Week 8: Chapter 9

Linear Momentum and Collisions

Linear Momentum

- The **linear momentum** of a particle, or an object that can be modeled as a particle, of mass *m* moving with a velocity $\vec{\mathbf{v}}$ is defined to be the product of the mass and velocity:
  - $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$
    - The terms momentum and linear momentum will be used interchangeably in the text

Linear Momentum, cont

- Linear momentum is a vector quantity
  - Its direction is the same as the direction of the velocity
- The dimensions of momentum are ML/T
- The SI units of momentum are kg · m / s
- Momentum can be expressed in component form:
  - $p_x = m\, v_x$ $p_y = m\, v_y$ $p_z = m\, v_z$

Newton' Law and Momentum

- Newton's Second Law can be used to relate the momentum of a particle to the resultant force acting on it

$$\Sigma\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{v}}}{dt} = \frac{d\left(m\vec{\mathbf{v}}\right)}{dt} = \frac{d\vec{\mathbf{p}}}{dt}$$

with constant mass

Conservation of Linear Momentum

- Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant
  - The momentum of the system is conserved, not necessarily the momentum of an individual particle
  - This also tells us that the total momentum of an isolated system equals its initial momentum

Conservation of Momentum, 2

- Conservation of momentum can be expressed mathematically in various ways
  - $\vec{p}_{total} = \vec{p}_1 + \vec{p}_2 =$ constant
  - $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$
- In component form, the total momenta in each direction are independently conserved
  - $p_{ix} = p_{fx}$ $p_{iy} = p_{fy}$ $p_{iz} = p_{fz}$
- Conservation of momentum can be applied to systems with any number of particles
- This law is the mathematical representation of the momentum version of the isolated system model

Conservation of Momentum, Archer Example

- The archer is standing on a frictionless surface (ice)
- Approaches:
  - Newton's Second Law – no, no information about *F* or *a*
  - Energy approach – no, no information about work or energy
  - Momentum – yes

Archer Example, 2

- Conceptualize
  - The arrow is fired one way and the archer recoils in the opposite direction
- Categorize
  - Momentum
    - Let the system be the archer with bow (particle 1) and the arrow (particle 2)
    - There are no external forces in the *x*-direction, so it is isolated in terms of momentum in the *x*-direction
- Analyze
  - Total momentum before releasing the arrow is 0

Archer Example, 3

- Analyze, cont.
  - The total momentum after releasing the arrow is $\vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f} = 0$
- Finalize
  - The final velocity of the archer is negative
    - Indicates he moves in a direction opposite the arrow
    - Archer has much higher mass than arrow, so velocity is much lower

Impulse and Momentum

- From Newton's Second Law, $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$
- Solving for $d\vec{\mathbf{p}}$ gives $d\vec{\mathbf{p}} = \sum \vec{\mathbf{F}} dt$
- Integrating to find the change in momentum over some time interval

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \int_{t_i}^{t_f} \vec{\mathbf{F}} dt = \vec{\mathbf{I}}$$

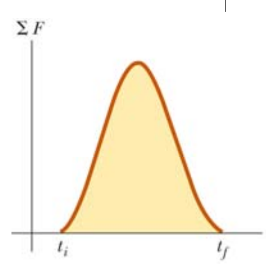
- The integral is called the *impulse,* $\vec{\mathbf{I}}$, of the force acting on an object over Δt

Impulse-Momentum Theorem

- This equation expresses the **impulse-momentum theorem**: The impulse of the force acting on a particle equals the change in the momentum of the particle
  - $\Delta\vec{\mathbf{p}} = \vec{\mathbf{I}}$
  - This is equivalent to Newton's Second Law

More About Impulse

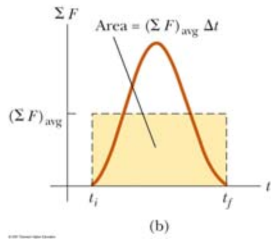
- Impulse is a vector quantity
- The magnitude of the impulse is equal to the area under the force-time curve
  - The force may vary with time
- Dimensions of impulse are M L / T
- Impulse is not a property of the particle, but a measure of the change in momentum of the particle



(a)

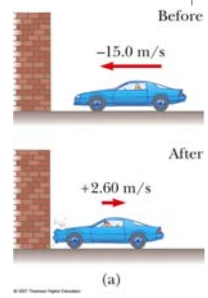
Impulse, Final

- The impulse can also be found by using the time averaged force
- $\vec{\mathbf{I}} = \sum \vec{\mathbf{F}} \Delta t$
- This would give the same impulse as the time-varying force does



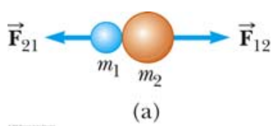
Impulse-Momentum: Crash Test Example

- Categorize
  - Assume force exerted by wall is large compared with other forces
  - Gravitational and normal forces are perpendicular and so do not effect the horizontal momentum
  - Can apply impulse approximation



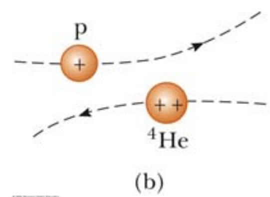
Collisions – Example 1

- Collisions may be the result of direct contact
- The impulsive forces may vary in time in complicated ways
  - This force is internal to the system
  - Observe the variations in the active figure
- Momentum is conserved



Collisions – Example 2

- The collision need not include physical contact between the objects
- There are still forces between the particles
- This type of collision can be analyzed in the same way as those that include physical contact



Types of Collisions

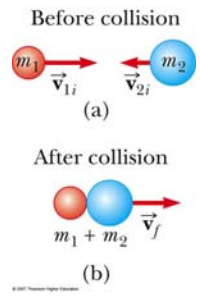
- In an ***elastic*** collision, momentum and kinetic energy are conserved
  - Perfectly elastic collisions occur on a microscopic level
  - In macroscopic collisions, only approximately elastic collisions actually occur
    - Generally some energy is lost to deformation, sound, etc.
- In an ***inelastic*** collision, kinetic energy is not conserved, although momentum is still conserved
  - If the objects stick together after the collision, it is a ***perfectly inelastic*** collision

Collisions, cont

- In an inelastic collision, some kinetic energy is lost, but the objects do not stick together
- Elastic and perfectly inelastic collisions are limiting cases, most actual collisions fall in between these two types
- Momentum is conserved in all collisions

Perfectly Inelastic Collisions

- Since the objects stick together, they share the same velocity after the collision
- $m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = (m_1 + m_2)\vec{\mathbf{v}}_f$



Clicker Question

In a perfectly inelastic one-dimensional collision between two moving objects, what condition alone is necessary so that the final kinetic energy of the system is zero after the collision?

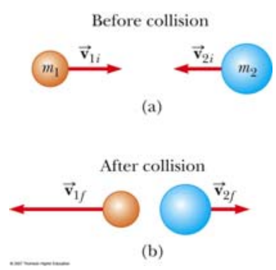
A. It is not possible
B. The objects must have momenta with the same magnitude but opposite directions.
C. The objects must have the same mass.
D. The objects must have the same velocity.
E. The objects must have the same speed, with velocity vectors in opposite directions.

Elastic Collisions

- Both momentum and kinetic energy are conserved

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}$$

$$\frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 = \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2$$



Elastic Collisions, cont

- Typically, there are two unknowns to solve for and so you need two equations
- The kinetic energy equation can be difficult to use
- With some algebraic manipulation, a different equation can be used

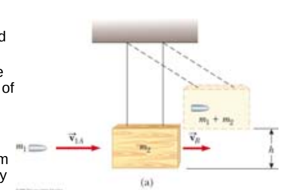
  $v_{1i} - v_{2i} = v_{1f} + v_{2f}$
- This equation, along with conservation of momentum, can be used to solve for the two unknowns
  - It can only be used with a one-dimensional, elastic collision between two objects

Elastic Collisions, final

- Example of some special cases
  - $m_1 = m_2$ – the particles exchange velocities
  - When a very heavy particle collides head-on with a very light one initially at rest, the heavy particle continues in motion unaltered and the light particle rebounds with a speed of about twice the initial speed of the heavy particle
  - When a very light particle collides head-on with a very heavy particle initially at rest, the light particle has its velocity reversed and the heavy particle remains approximately at rest

Collision Example – Ballistic Pendulum

- Conceptualize
  - Observe diagram
- Categorize
  - Isolated system of projectile and block
  - Perfectly inelastic collision – the bullet is embedded in the block of wood
  - Momentum equation will have two unknowns
  - Use conservation of energy from the pendulum to find the velocity just after the collision
  - Then you can find the speed of the bullet

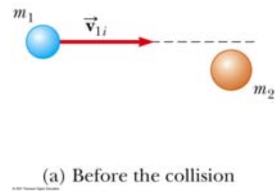


Two-Dimensional Collisions

- The momentum is conserved in all directions
- Use subscripts for
  - Identifying the object
  - Indicating initial or final values
  - The velocity components
- If the collision is elastic, use conservation of kinetic energy as a second equation
  - Remember, the simpler equation can only be used for one-dimensional situations

Two-Dimensional Collision, example

- Particle 1 is moving at velocity $\vec{\mathbf{v}}_{1i}$ and particle 2 is at rest
- In the *x*-direction, the initial momentum is m_1v_{1i}
- In the *y*-direction, the initial momentum is 0

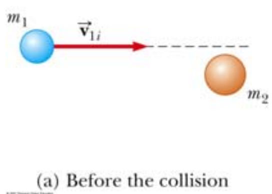


(a) Before the collision

Clicker Question

What is the direction of motion of m_2 after the collision?

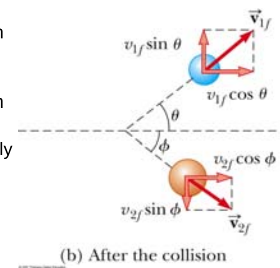
A. up-left
B. up-right
C. down-left
D. down-right
E. Right only



(a) Before the collision

Two-Dimensional Collision, example cont

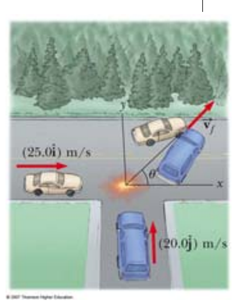
- After the collision, the momentum in the *x*-direction is $m_1v_{1f}\cos\theta + m_2v_{2f}\cos\phi$
- After the collision, the momentum in the *y*-direction is $m_1v_{1f}\sin\theta + m_2v_{2f}\sin\phi$
- If the collision is elastic, apply the kinetic energy equation
- This is an example of a *glancing collision*



(b) After the collision

Two-Dimensional Collision Example

- Conceptualize
  - See picture
  - Choose East to be the positive x-direction and North to be the positive y-direction
- Categorize
  - Ignore friction
  - Model the cars as particles
  - The collision is perfectly inelastic
    - The cars stick together



The Center of Mass

- There is a special point in a system or object, called the ***center of mass***, that moves as if all of the mass of the system is concentrated at that point
- The system will move as if an external force were applied to a single particle of mass *M* located at the center of mass
  - *M* is the total mass of the system

Center of Mass, Coordinates

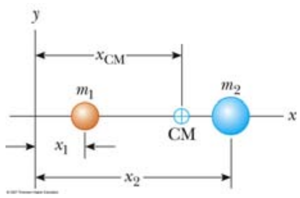
- The coordinates of the center of mass are

$$x_{CM} = \frac{\sum_i m_i x_i}{M}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{M}$$

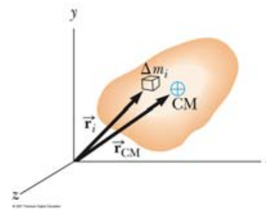
$$z_{CM} = \frac{\sum_i m_i z_i}{M}$$

  - M is the total mass of the system
    - Use the active figure to observe effect of different masses and positions



Center of Mass, Extended Object

- Similar analysis can be done for an extended object
- Consider the extended object as a system containing a large number of particles
- Since particle separation is very small, it can be considered to have a constant mass distribution



Center of Mass, position

- The center of mass in three dimensions can be located by its position vector, $\vec{\mathbf{r}}_{CM}$
  - For a system of particles,

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M}\sum_i m_i \vec{\mathbf{r}}_i$$

    - $\vec{\mathbf{r}}_i$ is the position of the i^{th} particle, defined by

$$\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

  - For an extended object,

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M}\int \vec{\mathbf{r}}\, dm$$

Finding Center of Mass, Irregularly Shaped Object

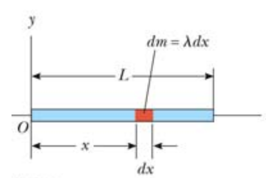
- Suspend the object from one point
- The suspend from another point
- The intersection of the resulting lines is the center of mass

Center of Gravity

- Each small mass element of an extended object is acted upon by the gravitational force
- The net effect of all these forces is equivalent to the effect of a single force $M\vec{\mathbf{g}}$ acting through a point called the **center of gravity**
  - If $\vec{\mathbf{g}}$ is constant over the mass distribution, the center of gravity coincides with the center of mass

Center of Mass, Rod

- Conceptualize
  - Find the center of mass of a rod of mass *M* and length *L*
  - The location is on the *x*-axis (or $y_{CM} = z_{CM} = 0$)
- Categorize
  - Analysis problem
- Analyze
  - Use equation for x_{cm}
  - $x_{CM} = L / 2$



Velocity and Momentum of a System of Particles

- The velocity of the center of mass of a system of particles is

$$\vec{\mathbf{v}}_{CM} = \frac{d\vec{\mathbf{r}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \vec{\mathbf{v}}_i$$

- The momentum can be expressed as

$$M\vec{\mathbf{v}}_{CM} = \sum_i m_i \vec{\mathbf{v}}_i = \sum_i \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_{tot}$$

- The total linear momentum of the system equals the total mass multiplied by the velocity of the center of mass

Acceleration of the Center of Mass

- The acceleration of the center of mass can be found by differentiating the velocity with respect to time

$$\vec{\mathbf{a}}_{CM} = \frac{d\vec{\mathbf{v}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \vec{\mathbf{a}}_i$$

Forces In a System of Particles

- The acceleration can be related to a force

$$M\vec{\mathbf{a}}_{CM} = \sum_i \vec{\mathbf{F}}_i$$

- If we sum over all the internal forces, they cancel in pairs and the net force on the system is caused only by the external forces

Newton's Second Law for a System of Particles

- Since the only forces are external, the net external force equals the total mass of the system multiplied by the acceleration of the center of mass:

$$\sum \vec{\mathbf{F}}_{ext} = M\vec{\mathbf{a}}_{CM}$$

- The center of mass of a system of particles of combined mass *M* moves like an equivalent particle of mass *M* would move under the influence of the net external force on the system

Impulse and Momentum of a System of Particles

- The impulse imparted to the system by external forces is

$$\vec{\mathbf{I}} = \int \sum \vec{\mathbf{F}}_{ext}dt = M\int d\vec{\mathbf{v}}_{CM} = \Delta\vec{\mathbf{p}}_{tot}$$

- The total linear momentum of a system of particles is conserved if no net external force is acting on the system

$$M\,\vec{\mathbf{v}}_{CM} = \vec{\mathbf{p}}_{tot} = constant \quad when \sum \vec{\mathbf{F}}_{ext} = 0$$

Motion of the Center of Mass, Example

- A projectile is fired into the air and suddenly explodes
- With no explosion, the projectile would follow the dotted line
- After the explosion, the center of mass of the fragments still follows the dotted line, the same parabolic path the projectile would have followed with no explosion
- Use the active figure to observe a variety of explosions

Deformable Systems

- To analyze the motion of a deformable system, use Conservation of Energy and the Impulse-Momentum Theorem

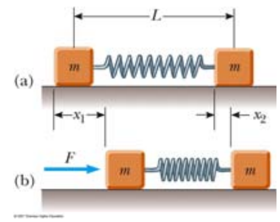
$$\Delta E_{system} = \sum T \rightarrow \Delta K + \Delta U = 0$$

$$\vec{\mathbf{I}} = \Delta \vec{\mathbf{p}}_{tot} \rightarrow \int \vec{\mathbf{F}}_{ext} dt = m\Delta\vec{\mathbf{v}}$$

  - If the force is constant, the integral can be easily evaluated

Deformable System (Spring) Example

- Conceptualize
  - See figure
  - Push on left block, it moves to right, spring compresses
  - At any given time, the blocks are generally moving with different velocities
  - The blocks oscillate back and forth with respect to the center of mass



Spring Example, cont

- Categorize
  - Non isolated system
    - Work is being done on it by the applied force
  - It is a deformable system
  - The applied force is constant, so the acceleration of the center of mass is constant
  - Model as a particle under constant acceleration
- Analyze
  - Apply impulse-momentum
  - Solve for v_{cm}

Spring Example, final

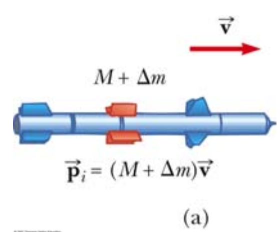
- Analyze, cont.
  - Find energies
- Finalize
  - Answers do not depend on spring length, spring constant, or time interval

Rocket Propulsion

- The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel

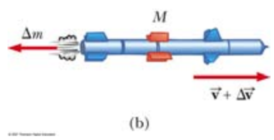
Rocket Propulsion, 2

- The initial mass of the rocket plus all its fuel is $M + \Delta m$ at time t_i and speed v
- The initial momentum of the system is $\vec{\mathbf{p}}_i = (M + \Delta m)\vec{\mathbf{v}}$



Rocket Propulsion, 3

- At some time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel, Δm has been ejected
- The rocket's speed has increased by Δv



Rocket Propulsion, 4

- Because the gases are given some momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction
- Therefore, the rocket is accelerated as a result of the "push" from the exhaust gases
- In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process

Rocket Propulsion, 5

- The basic equation for rocket propulsion is

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right)$$

- The increase in rocket speed is proportional to the speed of the escape gases (v_e)
  - So, the exhaust speed should be very high
- The increase in rocket speed is also proportional to the natural log of the ratio M_i/M_f
  - So, the ratio should be as high as possible, meaning the mass of the rocket should be as small as possible and it should carry as much fuel as possible

Thrust

- The thrust on the rocket is the force exerted on it by the ejected exhaust gases

$$thrust = M\frac{dv}{dt} = \left|v_e \frac{dM}{dt}\right|$$

- The thrust increases as the exhaust speed increases
- The thrust increases as the rate of change of mass increases
  - The rate of change of the mass is called the **burn rate**