Week 14: Chapter 15

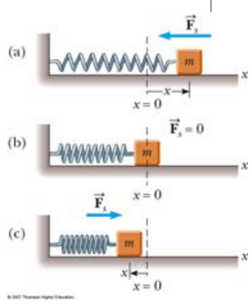
Oscillatory Motion

Periodic Motion

- ***Periodic motion*** is motion of an object that regularly returns to a given position after a fixed time interval
- A special kind of periodic motion occurs in mechanical systems when the force acting on the object is proportional to the position of the object relative to some equilibrium position
  - If the force is always directed toward the equilibrium position, the motion is called ***simple harmonic motion***

Motion of a Spring-Mass System

- A block of mass *m* is attached to a spring, the block is free to move on a frictionless horizontal surface
  - Use the active figure to vary the initial conditions and observe the resultant motion
- When the spring is neither stretched nor compressed, the block is at the ***equilibrium position***
  - $x = 0$

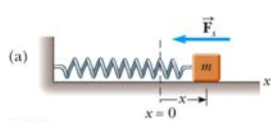


Hooke's Law

- Hooke's Law states $F_s = - kx$
  - F_s is the restoring force
    - It is always directed toward the equilibrium position
    - Therefore, it is always opposite the displacement from equilibrium
  - *k* is the force (spring) constant
  - *x* is the displacement

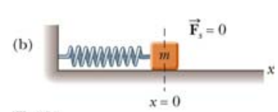
More About Restoring Force

- The block is displaced to the right of $x = 0$
  - The position is positive
- The restoring force is directed to the left



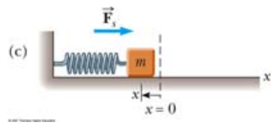
More About Restoring Force, 2

- The block is at the equilibrium position
  - $x = 0$
- The spring is neither stretched nor compressed
- The force is 0



More About Restoring Force, 3

- The block is displaced to the left of $x = 0$
  - The position is negative
- The restoring force is directed to the right



Acceleration

- The force described by Hooke's Law is the net force in Newton's Second Law

$$F_{\text{Hooke}} = F_{\text{Newton}}$$
$$-kx = ma_x$$
$$a_x = -\frac{k}{m}x$$

Motion of the Block

- The block continues to oscillate between –A and +A
  - These are turning points of the motion
- The force is conservative
- In the absence of friction, the motion will continue forever
  - Real systems are generally subject to friction, so they do not actually oscillate forever

Simple Harmonic Motion – Mathematical Representation

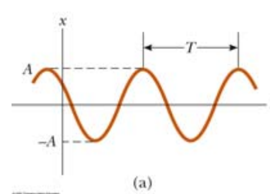
- Model the block as a particle
  - The representation will be **particle in simple harmonic motion model**
- Choose x as the axis along which the oscillation occurs
- Acceleration $a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$
- We let $\omega^2 = \frac{k}{m}$
- Then $a = -\omega^2 x$

Simple Harmonic Motion – Mathematical Representation, 2

- A function that satisfies the equation is needed
  - Need a function $x(t)$ whose second derivative is the same as the original function with a negative sign and multiplied by ω^2
  - The sine and cosine functions meet these requirements

Simple Harmonic Motion – Graphical Representation

- A solution is $x(t) = A \cos(\omega t + \phi)$
- A, ω, ϕ are all constants
- A cosine curve can be used to give physical significance to these constants



Simple Harmonic Motion – Definitions

- A is the amplitude of the motion
  - This is the maximum position of the particle in either the positive or negative direction
- ω is called the angular frequency
  - Units are rad/s
- ϕ is the phase constant or the initial phase angle

Simple Harmonic Motion, cont

- A and ϕ are determined uniquely by the position and velocity of the particle at t = 0
  - If the particle is at x = A at t = 0, then $\phi = 0$
- The phase of the motion is the quantity (ωt + ϕ)
- x (t) is periodic and its value is the same each time ωt increases by 2π radians

Period

- The ***period***, *T*, is the time interval required for the particle to go through one full cycle of its motion
  - The values of *x* and *v* for the particle at time *t* equal the values of *x* and *v* at *t* + *T*

$$T = \frac{2\pi}{\omega}$$

Frequency

- The inverse of the period is called the ***frequency***
- The frequency represents the number of oscillations that the particle undergoes per unit time interval

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

- Units are cycles per second = hertz (Hz)

Summary Equations – Period and Frequency

- The frequency and period equations can be rewritten to solve for ω

$$\omega = 2\pi f = \frac{2\pi}{T}$$

- The period and frequency can also be expressed as:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

An object of mass *m* is hung from a spring and set into oscillation. The period of the oscillation is measured and recorded as *T*. The object of mass *m* is removed and replaced with an object of mass 2*m*. When this object is set into oscillation, what is the period of the motion?

A. 2*T*
B. $\sqrt{2}T$
C. *T*
D. $T/\sqrt{2}$
E. *T*/2

Period and Frequency, cont

- The frequency and the period depend only on the mass of the particle and the force constant of the spring
- They do not depend on the parameters of motion
- The frequency is larger for a stiffer spring (large values of *k*) and decreases with increasing mass of the particle

Motion Equations for Simple Harmonic Motion

$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

- Simple harmonic motion is one-dimensional and so directions can be denoted by + or - sign
- Remember, simple harmonic motion is **not** uniformly accelerated motion

Maximum Values of v and a

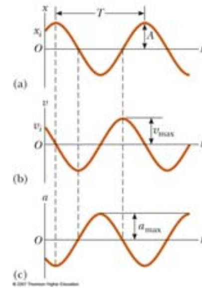
- Because the sine and cosine functions oscillate between ±1, we can easily find the maximum values of velocity and acceleration for an object in SHM

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}\,A$$

$$a_{max} = \omega^2 A = \frac{k}{m}A$$

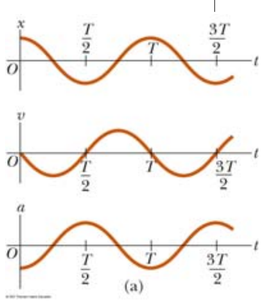
Graphs

- The graphs show:
  - (a) displacement as a function of time
  - (b) velocity as a function of time
  - (c) acceleration as a function of time
- The velocity is 90º out of phase with the displacement and the acceleration is 180º out of phase with the displacement



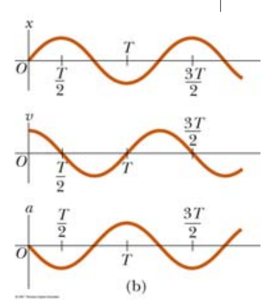
SHM Example 1

- Initial conditions at $t = 0$ are
  - $x(0) = A$
  - $v(0) = 0$
- This means $\phi = 0$
- The acceleration reaches extremes of $\pm\,\omega^2 A$ at A
- The velocity reaches extremes of $\pm\,\omega A$ at x = 0



SHM Example 2

- Initial conditions at $t = 0$ are
  - $x(0)=0$
  - $v(0) = v_i$
- This means $\phi = -\pi/2$
- The graph is shifted one-quarter cycle to the right compared to the graph of $x(0) = A$

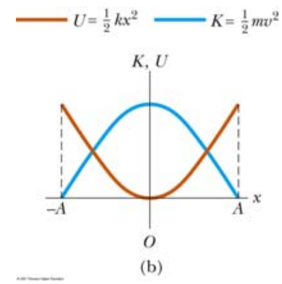


Energy of the SHM Oscillator

- Assume a spring-mass system is moving on a frictionless surface
- This tells us the total energy is constant
- The kinetic energy can be found by
  - $K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi)$
- The elastic potential energy can be found by
  - $U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$
- The total energy is $E = K + U = \frac{1}{2} kA^2$

Energy of the SHM Oscillator, cont

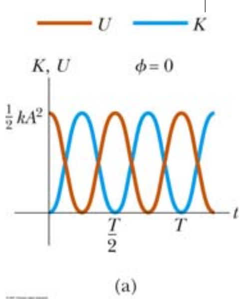
- The total mechanical energy is constant
- The total mechanical energy is proportional to the square of the amplitude
- Energy is continuously being transferred between potential energy stored in the spring and the kinetic energy of the block
  - Use the active figure to investigate the relationship between the motion and the energy



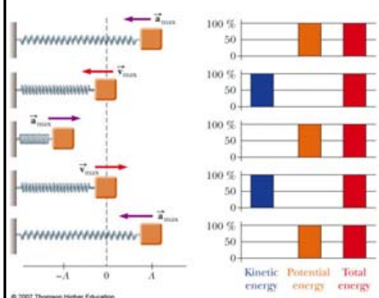
Energy of the SHM Oscillator, cont

- As the motion continues, the exchange of energy also continues
- Energy can be used to find the velocity

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$
$$= \pm\omega^2\sqrt{A^2 - x^2}$$



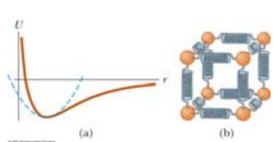
Energy in SHM, summary



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{T}{4}$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$\frac{T}{2}$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{3T}{4}$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

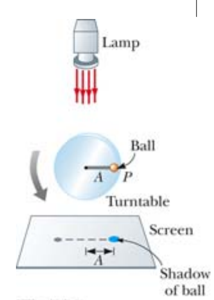
Importance of Simple Harmonic Oscillators

- Simple harmonic oscillators are good models of a wide variety of physical phenomena
- Molecular example
  - If the atoms in the molecule do not move too far, the forces between them can be modeled as if there were springs between the atoms
  - The potential energy acts similar to that of the SHM oscillator



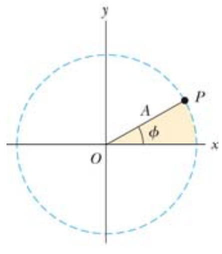
SHM and Circular Motion

- This is an overhead view of a device that shows the relationship between SHM and circular motion
- As the ball rotates with constant angular speed, its shadow moves back and forth in simple harmonic motion



SHM and Circular Motion, 2

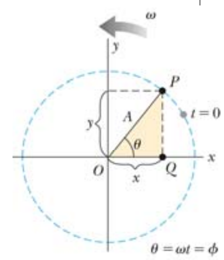
- The circle is called a ***reference circle***
- Line *OP* makes an angle ϕ with the *x* axis at $t = 0$
- Take *P* at $t = 0$ as the reference position



(a)

SHM and Circular Motion, 3

- The particle moves along the circle with constant angular velocity ω
- *OP* makes an angle θ with the *x* axis
- At some time, the angle between *OP* and the *x* axis will be $\theta = \omega t + \phi$



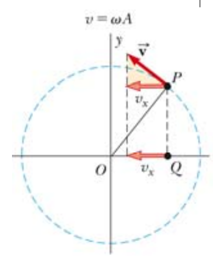
(b)

SHM and Circular Motion, 4

- The points *P* and *Q* always have the same *x* coordinate
- $x(t) = A \cos(\omega t + \phi)$
- This shows that point *Q* moves with simple harmonic motion along the *x* axis
- Point *Q* moves between the limits ±*A*

SHM and Circular Motion, 5

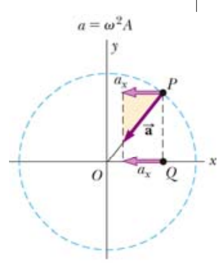
- The *x* component of the velocity of *P* equals the velocity of *Q*
- These velocities are
  - $v = -\omega A \sin(\omega t + \phi)$



(c)

SHM and Circular Motion, 6

- The acceleration of point *P* on the reference circle is directed radially inward
- *P* 's acceleration is $a = \omega^2 A$
- The *x* component is $-\omega^2 A \cos(\omega t + \phi)$
- This is also the acceleration of point *Q* along the *x* axis



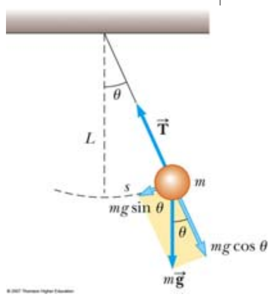
(d)

Simple Pendulum

- A simple pendulum also exhibits periodic motion
- The motion occurs in the vertical plane and is driven by gravitational force
- The motion is very close to that of the SHM oscillator
  - If the angle is <10°

Simple Pendulum, 2

- The forces acting on the bob are the tension and the weight
  - $\vec{T}$ is the force exerted on the bob by the string
  - $m\vec{g}$ is the gravitational force
- The tangential component of the gravitational force is a restoring force



Simple Pendulum, 3

- In the tangential direction,

$$F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

- The length, L, of the pendulum is constant, and for small values of θ

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta$$

- This confirms the form of the motion is SHM

Simple Pendulum, 4

- The function θ can be written as
  $\theta = \theta_{max}\cos(\omega t + \phi)$
- The angular frequency is

$$\omega = \sqrt{\frac{g}{L}}$$

- The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Simple Pendulum, Summary

- The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity
- The period is independent of the mass
- All simple pendula that are of equal length and are at the same location oscillate with the same period

Clicker Question

A grandfather clock depends on the period of a pendulum to keep correct time. Suppose a grandfather clock is calibrated correctly and then a mischievous child slides the bob of the pendulum downward on the oscillating rod. The grandfather clock will run:

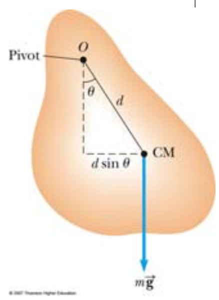
A. slow
B. fast
C. correctly
D. depending on the weight of the bob.

Physical Pendulum

- If a hanging object oscillates about a fixed axis that does not pass through the center of mass and the object cannot be approximated as a particle, the system is called a ***physical pendulum***
  - It cannot be treated as a simple pendulum

Physical Pendulum, 2

- The gravitational force provides a torque about an axis through *O*
- The magnitude of the torque is
  $mgd \sin \theta$
- *I* is the moment of inertia about the axis through *O*



Physical Pendulum, 3

- From Newton's Second Law,

$$-mgd \sin\theta = I\frac{d^2\theta}{dt^2}$$

- The gravitational force produces a restoring force
- Assuming θ is small, this becomes

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

Physical Pendulum,4

- This equation is in the form of an object in simple harmonic motion
- The angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}}$$

- The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Physical Pendulum, 5

- A physical pendulum can be used to measure the moment of inertia of a flat rigid object
  - If you know *d*, you can find *I* by measuring the period
- If $I = md^2$ then the physical pendulum is the same as a simple pendulum
  - The mass is all concentrated at the center of mass

Torsional Pendulum

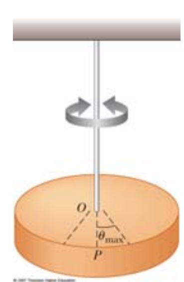
- Assume a rigid object is suspended from a wire attached at its top to a fixed support
- The twisted wire exerts a restoring torque on the object that is proportional to its angular position

Torsional Pendulum, 2

- The restoring torque is $\tau = -\kappa\theta$
  - κ is the *torsion constant* of the support wire
- Newton's Second Law gives

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$



Torsional Period, 3

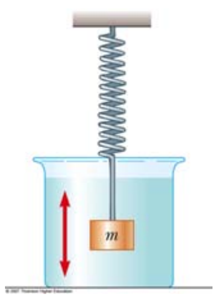
- The torque equation produces a motion equation for simple harmonic motion
- The angular frequency is $\omega = \sqrt{\frac{\kappa}{I}}$
- The period is $T = 2\pi\sqrt{\frac{I}{\kappa}}$
  - No small-angle restriction is necessary
  - Assumes the elastic limit of the wire is not exceeded

Damped Oscillations

- In many real systems, nonconservative forces are present
  - This is no longer an ideal system (the type we have dealt with so far)
  - Friction is a common nonconservative force
- In this case, the mechanical energy of the system diminishes in time, the motion is said to be ***damped***

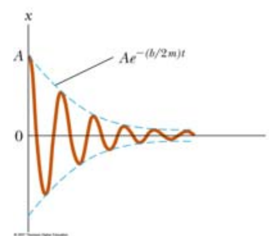
Damped Oscillation, Example

- One example of damped motion occurs when an object is attached to a spring and submerged in a viscous liquid
- The retarding force can be expressed as $\vec{\mathbf{R}} = -b\vec{\mathbf{v}}$ where b is a constant
  - b is called the ***damping coefficient***



Damped Oscillations, Graph

- A graph for a damped oscillation
- The amplitude decreases with time
- The blue dashed lines represent the ***envelope*** of the motion
- Use the active figure to vary the mass and the damping constant and observe the effect on the damped motion



Damping Oscillation, Equations

- The restoring force is $-kx$
- From Newton's Second Law

  $\Sigma F_x = -k\,x - bv_x = ma_x$
- When the retarding force is small compared to the maximum restoring force we can determine the expression for x
  - This occurs when b is small

Damping Oscillation, Equations, cont

- The position can be described by

$$x = Ae^{-\left(b/2m\right)t}\cos(\omega t + \phi)$$

- The angular frequency will be

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Damping Oscillation, Example Summary

- When the retarding force is small, the oscillatory character of the motion is preserved, but the amplitude decreases exponentially with time
- The motion ultimately ceases
- Another form for the angular frequency

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

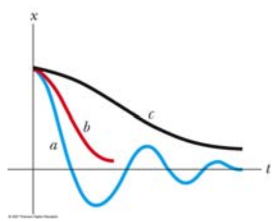
  - where ω_0 is the angular frequency in the absence of the retarding force and is called the **natural frequency** of the system

Types of Damping

- If the restoring force is such that $b/2m < \omega_o$, the system is said to be ***underdamped***
- When b reaches a critical value b_c such that $b_c / 2m = \omega_0$, the system will not oscillate
  - The system is said to be ***critically damped***
- If the restoring force is such that $bv_{max} > kA$ and $b/2m > \omega_o$, the system is said to be ***overdamped***

Types of Damping, cont

- Graphs of position versus time for
  - (a) an underdamped oscillator
  - (b) a critically damped oscillator
  - (c) an overdamped oscillator
- For critically damped and overdamped there is no angular frequency



Forced Oscillations

- It is possible to compensate for the loss of energy in a damped system by applying an external force
- The amplitude of the motion remains constant if the energy input per cycle exactly equals the decrease in mechanical energy in each cycle that results from resistive forces

Forced Oscillations, 2

- After a driving force on an initially stationary object begins to act, the amplitude of the oscillation will increase
- After a sufficiently long period of time, $E_{driving} = E_{lost\ to\ internal}$
  - Then a steady-state condition is reached
  - The oscillations will proceed with constant amplitude

Forced Oscillations, 3

- The amplitude of a driven oscillation is

$$A = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

  - ω_0 is the natural frequency of the undamped oscillator

Resonance

- When the frequency of the driving force is near the natural frequency ($\omega \approx \omega_0$) an increase in amplitude occurs
- This dramatic increase in the amplitude is called ***resonance***
- The natural frequency ω_0 is also called the resonance frequency of the system

Resonance, cont

- At resonance, the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum
  - The applied force and v are both proportional to sin ($\omega t + \phi$)
  - The power delivered is $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$
    - This is a maximum when the force and velocity are in phase
    - The power transferred to the oscillator is a maximum

Resonance, Final

- Resonance (maximum peak) occurs when driving frequency equals the natural frequency
- The amplitude increases with decreased damping
- The curve broadens as the damping increases
- The shape of the resonance curve depends on b