt{-1}$$

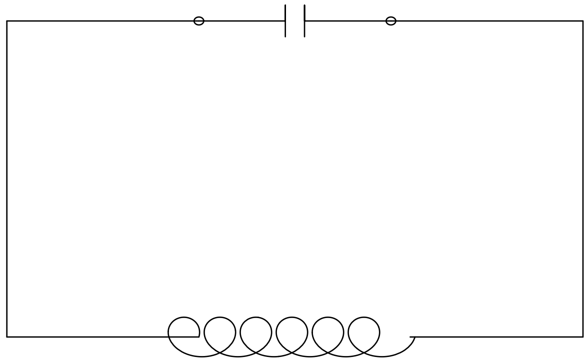
$$e^{\pm \boldsymbol{i}\alpha} = \cos\alpha \pm \boldsymbol{i}\sin\alpha \tag{28}$$

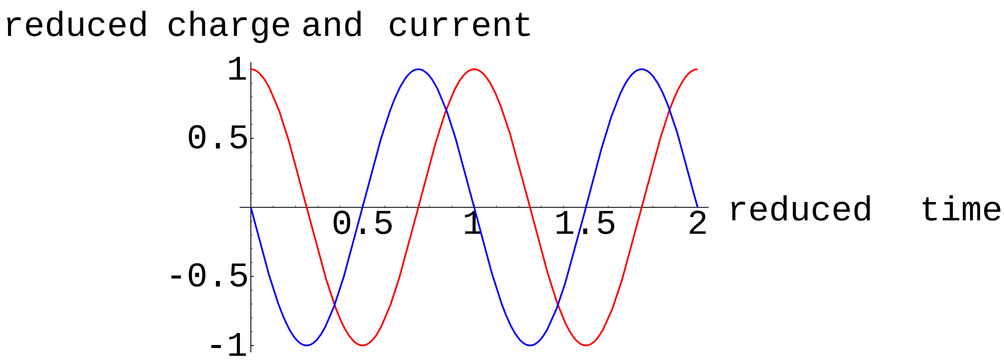
$$\cos\alpha = \frac{e^{\boldsymbol{i}\alpha} + e^{-\boldsymbol{i}\alpha}}{2}\ , \quad \sin\alpha = \frac{e^{\boldsymbol{i}\alpha} - e^{-\boldsymbol{i}\alpha}}{2\boldsymbol{i}} \tag{29}$$

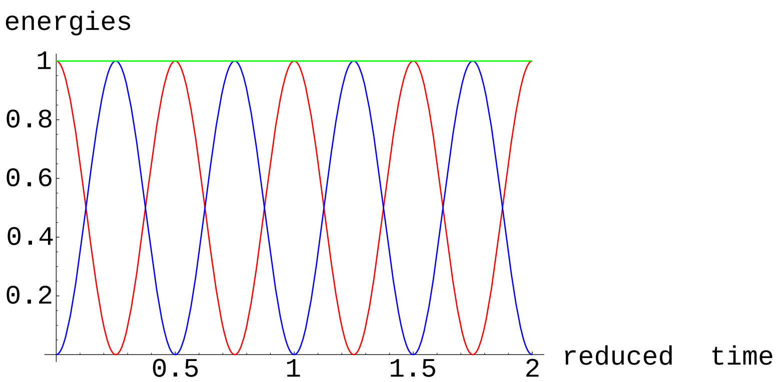
$$\frac{d}{dt}e^{\pm \boldsymbol{i}\omega t} = \pm \boldsymbol{i}\omega e^{\pm \boldsymbol{i}\omega t}\ , \quad \frac{d^2}{dt^2}e^{\pm \boldsymbol{i}\omega t} = -\omega^2 e^{\pm \boldsymbol{i}\omega t} \tag{30}$$

B. LC-circuit, free oscillations

The capacitor is originally charged with $Q = Q_{\max}$.

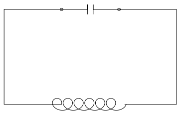






Free oscillations. Reduced time is t/T, with $T = 2\pi\sqrt{LC}$, the period of oscillations. Left - charge on capacitor $q(t)/Q_{\max}$ (red) and current in the inductor $I(t)/I_{\max}$ (blue). Note that when charge is maximum (or minimum) current is zero, and vice versa. Right - electric (red) and magnetic (blue) energies. Their sum (green line) is the total energy which remains constant (and all energies are reduced by this value).

1. Differential equation



From the loop rule:

$$\mathcal{E}_L + V_C = 0$$

$$\mathcal{E}_L = -L\frac{dI}{dt} = L\frac{d^2q}{dt^2} \;\text{ and } V_C = q/C$$

$$L\frac{d^2q}{dt^2} + q/C = 0\,, \;\text{ a differential equation for } q(t)$$

Look for a solution

$$q(t) = Q\cos(\omega t)\,,\; \omega - ? \;\; (\text{or, } q(t) = Qe^{\boldsymbol{i}\omega t})$$

$$d^2q/dt^2 = -\omega^2 q\,,\; \Rightarrow\; q(t)\left(-L\omega^2 + 1/C\right) = 0,\; \Rightarrow$$

$$\boxed{\omega = 1/\sqrt{LC}} \tag{31}$$

Checkpoint. Calculate explicitly the electric and magnetic energy, and make sure their sum remains constant.

If the resistance of the LC circuit is non-negligible, the electromagnetic energy will not be conserved, and the amplitude will get smaller with time - see Fig. 9. One can get the description mathematically by adding a term Rdq/dt into the l.h.s. of the above differential equation.

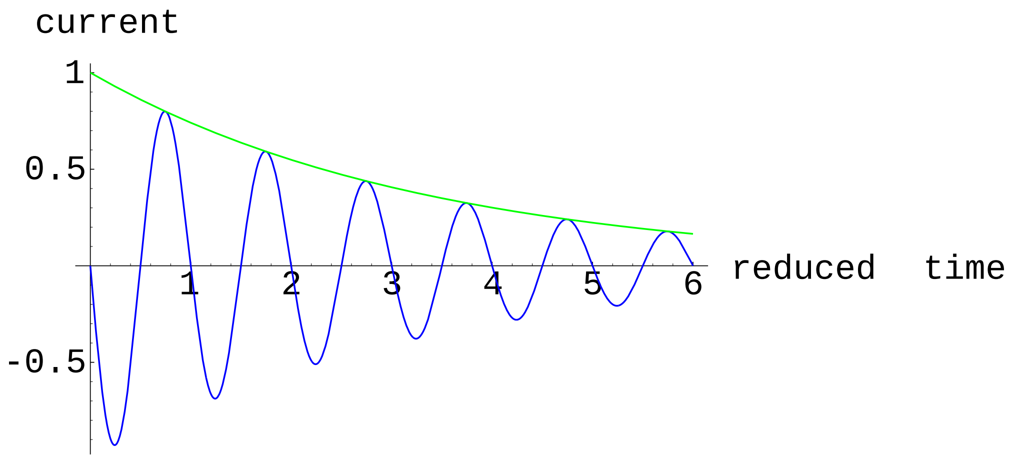


FIG. 9: Free oscillations in an RLC-circuit. The upper (green) line shows the decaying amplitude due to dissipation of energy in the resistance R. For a sufficiently large R ("overdamped case") there would be no oscillations at all.

Advanced: Loop equation.

$$q/C + \mathcal{E}_L - I(t)R = 0\,,\ I(t) \sim e^{\boldsymbol{i}\omega t}$$

$$q(t) = -\int I\,dt \sim -\frac{1}{\boldsymbol{i}\omega}e^{\boldsymbol{i}\omega t}\,,\ \mathcal{E}_L = -L\frac{dI}{dt} \sim \boldsymbol{i}\omega L e^{\boldsymbol{i}\omega t}$$

$$-\frac{1}{\boldsymbol{i}\omega C} - \boldsymbol{i}\omega L - R = 0$$

$$\omega^2 LC - \boldsymbol{i}\omega RC - 1 = 0$$

$$\omega_0^2 \equiv \frac{1}{LC}\,,\ x \equiv \frac{\omega}{\omega_0}:\ x^2 - ixR\sqrt{\frac{C}{L}} - 1 = 0$$

$$x = \frac{1}{2}\boldsymbol{i}R\sqrt{\frac{C}{L}} \pm \sqrt{1 - R^2C/4L}$$

$$\omega = \pm\Omega + \boldsymbol{i}\gamma$$

$$\Omega = \omega_0\sqrt{1 - R^2C/4L}\,,\ \gamma = \omega_0\frac{1}{2}R\sqrt{C/L}$$

$$I(t) \sim e^{\boldsymbol{i}\omega t} = e^{\pm\boldsymbol{i}\Omega t}e^{-\gamma t}$$

C. Driven oscillations and resonance

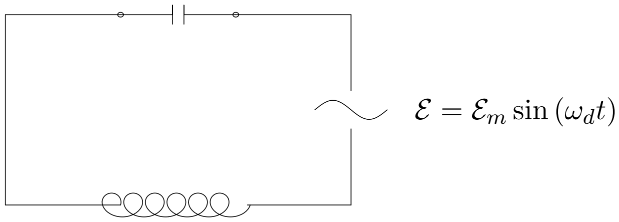


FIG. 10: LC-circuit. Driven oscillations.

1. *Negligible R*

Consider Fig. 10 with the driving EMF given by

$$\mathcal{E} = \mathcal{E}_m \sin(\omega_d t) \text{ with } \omega_d \text{ different from } \omega_0 = \frac{1}{\sqrt{LC}}$$

This should be added into the r.h.s. of the differential equation:

$$L\frac{d^2q}{dt^2} + q/C = \mathcal{E}_m \sin(\omega_d t) \text{ or } \frac{d^2q}{dt^2} + \omega_0^2 q = \frac{\mathcal{E}_m}{L}\sin(\omega_d t)$$

$$\text{look for a solution } q(t) = Q_m \sin(\omega_d t)$$

Now the frequency is known, but Q_m is the maximum, yet unknown charge to appear on the capacitor. Similarly to free oscillations

$$d^2q/dt^2 = -\omega_d^2 q \;\Rightarrow\; -\omega_d^2 q + \omega_0^2 q = \frac{\mathcal{E}_m}{L}\sin(\omega_d t)$$

or with the definition of ω_0 for *free* oscillations

$$Q_m \sin(\omega_d t)\left(-\omega_d^2 + \omega_0^2\right) = \mathcal{E}_m/L \cdot \sin(\omega_d t)$$

From here

$$Q_m = \frac{\mathcal{E}_m}{L} \frac{1}{\omega_0^2 - \omega_d^2}$$

The amplitude for current is similar, and is given by $I_m = \omega_d Q_m$. Note a dramatic increase of the amplitude when $\omega_d \to \omega_0$. This is the *resonance* - see Fig. 11

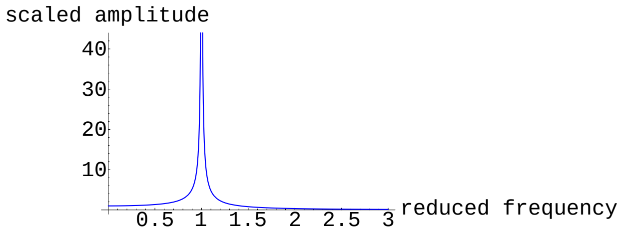
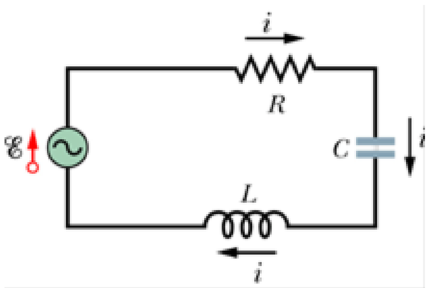


FIG. 11: Resonance in driven oscillations. When the driving frequency ω_d is close to the natural frequency $\omega_0 = 1/\sqrt{LC}$ there is an enormous increase in the amplitude.

2. $R \neq 0$



Loop:

$$\mathcal{E} - \frac{q}{C} + \mathcal{E}_L - I(t)R = 0$$

$$\mathcal{E} = \mathcal{E}_m e^{\boldsymbol{i}\omega_d t}\,,\ I(t) = I_m e^{\boldsymbol{i}\omega_d t}$$

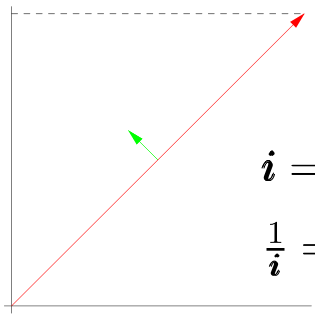
$$q = \int I(t)\,dt = \frac{1}{\boldsymbol{i}\omega_d} I_m e^{\boldsymbol{i}\omega_d t}\,,\ \mathcal{E}_L = -L\boldsymbol{i}\omega_d I_m e^{\boldsymbol{i}\omega_d t}$$

$$\mathcal{E}_m - \frac{1}{\boldsymbol{i}\omega_d C} I_m - \boldsymbol{i}\omega_d L I_m - I_m R = 0$$

$$I_m \left[R + \boldsymbol{i} \left(\omega_d L - \frac{1}{\omega_d C} \right) \right] = \mathcal{E}_m$$

$$|I_m|\, Z = |\mathcal{E}_m| \ , \ Z = \sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C} \right)^2}$$

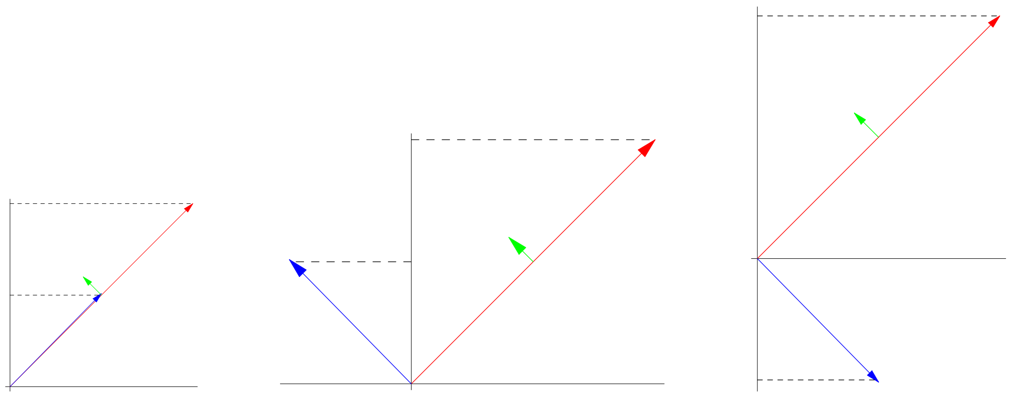
3. Phasors



$$\boldsymbol{i} = e^{\boldsymbol{i}\frac{\pi}{2}} \ \Rightarrow \ \boldsymbol{i} e^{\boldsymbol{i}\omega t} = e^{\boldsymbol{i}(\omega t + \frac{\pi}{2})}$$

$$\frac{1}{\boldsymbol{i}} = e^{-\boldsymbol{i}\frac{\pi}{2}} \ \Rightarrow \ \frac{1}{\boldsymbol{i}} e^{\boldsymbol{i}\omega t} = e^{\boldsymbol{i}(\omega t - \frac{\pi}{2})}$$

FIG. 12: A phasor. It spins in the counter-clockwise direction with the angular frequency ω_d and its vertical projection determines the physical quantity, e.g. voltage or current.



Phasors for voltage (red) and current (blue) for 3 different elements: resistor (left), capacitor (middle) and inductor (right). Note that for a resistor the voltage and current have the same phase, for the capacitor current leads the voltage by 90^o, and for the inductor the current lags by 90^o.

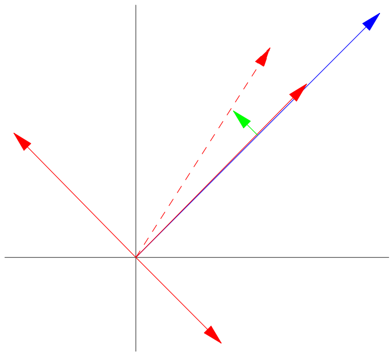


FIG. 13: Solving a driven RLC circuit using phasors. The current (blue) is identical since all elements are in series. Voltages (red) are to be added as vectors, which can be done using Pythagorean theorem. The resultant (dashed) should correspond to the driving EMF. In case of a resonance the voltage on the inductor (leading voltage - upper left) and the voltage on capacitor (lagging voltage - lower right) would completely cancel each other.

$$\text{Capacitive reactance: } X_C = \frac{1}{\omega_d C}$$

$$\text{Inductive reactance: } X_L = \omega_d L$$

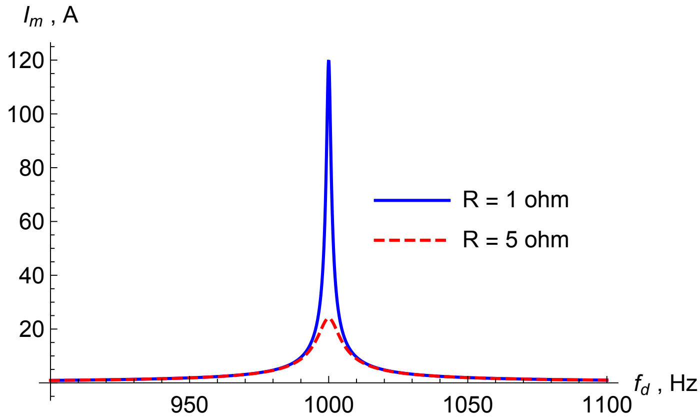
$$\text{Impedance: } Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Amplitude of current and phase angle:

$$I = \frac{\mathcal{E}_m}{Z}, \ \tan\phi = \frac{X_L - X_C}{R}$$

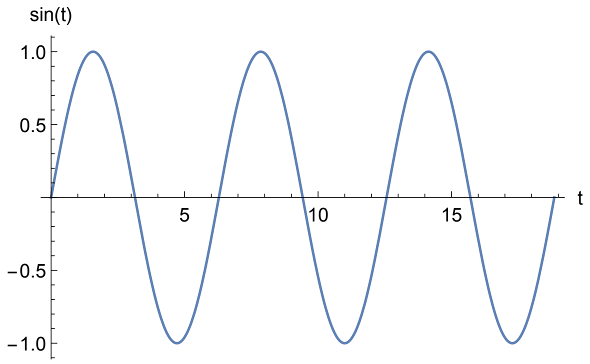
$$\text{resonance: } \omega_d = \omega_0\,,\;\; X_L = X_C$$

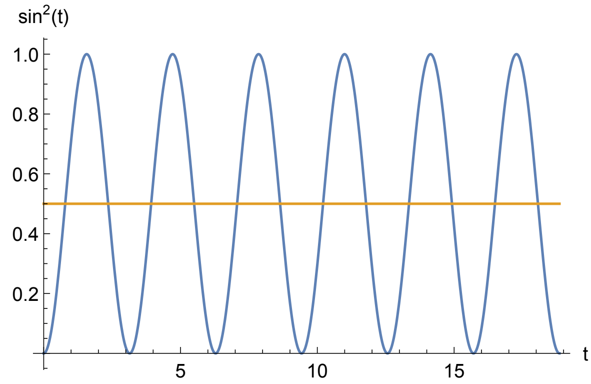
$$Z = R = \min\,,\;\; I = \mathcal{E}_m/R = \max$$



Amplitude of current for a variable driving frequency f_d with fixed $L = 0.1\,H$, $C = 0.253\,\mu F$ (with $f_{res} = 1000\,Hz$), $\mathcal{E} = 120\,V$ and $R = 1\,\Omega$ (blue) or $R = 5\,\Omega$ (red-dashed).

D. Power





$$I(t) = I_m \sin(\omega_d t) \;\Rightarrow\; \bar{I} = 0\ ???$$

$$I^2(t) = I_m^2 \sin^2(\omega_d t) = I_m^2 \frac{1}{2}(1 - \cos(2\omega_d t)) \;\Rightarrow\; \bar{I^2} = \frac{1}{2} I_m^2$$

$$I_{RMS} = \sqrt{\bar{I^2}} = \frac{1}{\sqrt{2}} I_m \,.\ \text{Note:}\ \boxed{I_{RMS} = \frac{V_{RMS}}{Z}}$$

$$\bar{P} = \langle I^2(t) R \rangle = I_{RMS}^2 R \tag{32}$$

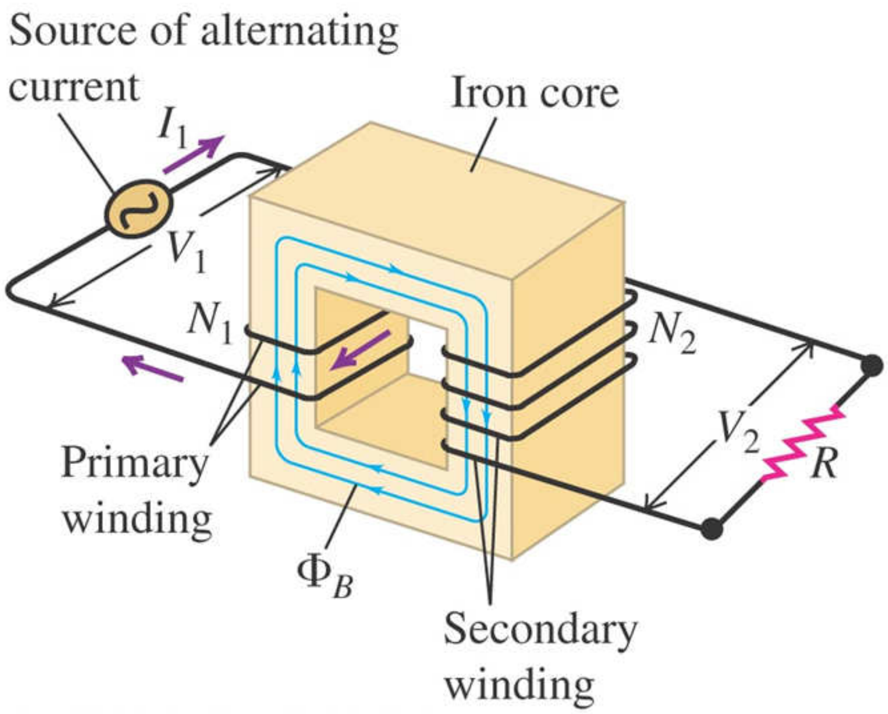
or

$$\bar{P} = I_{RMS} V_{RMS} \cos\phi \,,\;\; \cos\phi = R/Z$$

E. Transformers

The induced emf *per turn* is the same in both coils, so we adjust the ratio of terminal voltages by adjusting the ratio of turns:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$



$$\Phi_1 = N_1 BA\,,\ \Phi_2 = N_2 BA$$

$$V_1 = -\frac{d\Phi_1}{dt}\,,\ V_2 = -\frac{d\Phi_2}{dt}$$

$$V_2 = V_1 \frac{N_2}{N_1}$$

$N_2 > N_1$ - "step up", $N_2 < N_1$ - "step down"

Current: Consider power

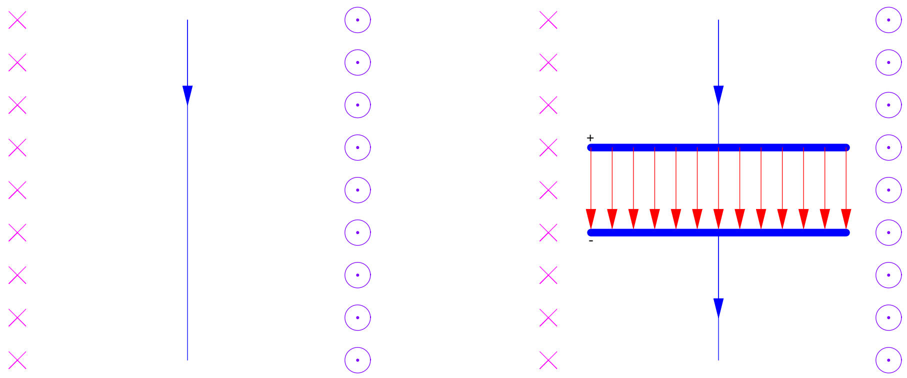
$$P_1 = I_1 V_1 \cos\phi_1\,,\ P_2 = I_2 V_2 \cos\phi_2$$

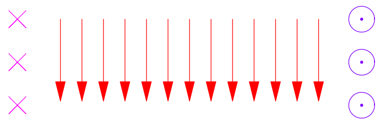
Ideal: matched ϕ_1 and ϕ_2:

$$I_1 V_1 = I_2 V_2\,,\ I_2 = I_1 \frac{N_1}{N_2}$$

Transmission of energy. Power plant (step up): $V = \max$, $I = \min$ - minimal losses I^2R. Consumer (step down): $V = 110\, volt$ ...

VII. MAXWELL EQUATIONS AND ELECTROMAGNETIC WAVES





The idea of Maxwell's construction. Upper left: a standard wire with current (a 2-dimensional cross-section); magnetic field is determined from the Ampere's circulation theorem,

$$\oint \vec{B} \cdot \vec{s} = \mu_0 i$$

Upper right: the wire is cut and a parallel plate capacitor with plate area A is inserted. The current, however, keeps coming in the original direction, charging the upper plate positively and the lower negatively. There is an electric field between the plates $E(t) = Q(t)/A\epsilon_0$, with a time-dependent electric flux,

$$d\Phi_E/dt = d\,(AE)\,/dt = d\,(Q/\epsilon_0)\,/dt = i/\epsilon_0$$

The magnetic field is expected *not* to change.

Lower figure: the wire and the capacitor are removed (as scaffolding). The magnetic field is due to the changing electric flux:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 d\Phi_E/dt$$

$$\epsilon_0 = \frac{1}{4\pi k} \simeq \frac{1}{4\pi \cdot 9 \cdot 10^9}\,,\ \mu_0 = 4\pi 10^{-7}\,,\ \mu_0\epsilon_0 = \ldots$$

A. The Maxwell equations

$$\oint \vec{E} \cdot d\vec{A} = q_{enc}/\epsilon_0 \tag{33}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \tag{34}$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \tag{35}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} + \frac{1}{c^2}\frac{d\Phi_E}{dt} \tag{36}$$

with $c = 1/\sqrt{\mu_0\epsilon_0} \simeq 3 \cdot 10^8\ m/s$, the speed of light.

B. And then there was Light!

Maxwell's equations in empty space

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \frac{1}{c^2}\frac{d\Phi_E}{dt}$$

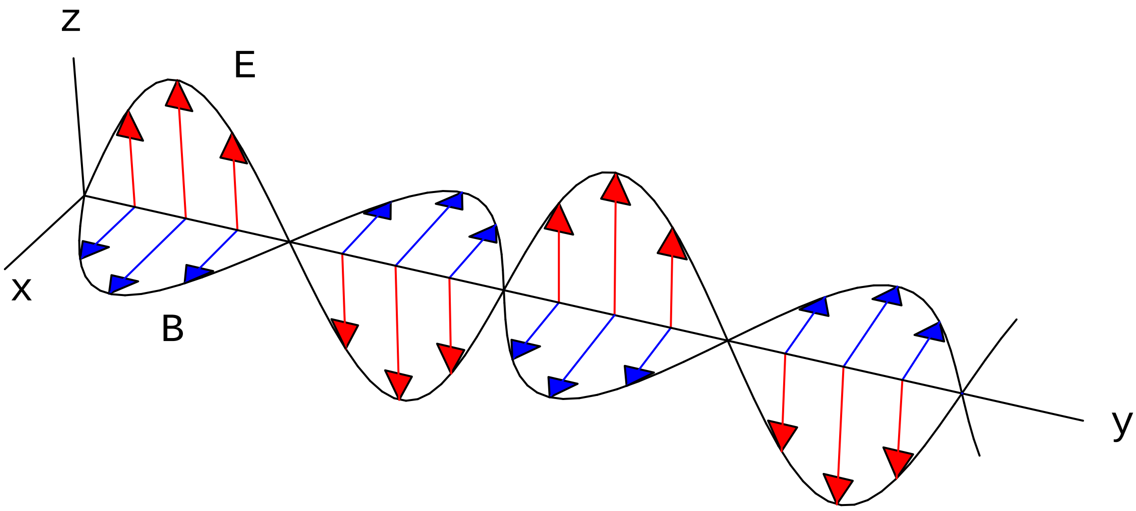


FIG. 14: Structure of an electromagnetic wave. The electric field (red) is in the $\pm z$-direction, the magnetic field (blue) is in the $\pm x$-direction and the wave propagates in the y-direction. (This is known as a plane, polarized wave). The distance in the y-direction in which the oscillations start to repeat themselves is the *wavelength*.

For example, the wave in Fig. 14 is described by

$$\vec{E} = \vec{E}_0 \sin\left\{2\pi\left(\frac{y}{\lambda} - ft\right)\right\}$$

$$\vec{B} = \vec{B}_0 \sin\left\{2\pi\left(\frac{y}{\lambda} - ft\right)\right\}$$

with

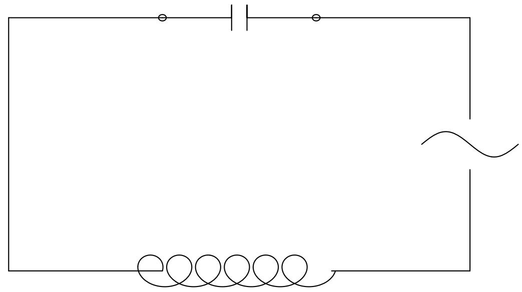
$$\left|\vec{E}_0\right| = c\left|\vec{B}_0\right|$$

f is known as *frequency* and λ as wavelength. One has

$$f = \frac{c}{\lambda}$$

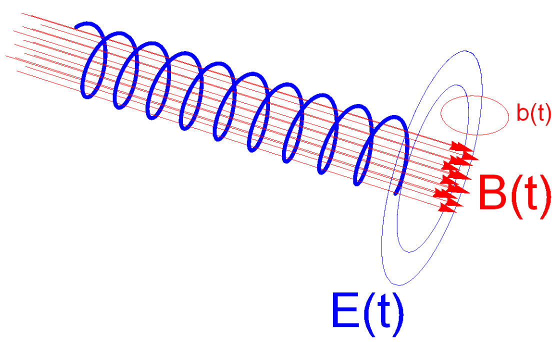
for *any* wavelength in empty space.

Consider $R \to 0$:



$$I_m = \frac{\mathcal{E}_m}{Z}\,,\ Z = \lim_{R\to 0}\sqrt{R^2 + (X_L - X_C)^2} = |X_L - X_C|$$

$$X_L = \omega_d L\,,\ X_C = \frac{1}{\omega_d C}\,,\ \omega_0 = \frac{1}{\sqrt{LC}} \ \Rightarrow\ I_m \propto \frac{1}{|\omega_d^2 - \omega_0^2|}$$

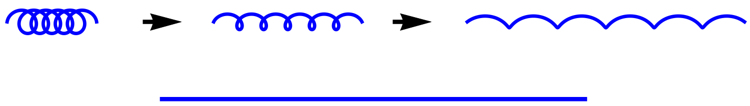


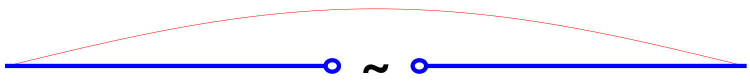
Radiation power $\propto\ \omega^4$

power $=$ max $\ \Rightarrow\ \omega =$ max

$\omega \simeq \frac{1}{\sqrt{LC}}\ \Rightarrow\ LC = \min$

$$L/l = \mu_0 A \left(\frac{N}{l}\right)^2 \;,\; L/l = \min \;\Rightarrow\; l = \max$$





Linear antenna. Red line shows a snapshot of the current density. (Antenna works as a time-dependent dipole). If length l is adjusted to correspond to half a wavelength at the driving frequency, $l = \pi c/\omega_d$ the antenna is "in resonance" and will radiate intensely.