PHYS 777
Plasma Physics and Magnetohydrodynamics

2004 Fall
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Lecture 3

Magneto hydrostatics

Force Equation (2-21)

$$\rho \frac{\nabla \vec{v}}{\nabla t} = -\nabla \vec{P} + \vec{j} \times \vec{B} + \rho \vec{g}$$

magnetohydrostatics means $\frac{\nabla \vec{v}}{\nabla t} = 0$

$$-\nabla P + \vec{j} \times \vec{B} + \rho g = 0$$

$$j = \frac{\vec{\nabla} \times \vec{B}}{\mu} \qquad \nabla \cdot \vec{B} = 0 \qquad \rho = \frac{mP}{kT}$$

define a coordinate along magnetic field lines (s)

$$\vec{j} \times \vec{B} = 0 \qquad -\frac{dP}{ds} = \rho g \cos\theta$$

$$\delta s \cos\theta = \delta z \qquad \frac{dP}{dz} = -\rho g$$

$$P = P_o \exp \int_0^z \frac{dz}{\Lambda(z)} \qquad P_o = P(z = 0)$$

$$\Lambda(z) = \frac{kT(z)}{mg} = \frac{P}{\rho g} \qquad \text{Scale Height}$$

if Λ is a constant $\quad P = P_o e^{-z/\Lambda}$

$$g = g_\Theta \frac{r_\Theta^2}{r^2} \qquad g_\Theta = 274 \quad m/s^2$$

$$r_\Theta = 6.96 \times 10^8 \quad m$$

$$\Lambda = 50\,T\left(\frac{r}{r_\Theta}\right)^2 \quad m$$

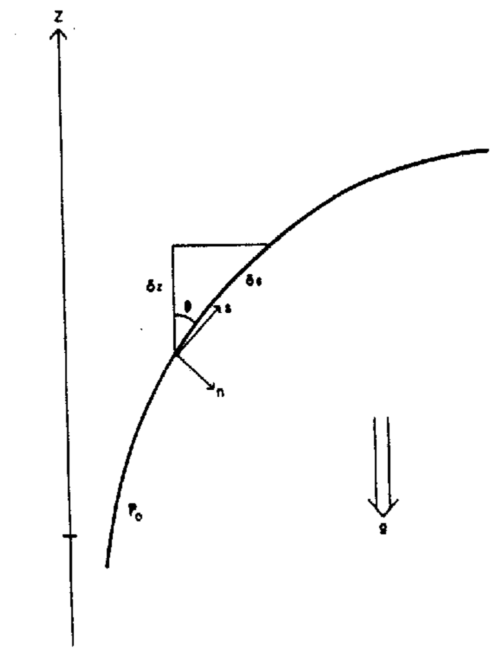


Fig. 3.1. A magnetic field line inclined at θ to the vertical z-axis. Distance s is measured along the field line and p_0 is the pressure at the reference height $z = 0$.

where p_0 is the base pressure (at $z = 0$) which may vary from one field line to another; also

$$\Lambda(z) = \frac{k_B T(z)}{mg} \left(= \frac{p}{\rho g} = \frac{\tilde{R} T(z)}{\tilde{\mu} g} \right) \tag{3.7}$$

is the (*pressure*) *scale-height*, which represents the vertical distance over which the pressure falls by a factor *e*. In terms of density Equation (3.6) becomes

$$\frac{\rho}{\rho_0} = \frac{T_0}{T(z)} \exp - \int_0^z \frac{1}{\Lambda(z)} \, dz. \tag{3.8}$$

Equation (3.6) shows that the pressure along a given magnetic field line decreases exponentially with height. The rate of decrease depends on the temperature structure as determined by the energy equation. It is here that the magnetic field enters implicitly, since the length of the field line depends on the magnetic structure and may influence both the conductive and heating terms in the energy balance. The corresponding density variation follows from Equation (3.8). When the temperature increases with height the density decreases faster than the pressure; but, when the temperature falls with height, the density may either increase or decrease locally depending on whether the factor T^{-1} or the exponential dominates in Equation (3.8).

For the particular case when the temperature is uniform along a field line (due to.

$e.g.\quad T = 10^4\,\mathrm{k} \qquad r = r_\Theta \qquad \Lambda = 500\mathrm{km}$

$T = 10^6\,\mathrm{k} \qquad \Lambda = 5\times 10^7 (\frac{r}{r_\Theta})^2 \;\mathrm{m}$

if structure height Scale height

ρ term can be ignored

if $\beta = \frac{2\mu P_0}{B_0}$ 1, pressure term can be dropped as well

then we have $\vec{j}\times\vec{B} = 0$ force - free condition

pure vertical fields

$$0 = -\frac{\partial}{\partial x}(P + \frac{B^2}{2\mu})$$ solutions are same as above

$$P + \frac{B^2}{2\mu} = f(z) \qquad \frac{\mathrm{dP}}{\mathrm{dz}} = -\rho g$$ density is not a function of x

Horizontal field

$$B = B(z)\vec{x}$$

$$0 = -\frac{d}{dz}(P + \frac{B^2}{2\mu}) - \rho g$$

$$P = [P_0 - \int_0^z e^{z/\Lambda} \frac{d}{dz}(\frac{B^2}{2\mu})dz]e^{-z/\Lambda}$$

$$P = P_0 e^{-z/\Lambda_B} \qquad \rho = \rho_0 e^{-z/\Lambda_B} \qquad \mathrm{B} = \mathrm{B}_0 e^{-z/2\Lambda_B}$$

$$\Lambda_B = (P_0 + B_0^2 / 2\mu) / \rho_0 g$$ modified scale height

vertical fields

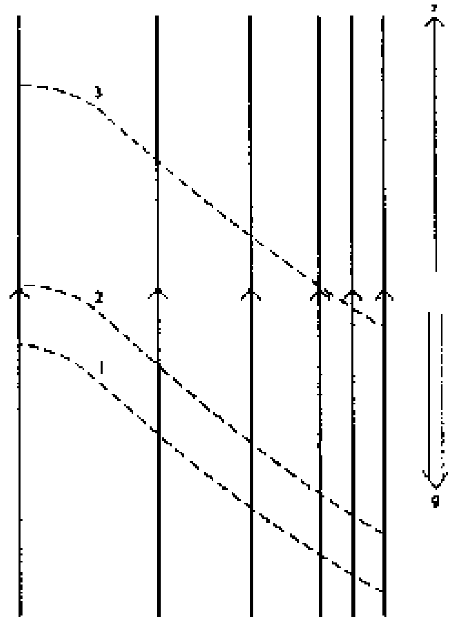
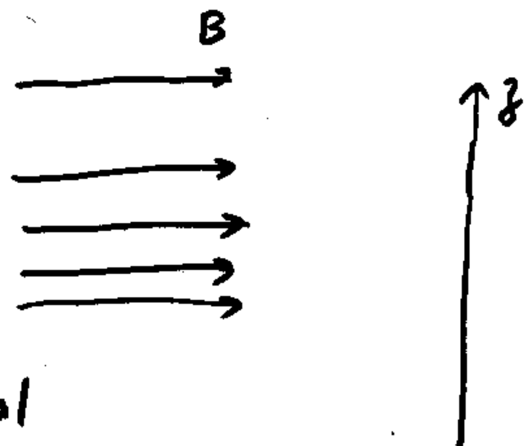


Fig. 3.2. For plasma situated in a vertical magnetic field, the isodensity contours are horizontal, while the constant-pressure contours (dashed) are inclined as shown, with the labels 1, 2, 3 indicating successively lower pressure values.



horizontal fields

Cylindrically Symmetric Fields

$$\vec{B} = (0, B_\Phi(R), B_Z(R))$$

$$\vec{j} = (0, -\frac{1}{\mu}\frac{dB_Z}{dR}, \frac{1}{\mu R}\frac{d}{dR}(RB_\Phi))$$

ignore g in force equation

$$\frac{dP}{dR} + \frac{d}{dR}(\frac{B_\Phi^2 + B_Z^2}{2\mu}) + \frac{B_\Phi^2}{\mu R} = 0 \qquad \text{(3-16)}$$

Field lines are given by

$$\frac{Rd\Phi}{B_\Phi} = \frac{dZ}{B_Z}$$

twist $\quad \Phi = \int d\Phi = \int_0^{2L} \frac{B_\Phi}{RB_Z} dZ$

$$\Phi(R) = \frac{B_\Phi \, 2L}{R\, B_Z(R)} \qquad \frac{4\pi L}{\Phi}: \text{ pitch of field}$$

Pure Axial Fields

$$B_\Phi = \begin{cases} \dfrac{\mu I R}{2\pi a^2} & R<a \\[2ex] \dfrac{\mu I}{2\pi R} & R>a \end{cases}$$

$$P = \begin{cases} P_\infty + \dfrac{1}{4}\mu(\dfrac{I}{\pi a^2})^2 (a^2 - R^2) & R<a \\[2ex] P_\infty & R>a \end{cases}$$

R<a   gas pressure balances with magnetic pressure and tension

R>a   gas pressure balances with tension

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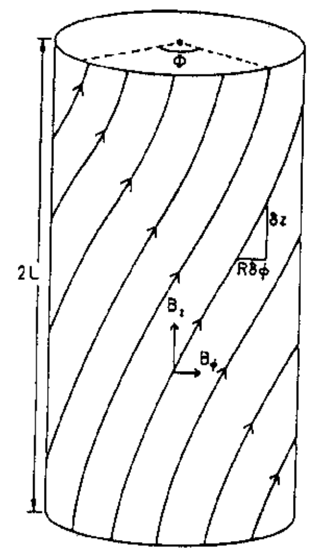

Fig. 3.3.   The notation for a cylindrically symmetric flux tube of length $2L$.

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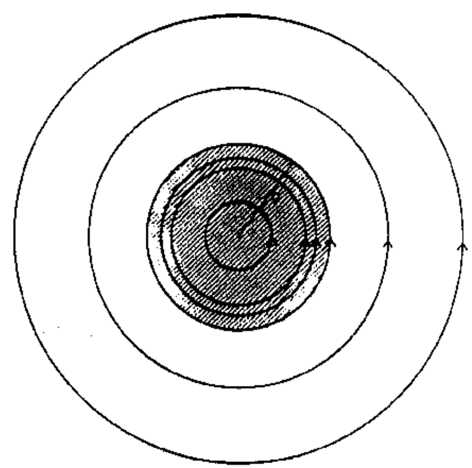

Fig. 3.4.   The purely azimuthal magnetic field lines in a section across a column of uniform current and radius a.

In lab, this configuration is called linear Pitch Relationship beween current I and number of particles / unit length

$$I = \int_0^{R_0} j_z 2\pi R dR$$

$$N = \int_0^{R_0} n 2\pi R dR$$

$$\int_0^{R_0} R^2 dP = -\int_0^{R_0} \frac{RB_\Phi}{\mu} d(RB_\Phi)$$

$$T = \frac{P}{nK}$$

$$I^2 = \frac{8\pi}{\mu} KTN$$ —— Bennet's Relation

Force – Free Field $\mu \vec{j} = \alpha \vec{B}$

- linear force free field

$$\frac{d}{dR}\left(\frac{B_\Phi^2 + B_Z^2}{2\mu}\right) + \frac{B_\Phi^2}{\mu R} = 0 \qquad (3-16, P=0)$$

if α is constant, —— linear force free

$$\vec{j} = \frac{\nabla \times \vec{B}}{\mu} \qquad -\frac{dB_Z}{dR} = \alpha B_\Phi$$

$$B_\Phi = B_0 J_1(\alpha R) \qquad B_Z = B_0 J_0(\alpha R)$$

J_0 and J_1 are Bessel functions

- Non-linear if $\mathrm{B} = \mathrm{f(R)}$ $\quad \mathrm{B}_\Phi^2 = -\frac{1}{2} R \frac{df}{dR}$

$$\mathrm{B}_z^2 = B^2 - B_\Phi^2 \qquad \mathrm{f} = \frac{1}{\mathrm{R}^2}$$ gives pure azimuthal fields

Another example: Uniform - twist field

$$B_\Phi = \frac{B_0 \Phi R / 2L}{1 + \Phi^2 R^2 / (2L)^2}$$

$$B_Z = \frac{B_0}{1 + \Phi^2 R^2 / (2L)^2}$$

Magnetostatic Fields

$$\frac{dP}{dR} + \frac{d}{dR}\left(\frac{B_\Phi^2 + B_Z^2}{2\mu}\right) + \frac{B_\Phi^2}{R\mu} = 0$$

one simple solution --- uniform twist

$$B_Z = \frac{B_0}{1 + R^2 / a^2} \qquad B_\Phi = \frac{\Phi R B_Z}{2L}$$

$$P(R) = P_\infty + \frac{[\Phi a^2 / (2L)^2 - 1] B_0^2}{(1 + R^2 / a^2) 2\mu}$$

Another simple solution --- non - uniform twist

$$\mathrm{B}_\mathrm{Z} = B_0$$

$$\Phi(R) = \frac{2LB_\Phi}{RB_0} = \frac{\Phi_0}{1 + R^2 / a^2}$$

$$P(R) = P_\infty + \frac{\Phi^2 B_0^2 a^2}{8\mu L^2}$$

Effect of Expanding a tube: $\frac{\mathrm{B}_\Phi}{\mathrm{B}_\mathrm{Z}}$ increases.

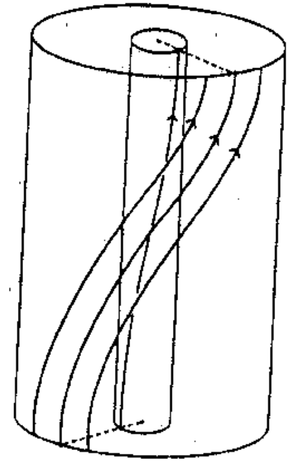

Fig. 3.5. Magnetic field lines at two radii for the uniform-twist field.

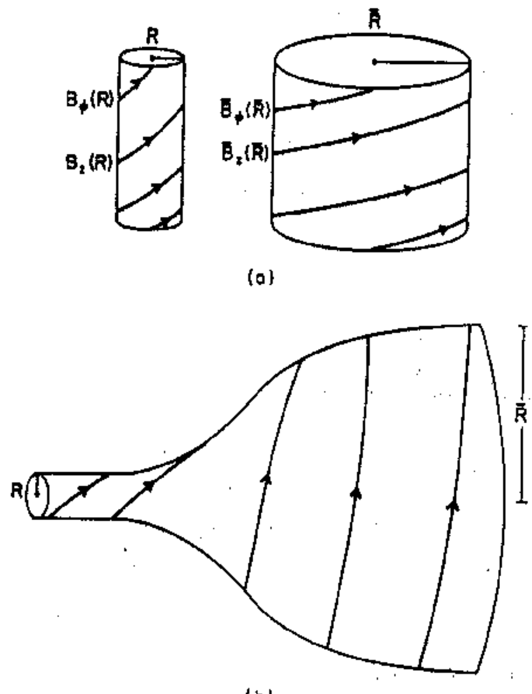

Fig. 3.6. (a) The radial expansion of a twisted flux tube from radius a to radius $\bar{a}$. (b) The concentration of azimuthal flux in the widest part of a flux tube.

Current – Free Fields (potential fields)

$$\vec{J} = 0 = \nabla \times \vec{B} / \mu$$

$\nabla^2 \vec{B} = 0$ $\quad$ $\vec{B} = \nabla \Psi$ $\quad$ Ψ scalar magnetic potential

$\nabla^2 \Psi = 0$ --- Laplace Equation

this is a very likely situation on the Sun

general solution

$$\Psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (a_{lm} r^l + b_{lm} r^{-(l+1)}) P_l^m (Cos\,\theta) e^{im\phi}$$

In term of associated Legendre Polynomial

Constants are determined by boundary conditions. Many codes have been developed. e.g. Fig 3.7.
Use observed surface magnetic fields as boundary condition.

Force-Free Fields

force balance $\quad \vec{j} \times \vec{B} = 0 \quad \vec{j} = \dfrac{\vec{\nabla} \times \vec{B}}{\mu}$

$$\boxed{\nabla \times \vec{B} = \alpha \vec{B}}$$

$\alpha = 0$, current - free

take the divergence

$(\vec{B} \cdot \nabla)\alpha = 0$ $\quad$ $\vec{B}$ lies on surface of constant α

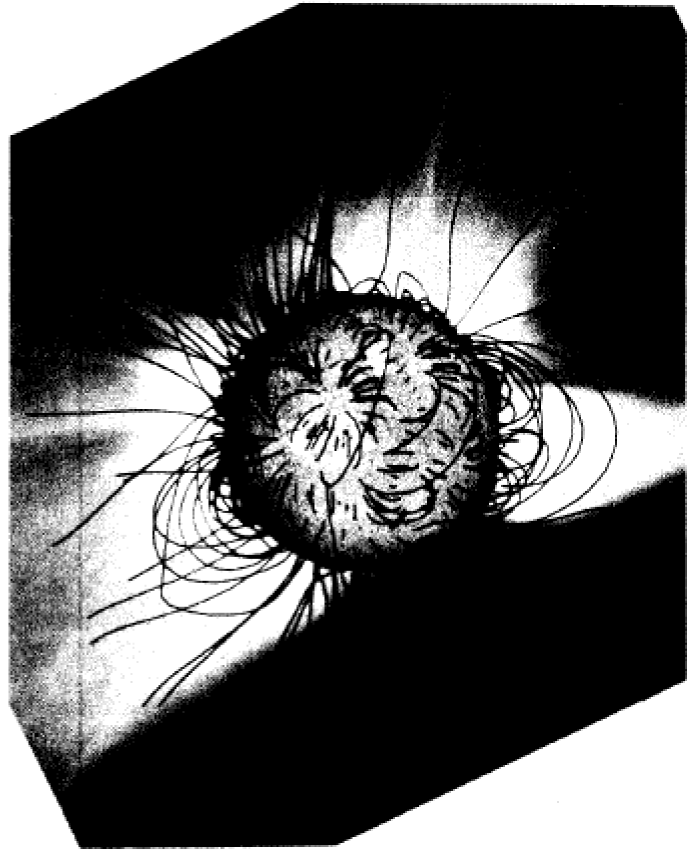

Fig. 3.7. A 'hairy ball', indicating potential magnetic field lines in the solar corona for 12 November 1966, calculated by the Altschuler–Newkirk code and superimposed on the eclipse photograph (courtesy G. Newkirk, High Altitude Observatory).

Pneuman (1976) employing a finite-difference method. Both methods yield similar solutions for the large-scale field at two solar radii. The Adams–Pneuman code possesses uniform accuracy at all heights, but, in order to treat small-scale surface features with the same accuracy, the Altschuler–Newkirk program needs to employ many more than the original nine polynomials. A much faster code has been developed by Riesebieter and Neubauer (1979), who use orthogonality relations of the spherical harmonics to determine recursion formulae for the harmonic coefficients. (The effect of a non-spherical source surface has been incorporated by Levine *et al.* (1982).) Limitations on these global calculations are the poor quality of data near the poles and the fact that variations on a time shorter than the solar rotation period cannot be studied.

Another ingenious code has been developed by Sakurai and Uchida (1977) for modelling the global field from several active regions; it models them by a series of when α is constant - - linear force free

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \alpha \vec{\nabla} \times \vec{B}$$

$$-\nabla^2 B = \alpha^2 \qquad (\nabla^2 + \alpha^2)\vec{B} = 0$$

General remarks

It can be shown that (Cowling 1976) if fields have minimum energy, must be force free. However, force-free does not mean minimum energy.

Virial theorem

$$\int_V \frac{B^2}{2\mu} dV = \int_s \frac{(xB_x + yB_y)B_z}{\mu} dxdy$$

if one knows $\vec{\mathrm{B}}$ in a surface, magnetic energy can be derived for a 3 - D space

if $\quad \vec{j} \times \vec{B} = 0 \qquad \vec{B} \equiv 0$

force - free field must be anchored down some where in surface

An axisymmetric force free poloidal field must be current free

Simple constant α force free solutions

Simplest solution $\vec{B} = (0, B_y(x), B_z(x))$

$$B_y^2 + B_z^2 = B_0^2$$

$$B_y = B_0 Sindx \qquad B_z = B_0 Cos\alpha x$$

other more complicated form

$$\begin{cases} B_x = A_1\ Coskx\ e^{-lz} \\ B_y = A_2\ Coskx\ e^{-lz} \\ B_z = B_0\ Sinkx\ e^{-lz} \end{cases} \qquad A_1 = -\frac{l}{k} B_0 \qquad A_2 = -(1-\frac{l^2}{K^2})^{1/2} B_0$$

in cylindrical polars (R, ϕ, Z)

$$\begin{cases} B_R = \frac{l}{k} B_0 J_1(KR)\, e^{-lz} \\ B_\phi = (1-\frac{l^2}{K^2})^{1/2} B_0 J_1(KR)\, e^{-lz} \\ B_z = B_0 J_0(KR)\, e^{-lz} \end{cases}$$

J: Bessel function

Non-constant α force free fields

$\vec{j} \times \vec{B} = 0, \quad \nabla \cdot \vec{B} = 0$ difficult non-linear equations

one approximation. $\vec{B}$ independent of y.

$$B_x = \frac{\partial A}{\partial z} \qquad B_y \qquad B_z = -\frac{\partial A}{\partial x} \qquad \text{A : flux function}$$

$$\nabla^2 A \frac{\partial A}{\partial x} + B_y \frac{\partial B_y}{\partial x} = 0 \qquad \frac{\partial B_y}{\partial z}\frac{\partial A}{\partial x} - \frac{\partial B_y}{\partial x}\frac{\partial A}{\partial z} = 0$$

$$\nabla^2 A \frac{\partial A}{\partial z} + B_y \frac{\partial B_y}{\partial z} = 0 \qquad \nabla^2 A + \frac{d}{dA}(\frac{1}{2}B_z^2) = 0$$

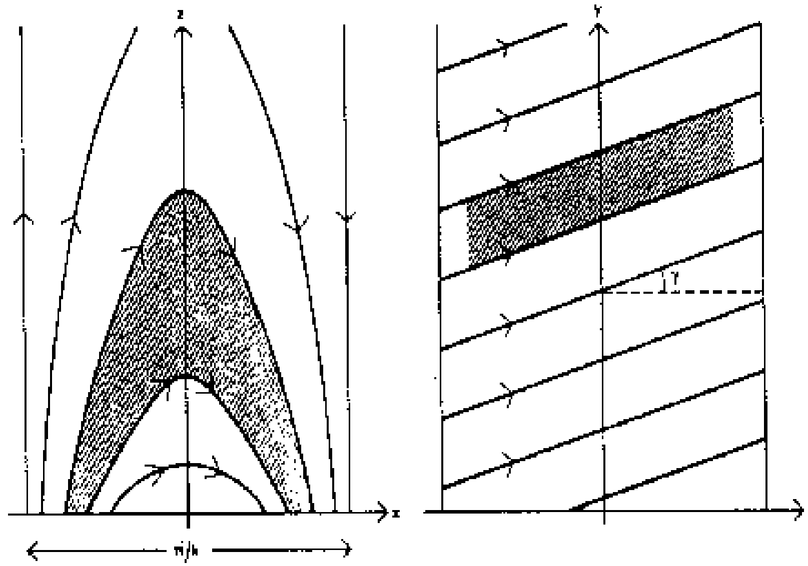

Fig. 3.8. Vertical and horizontal sections through a magnetic configuration described by Equation (3.44) with $B_0 < 0$. It may be used to model a coronal arcade. The shaded loop possesses a pressure that is enhanced at the base and therefore also at all heights (Section 3.2).

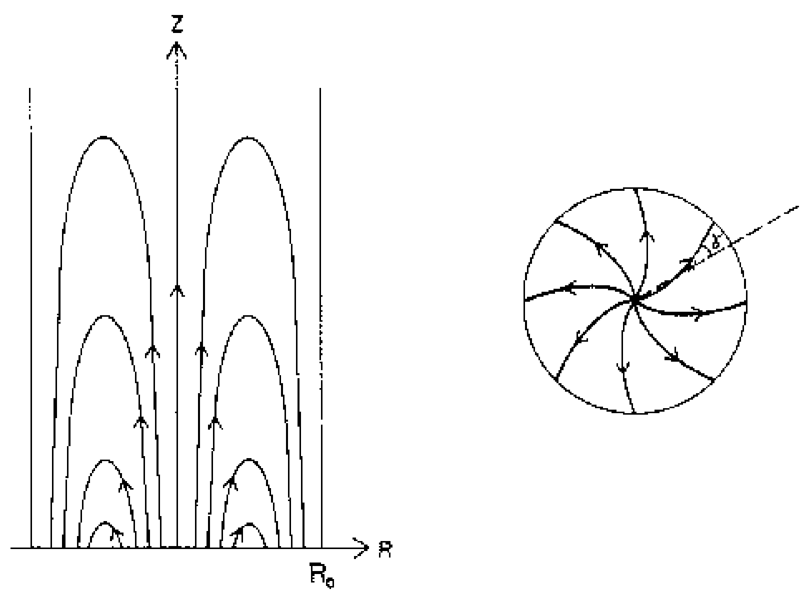

Fig. 3.9. Vertical and horizontal sections through a magnetic structure described by Equation (3.45). It may model the field above a sunspot.

in cylindrical polar system

$$(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial z^2})A + \frac{d}{dA}(\frac{1}{2}b_\phi^2) = 0$$

$$B_R = -\frac{1}{R}\frac{\partial A}{\partial z} \qquad B_\phi = \frac{b_\phi}{R} \qquad B_z = \frac{1}{R}\frac{\partial A}{\partial R}$$

Magnetic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + \eta \nabla^2 B$$

$$\nabla \times \vec{B} = \alpha \vec{B} \qquad \nabla \cdot B = 0 \qquad (\vec{B} \cdot \nabla)\alpha = 0$$

if medium is stationary $V \equiv 0$, & $\alpha = const$

then $\nabla^2 \vec{B} = -\alpha^2 \vec{B}$

$$\frac{\partial \vec{B}}{\partial t} = -\eta \alpha^2 \vec{B} \quad \to \text{ diffusion}$$

$$B = B_0\, e^{-\eta\alpha^2 t} \qquad j = j_0\, e^{-\eta\alpha^2 t}$$

For non-constant α, Sample solution:

$$B_y = B_0 Cos\phi \qquad B_z = B_0 Sin\phi \qquad \alpha = -\frac{\partial \phi}{\partial x}$$

ϕ and V are determined by

$$\frac{\partial \phi}{\partial t} - \eta \frac{\partial^2 \phi}{\partial x^2} + V_x \frac{\partial \phi}{\partial x} = 0$$

$$\eta (\frac{\partial \phi}{\partial x})^2 + \frac{\partial V_x}{\partial x} = 0$$

Homework

1. Estimate the scale heights in solar photosphere, transition region and corona.
2. A vertical magnetic flux tube should expand from photosphere to corona. If 1% of photospheric area is occupied by the magnetic fields, at what height, 100% of the surface area is occupied by magnetic fields?
3. Derive Bennett's relation, starting from fundamental magnetohydrostatics equation with cylindrical symmetry.