

Week 11: Chapter 11

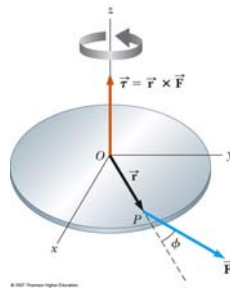
Angular Momentum

The Vector Product

- There are instances where the product of two vectors is another vector
 - Earlier we saw where the product of two vectors was a scalar
 - This was called the dot product
- The vector product of two vectors is also called the cross product

The Vector Product and Torque

- The torque vector lies in a direction perpendicular to the plane formed by the position vector and the force vector
- $\vec{\tau} = \vec{r} \times \vec{F}$
- The torque is the vector (or cross) product of the position vector and the force vector

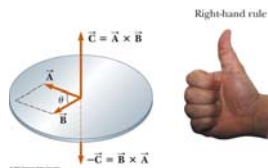


The Vector Product Defined

- Given two vectors, \vec{A} and \vec{B}
- The vector (cross) product of \vec{A} and \vec{B} is defined as a *third vector*, $\vec{C} = \vec{A} \times \vec{B}$
 - \vec{C} is read as “A cross B”
- The magnitude of vector C is $AB \sin \theta$
 - θ is the angle between \vec{A} and \vec{B}

More About the Vector Product

- The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by \vec{A} and \vec{B}
- The direction of \vec{C} is perpendicular to the plane formed by \vec{A} and \vec{B}
- The best way to determine this direction is to use the right-hand rule



Properties of the Vector Product

- The vector product is *not* commutative. The order in which the vectors are multiplied is important
 - To account for order, remember

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$
- If \vec{A} is parallel to \vec{B} ($\theta = 0^\circ$ or 180°), then $\vec{A} \times \vec{B} = 0$
 - Therefore $\vec{A} \times \vec{A} = 0$

More Properties of the Vector Product



- If \vec{A} is perpendicular to \vec{B} , then $|\vec{A} \times \vec{B}| = AB$
- The vector product obeys the distributive law
 - $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Final Properties of the Vector Product



- The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

where it is important to preserve the multiplicative order of \vec{A} and \vec{B}

Vector Products of Unit Vectors



$$\begin{aligned}\hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{i} \times \hat{j} &= -\hat{j} \times \hat{i} = \hat{k} \\ \hat{j} \times \hat{k} &= -\hat{k} \times \hat{j} = \hat{i} \\ \hat{k} \times \hat{i} &= -\hat{i} \times \hat{k} = \hat{j}\end{aligned}$$

Vector Products of Unit Vectors, cont



- Signs are interchangeable in cross products
 - $\vec{A} \times (-\vec{B}) = -\vec{A} \times \vec{B}$
 - and $\hat{i} \times (-\hat{j}) = -\hat{i} \times \hat{j}$

Using Determinants



- The cross product can be expressed as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

- Expanding the determinants gives

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} - (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

Vector Product Example



- Given $\vec{A} = 2\hat{i} + 3\hat{j}$, $\vec{B} = -\hat{i} + 2\hat{j}$
- Find $\vec{A} \times \vec{B}$
- Result

$$\begin{aligned}\vec{A} \times \vec{B} &= (2\hat{i} + 3\hat{j}) \times (-\hat{i} + 2\hat{j}) \\ &= 2\hat{i} \times (-\hat{i}) + 2\hat{i} \times 2\hat{j} + 3\hat{j} \times (-\hat{i}) + 3\hat{j} \times 2\hat{j} \\ &= 0 + 4\hat{k} + 3\hat{k} + 0 = 7\hat{k}\end{aligned}$$

Torque Vector Example

- Given the force and location

$$\vec{F} = (2.00\hat{i} + 3.00\hat{j}) \text{ N}$$

$$\vec{r} = (4.00\hat{i} + 5.00\hat{j}) \text{ m}$$

- Find the torque produced

$$\vec{\tau} = \vec{r} \times \vec{F} = [(4.00\hat{i} + 5.00\hat{j})\text{N}] \times [(2.00\hat{i} + 3.00\hat{j})\text{m}]$$

$$= [(4.00)(2.00)\hat{i} \times \hat{i} + (4.00)(3.00)\hat{i} \times \hat{j}$$

$$+ (5.00)(2.00)\hat{j} \times \hat{i} + (5.00)(3.00)\hat{j} \times \hat{j}]$$

$$= 2.0\hat{k} \text{ N}\cdot\text{m}$$

Angular Momentum

- Consider a particle of mass m located at the vector position \vec{r} and moving with linear momentum \vec{p}

- Find the net torque

$$\vec{r} \times \sum \vec{F} = \sum \vec{r} \times \frac{d\vec{p}}{dt}$$

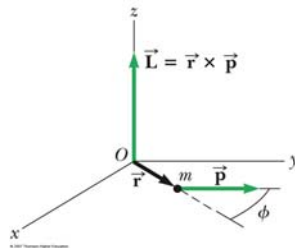
Add the term $\frac{d\vec{r}}{dt} \times \vec{p}$ (since it = 0)

$$\sum \vec{\tau} = \frac{d(\vec{r} \times \vec{p})}{dt}$$

Angular Momentum, cont

- The instantaneous angular momentum \vec{L} of a particle relative to the origin O is defined as the cross product of the particle's instantaneous position vector \vec{r} and its instantaneous linear momentum \vec{p}

$$\vec{L} = \vec{r} \times \vec{p}$$



Torque and Angular Momentum

- The torque is related to the angular momentum
 - Similar to the way force is related to linear momentum

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

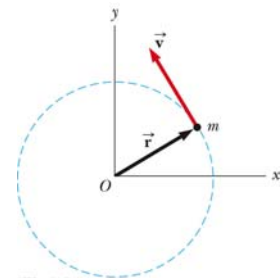
- The torque acting on a particle is equal to the time rate of change of the particle's angular momentum
- This is the rotational analog of Newton's Second Law
 - $\sum \vec{\tau}$ and \vec{L} must be measured about the same origin
 - This is valid for any origin fixed in an inertial frame

More About Angular Momentum

- The SI units of angular momentum are $(\text{kg}\cdot\text{m}^2)/\text{s}$
- Both the magnitude and direction of the angular momentum depend on the choice of origin
- The magnitude is $L = mvr \sin \phi$
 - ϕ is the angle between \vec{p} and \vec{r}
- The direction of \vec{L} is perpendicular to the plane formed by \vec{r} and \vec{p}

Angular Momentum of a Particle, Example

- The vector $\vec{L} = \vec{r} \times \vec{p}$ is pointed out of the diagram
- The magnitude is $L = mvr \sin 90^\circ = mvr$
 - $\sin 90^\circ$ is used since v is perpendicular to r
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path



Angular Momentum of a System of Particles

- The total angular momentum of a system of particles is defined as the vector sum of the angular momenta of the individual particles

- $\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_n = \sum_i \vec{L}_i$

- Differentiating with respect to time

$$\frac{d\vec{L}_{tot}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i$$



Angular Momentum of a System of Particles, cont

- Any torques associated with the internal forces acting in a system of particles are zero

- Therefore, $\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt}$

- The net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin
- This is the mathematical representation of the angular momentum version of the nonisolated system model.



Angular Momentum of a System of Particles, final

- The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass

- This applies even if the center of mass is accelerating, provided $\vec{\tau}$ and \vec{L} are evaluated relative to the center of mass

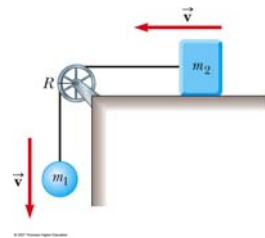


System of Objects, Example

- The masses are connected by a light cord that passes over a pulley; find the linear acceleration

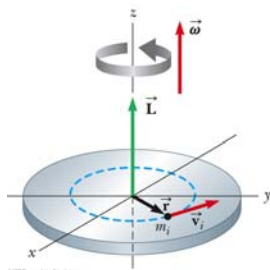
- Conceptualize**

- The sphere falls, the pulley rotates and the block slides
- Use angular momentum approach



Angular Momentum of a Rotating Rigid Object

- Each particle of the object rotates in the xy plane about the z axis with an angular speed of ω
- The angular momentum of an individual particle is $L_i = m_i r_i^2 \omega$
- \vec{L} and $\vec{\omega}$ are directed along the z axis



Angular Momentum of a Rotating Rigid Object, cont

- To find the angular momentum of the entire object, add the angular momenta of all the individual particles

$$L_z = \sum_i L_i = \sum_i (m_i r_i^2) \omega = I \omega$$

- This also gives the rotational form of Newton's Second Law

$$\sum \vec{\tau}_{ext} = \frac{dL_z}{dt} = I \frac{d\omega}{dt} = I \alpha$$

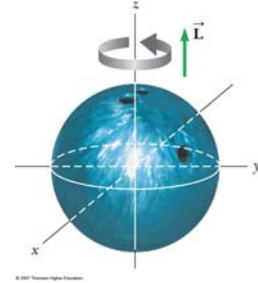


Angular Momentum of a Rotating Rigid Object, final

- The rotational form of Newton's Second Law is also valid for a rigid object rotating about a moving axis provided the moving axis:
 - (1) passes through the center of mass
 - (2) is a symmetry axis
- If a symmetrical object rotates about a fixed axis passing through its center of mass, the vector form holds: $\vec{L} = I\vec{\omega}$
 - where \vec{L} is the total angular momentum measured with respect to the axis of rotation

Angular Momentum of a Bowling Ball

- The moment of inertia of the ball is $\frac{2}{5}MR^2$
- The angular momentum of the ball is $L_z = I\omega$
- The direction of the angular momentum is in the positive z direction



Conservation of Angular Momentum

- The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero
 - Net torque = 0 -> means that the system is isolated
- $\vec{L}_{\text{tot}} = \text{constant}$ or $\vec{L}_i = \vec{L}_f$
- For a system of particles, $\vec{L}_{\text{tot}} = \sum \vec{L}_n = \text{constant}$

Conservation of Angular Momentum, cont

- If the mass of an isolated system undergoes redistribution, the moment of inertia changes
 - The conservation of angular momentum requires a compensating change in the angular velocity
 - $I_i \omega_i = I_f \omega_f = \text{constant}$
 - This holds for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system
 - The net torque must be zero in any case

Clicker Question

A person sitting on a rotating chair is holding two heavy dumbbells in his two hands. Initially, the position of dumbbells is in front of his chest. Suddenly, he extends his arms with dumbbells in hands. What will happen to this rotating system (chair-man-dumbbells)? Friction force is ignored.

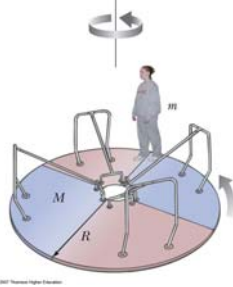
- It will rotate faster.
- It will rotate more slowly.
- It will stop rotating.
- Rotation speed will not change.
- Need more conditions to decide.

Conservation Law Summary

- For an isolated system -
 - Conservation of Energy:
 - $E_i = E_f$
 - Conservation of Linear Momentum:
 - $\vec{p}_i = \vec{p}_f$
 - Conservation of Angular Momentum:
 - $\vec{L}_i = \vec{L}_f$

Conservation of Angular Momentum: The Merry-Go-Round

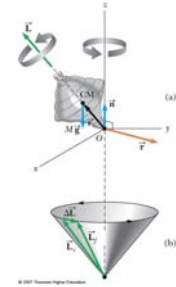
- The moment of inertia of the system is the moment of inertia of the platform plus the moment of inertia of the person
 - Assume the person can be treated as a particle
- As the person moves toward the center of the rotating platform, the angular speed will increase
 - To keep the angular momentum constant



Motion of a Top

- The only external forces acting on the top are the normal force and the gravitational force
- The direction of the angular momentum is along the axis of symmetry
- The right-hand rule indicates that the torque is in the xy plane

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times M\vec{g}$$



Motion of a Top, cont

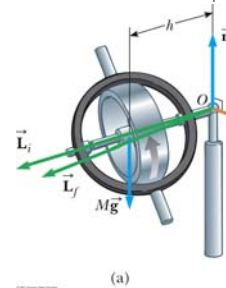
- The net torque and the angular momentum are related:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

- A non-zero torque produces a change in the angular momentum
- The result of the change in angular momentum is a precession about the z axis
- The direction of the angular momentum is changing
- The precessional motion is the motion of the symmetry axis about the vertical
- The precession is usually slow relative to the spinning motion of the top

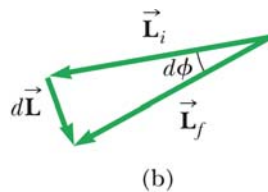
Gyroscope

- A gyroscope can be used to illustrate precessional motion
- The gravitational force produces a torque about the pivot, and this torque is perpendicular to the axle
- The normal force produces no torque



Gyroscope, cont

- The torque results in a change in angular momentum in a direction perpendicular to the axle.
 - The axle sweeps out an angle $d\phi$ in a time interval dt .
- The direction, not the magnitude, of the angular momentum is changing
- The gyroscope experiences precessional motion



Gyroscope, final

- To simplify, assume the angular momentum due to the motion of the center of mass about the pivot is zero
 - Therefore, the total angular momentum is due to its spin
 - This is a good approximation when $\vec{\omega}$ is large

Precessional Frequency

- Analyzing the previous vector triangle, the rate at which the axle rotates about the vertical axis can be found

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega}$$

- ω_p is the **precessional frequency**
 - This is valid only when $\omega_p \ll \omega$

Gyroscope in a Spacecraft

- The angular momentum of the spacecraft about its center of mass is zero
- A gyroscope is set into rotation, giving it a nonzero angular momentum
- The spacecraft rotates in the direction opposite to that of the gyroscope
- So the total momentum of the system remains zero



New Analysis Model 1

- Nonisolated System (Angular Momentum)
 - If a system interacts with its environment in the sense that there is an external torque on the system, the net external torque acting on the system is equal to the time rate of change of its angular momentum:

$$\sum \vec{\tau} = \frac{d\vec{L}_{tot}}{dt}$$

New Analysis Model 2

- Isolated System (Angular Momentum)
 - If a system experiences no external torque from the environment, the total angular momentum of the system is conserved:

$$\vec{L}_i = \vec{L}_f$$

- Applying this law of conservation of angular momentum to a system whose moment of inertia changes gives

$$I_i \omega_i = I_f \omega_f = \text{constant}$$