Biometrika Advance Access published January 19, 2016

Biometrika (2016), *pp.* 1–7
Printed in Great Britain

doi: 10.1093/biomet/asv063

Improving Holm's procedure using pairwise dependencies

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Summary

Seneta & Chen (2005) tightened the familywise error rate control of Holm's procedure by sharpening its critical values using pairwise dependencies of the p-values. In this paper we further sharpen these critical values in the case where the distribution functions of the pairwise maxima of null p-values are convex, a property shown to hold in some applications of Holm's procedure. The newer critical values are uniformly larger, providing tighter familywise error rate control than the approach of Seneta & Chen (2005), significantly so under high pairwise positive dependencies. The critical values can be further improved under exchangeable null p-values.

Some key words: Convexity; Familywise error rate; Kounias inequality; Multiple testing.

1. Introduction

Control of the familywise error rate, the probability of falsely rejecting at least one true null hypothesis, is commonly undertaken when testing multiple hypotheses. Among procedures for controlling this error rate, that of Holm (1979) is one of the most popular. Seneta & Chen (2005) attempted to improve upon Holm's procedure in situations where pairwise dependencies among the p-values can be quantified. They applied an inequality of Kounias (1968) to obtain an upper bound for the distribution function of the minimum of a set of null p-values which is tighter than that provided by Bonferroni's inequality, while modifying Holm's critical values. The modification tightens the familywise error rate control of Holm's procedure, and can be more powerful than the original step-up procedure of Hochberg (1988), as Seneta & Chen (2005) showed for some multiple testing problems associated with normally distributed test statistics with known correlations.

We propose two improved versions of Holm's step-down procedure for p-values such that the pairwise maxima of the null p-values have known convex distribution functions. The different versions depend on whether the null p-values are exchangeable, and each provides uniformly larger critical values than the method of Seneta & Chen (2005). The convexity of p-values is shown for some commonly used multivariate distributions. Numerical and simulation studies reveal that each of our proposed procedures is a better choice than the method of Seneta & Chen (2005), especially for small-scale multiple testing with high pairwise dependencies among the test statistics.

2. Seneta–Chen modified Holm procedure

Let $P_{(1)} \leqslant \cdots \leqslant P_{(n)}$ be ordered p-values available for testing n null hypotheses, with $H_{(i)}$ being the null hypothesis corresponding to $P_{(i)}$ $(i = 1, \ldots, n)$. Let each original null p-value be distributed as Un(0, 1). Holm's procedure for controlling the familywise error rate at a prespecified level α is a step-down procedure with critical values $\alpha_i = \alpha/(n-i+1)$ $(i = 1, \ldots, n)$, i.e., it rejects $H_{(i)}$ for all $i \leqslant R = \max\{i : P_{(j)} \leqslant \alpha_j = \alpha/(n-j+1),\ j \leqslant i\}$, provided the maximum exists; otherwise, it rejects none of the null hypotheses.

With $n_0 \geqslant 1$ true null hypotheses, the familywise error rate of a step-down procedure with any critical values $\alpha_1 \leqslant \cdots \leqslant \alpha_n$ satisfies

$$\text{familywise error rate} \leqslant \max_{I_{n_0} \in C_{n_0}} \text{pr}\left(\min_{j \in I_{n_0}} P_j \leqslant \alpha_{n-n_0+1}\right),$$

where I_{n_0} is the set of indices of the n_0 true null hypotheses and C_{n_0} is the collection of all such sets. Hence, the $\{\alpha_i\}$ providing familywise error rate control by this procedure at level α can be determined by finding, for each $1 \leqslant n_0 \leqslant n$, α_{n-n_0+1} such that $\max_{I_{n_0} \in C_{n_0}} \text{pr}(\min_{j \in I_{n_0}} P_j \leqslant \alpha_{n-n_0+1}) \leqslant \alpha$. The Bonferroni inequality gives $\text{pr}(\min_{j \in I_{n_0}} P_j \leqslant \alpha_{n-n_0+1}) \leqslant n_0 \alpha_{n-n_0+1}$, and when the right-hand side is bounded from above by α, this yields $\alpha_{n-n_0+1} = \alpha/n_0$ for $n_0 = 1, \ldots, n$, i.e, $\alpha_i = \alpha/(n-i+1)$ for $i = 1, \ldots, n$. These are the original critical values of Holm (1979).

Seneta & Chen (2005) sharpened these critical values by using the following inequality due to Kounias (1968) in terms of the pairwise distributions of the p-values:

$$\max_{I_{n_0} \in C_{n_0}} \text{pr}\left(\min_{j \in I_{n_0}} P_j \leqslant \alpha_{n-n_0+1}\right) \leqslant n_0 \alpha_{n-n_0+1} - (n_0 - 1)\beta_{n_0}(\alpha_{n-n_0+1}),$$

where $\beta_{n_0}(\alpha_{n-n_0+1})$ equals

$$\frac{1}{n_0 - 1} \min_{I_{n_0} \in C_{n_0}} \max_{j \in I_{n_0}} \left\{ \sum_{k \in I_{n_0},\, k \neq j} \text{pr}(P_j \leqslant \alpha_{n-n_0+1},\ P_k \leqslant \alpha_{n-n_0+1}) \right\} \quad (n_0 = 2, \ldots, n).$$

Let

$$G_{n_0}(\alpha_{n-n_0+1}) = n_0 \alpha_{n-n_0+1} - (n_0 - 1)\beta_{n_0}(\alpha_{n-n_0+1}).$$

Seneta & Chen (2005) suggested using $\alpha_{n-n_0+1} = \{\alpha + (n_0 - 1)\beta_{n_0}(\alpha/n_0)\}/n_0$ as a solution to the inequality $G_{n_0}(\alpha_{n-n_0+1}) \leqslant \alpha$, which is tighter than Bonferroni's inequality, for each $n_0 = 1, \ldots, n$; they then proposed a modified version of Holm's critical values that maintains the nondecreasing property,

$$\alpha_i = \min\left\{ \frac{\alpha + (n-i)\beta_{n-i+1}(\frac{\alpha}{n-i+1})}{n-i+1},\ \frac{\alpha}{n-i} \right\} \quad (i = 1, \ldots, n). \tag{1}$$

3. PROPOSED MODIFICATIONS OF HOLM'S PROCEDURE

We find solutions of the form $u = c\alpha/n_0$ to the inequality $G_{n_0}(u) \leqslant \alpha$ that are larger than $c = [\alpha + (n_0 - 1)\beta_{n_0}(\alpha/n_0)]/\alpha$, the Seneta–Chen solution, under the following assumption.

Assumption 1. The probability $\text{pr}\{\max(P_j, P_k) \leqslant u\}$ is convex in $u \in (0, 1)$ for all $j, k \in I_{n_0}$ such that $j \neq k$.

This assumption ensures the convexity of $\beta_{n_0}(u)$ for each fixed $n_0 \geqslant 2$, since the sum and maximum of multiple convex functions are also convex, implying the concavity of $G_{n_0}(u)$. The concavity of G_{n_0} facilitates our finding the desired c.

We propose two types of modification of Holm's procedure under Assumption 1. One imposes no additional conditions on the null p-values, while the other assumes that they are exchangeable. For the first modification, the concavity of $G_{n_0}(u)$ in $u \in (0, 1)$, along with the fact that $G_{n_0}(0) = 0$, means that $G_{n_0}(c\alpha/n_0) \leqslant cG_{n_0}(\alpha/n_0)$ for all $c \geqslant 1$; hence $c = \alpha/G_{n_0}(\alpha/n_0)$ gives us a solution to the inequality $G_{n_0}(c\alpha/n_0) \leqslant \alpha$. This value of c equals $\alpha/\{\alpha - (n_0 - 1)\beta_{n_0}(\alpha/n_0)\}$, which is clearly larger than

$\{\alpha + (n_0 - 1)\beta_{n_0}(\alpha/n_0)\}/\alpha$, as desired. Thus, our proposed solution to the inequality $G_{n_0}(\alpha_{n-n_0+1}) \leqslant \alpha$ is $\alpha_{n-n_0+1} = \{\alpha^2/n_0 G_{n_0}(\alpha/n_0)\}$, which leads to the following theorem.

THEOREM 1. *Let*

$$\tilde{\alpha}_i = \frac{\alpha^2/(n-i+1)}{G_{n-i+1}\{\alpha/(n-i+1)\}} \quad (i = 1, \ldots, n), \tag{2}$$

and define $\alpha'_i = \min(\tilde{\alpha}_i, \alpha'_{i+1})$ *for* $i = 1, \ldots, n-1$, *with* $\alpha'_n = \tilde{\alpha}_n$. *Under Assumption 1, the step-down procedure based on the critical values* $\alpha'_1, \ldots, \alpha'_n$ *provides tighter control of the familywise error rate than that based on Seneta and Chen's proposed critical values given in (1).*

Theorem 1 gives one of our proposed modifications of Holm's procedure. Although $\tilde{\alpha}_i > \alpha/(n-i+1)$, $\tilde{\alpha}_i$ may not be nondecreasing in i, and so $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ themselves cannot be used as the critical values in our proposed step-down procedure. The nondecreasing sequence of critical values $\alpha'_1, \ldots, \alpha'_n$ satisfying $\alpha'_i \geqslant \alpha/(n-i+1)$ constructed from these $\tilde{\alpha}_i$ is used to formulate our procedure. However, $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ are the critical values if the null p-values are exchangeable, since in this case $\tilde{\alpha}_i$ is nondecreasing in i, as shown in the Appendix.

Remark 1. With exchangeable null p-values having a common known distribution function H for the pairwise maxima, the step-down procedure using the critical values in (2) with $G_{n_0}(u) = n_0 u - (n_0 - 1)H(u)$ is proposed as our improved Holm's procedure, instead of that described in Theorem 1, under Assumption 1.

In fact, under exchangeability of the null p-values, we can obtain a solution of the form $c\alpha/n_0$, with $c \geqslant 1$, to the inequality $G_{n_0}(\alpha_{n-n_0+1}) \leqslant \alpha$, which is larger than what we consider in constructing the step-down procedure in Remark 1. Using a Taylor expansion of $G_{n_0}(c\alpha/n_0)$ about $c = 1$, we get

$$\begin{aligned} G_{n_0}(c\alpha/n_0) &\leqslant G_{n_0}(\alpha/n_0) + (c-1)G'_{n_0}(\alpha/n_0)\alpha/n_0 \\ &= \alpha - (n_0 - 1)H(\alpha/n_0) + (c-1)\{n_0 - (n_0 - 1)h(\alpha/n_0)\}\alpha/n_0 \quad (c \geqslant 1), \end{aligned} \tag{3}$$

where G'_{n_0} is the derivative of G_{n_0} and h the density of H. Upon equating the right-hand side of (3) to α and solving the resulting equation in c, we obtain the following solution to $G_{n_0}(\alpha_{n-n_0+1}) \leqslant \alpha$:

$$\alpha_{n-n_0+1} = \frac{\alpha}{n_0} + \frac{(n_0 - 1)H(\alpha/n_0)/n_0}{1 - (n_0 - 1)h(\alpha/n_0)/n_0}. \tag{4}$$

Since $H(u)$ is convex in $u \in (0, 1)$ and $H(0) = 0$, we have $H(u) \leqslant uh(u)$ for all $u \in (0, 1)$, and hence the solution in (4) is greater than or equal to

$$\frac{\alpha}{n_0} + \frac{(n_0 - 1)H(\alpha/n_0)/n_0}{1 - (n_0 - 1)H(\alpha/n_0)/\alpha} = \frac{\alpha}{\alpha - (n_0 - 1)H(\alpha/n_0)} \frac{\alpha}{n_0},$$

which is the solution in (2), as mentioned following Remark 1. Moreover, as we show in the Appendix, the critical values

$$\alpha^*_i = \frac{\alpha}{n-i+1} + \frac{(n-i)H(\alpha/n-i+1)/(n-i+1)}{1 - (n-i)h(\alpha/n-i+1)/(n-i+1)} \quad (i = 1, \ldots, n) \tag{5}$$

suggested by the solution in (4) are increasing in i if the following condition holds.

Condition 1. The level α for the familywise error rate control satisfies $h(\alpha) \leqslant 1$.

Thus, we have our next main result.

THEOREM 2. *Let the null p-values be exchangeable, with their pairwise maxima having common distribution function H with density h. If Condition 1 holds, the step-down procedure based on the critical values in (5) provides tighter control of the familywise error rate than that mentioned in Remark 1 under Assumption 1.*

Remark 2. We show in § 4 that Condition 1 holds for commonly chosen values of α in some multiple testing problems. If it is not satisfied, the critical values of the procedure in Theorem 2 can be reconstructed as in Theorem 1 to ensure monotonicity.

4. EXAMPLES

Suppose that the p-values are generated from continuous test statistics $X_1, \ldots, X_n$ through their common marginal null distribution function F with density f. Let $P_i = 1 - F(X_i)$; that is, we have right-tailed tests based on the X_i. Then, with $x(t) = F^{-1}(1-t)$, we have $H_{jk}(t) = \text{pr}\{\max(P_j, P_k) \leqslant t\} = \text{pr}\{X_j \geqslant x(t),\ X_k \geqslant x(t)\}$, and so the density $h_{jk}(t) = \mathrm{d}H_{jk}(t)/\mathrm{d}t$ of $\max(P_j, P_k)$ is given by $-\text{pr}\{X_j \geqslant x(t) \mid X_k = x(t)\} f\{x(t)\} x'(t) - \text{pr}\{X_k \geqslant x(t) \mid X_j = x(t)\} f\{x(t)\} x'(t)$. Since $f\{x(t)\}x'(t) = -1$, it can be seen that $h_{jk}(t) = \text{pr}\{X_j \geqslant x(t) \mid X_k = x(t)\} + \text{pr}\{X_k \geqslant x(t) \mid X_j = x(t)\}$. Therefore, the underlying convexity condition holds for such p-values when the X_i have a multivariate distribution that satisfies the following condition.

Property 1. The conditional probability $\text{pr}(X_j \geqslant x \mid X_k = x)$ is decreasing in x under the joint null distribution of (X_j, X_k), for any $1 \leqslant j < k \leqslant n$.

Remark 3. If the p-values correspond to left-tailed tests based on the X_i, that is, if $P_i = F(X_i)$, then Assumption 1 still holds under Property 1.

The following property ensures that Condition 1 holds for exchangeable null test statistics with common density of the pairwise maxima of the null p-values given by

$$h(t) = 2\,\text{pr}\{X_1 \geqslant F^{-1}(1-t) \mid X_2 = F^{-1}(1-t)\},$$

where (X_1, X_2) is any pair of these statistics.

Property 2. There exists an $\alpha_0 \in (0, 1)$ such that $h(\alpha) \leqslant 1$ for all $0 < \alpha \leqslant \alpha_0$.

We now give examples of multivariate distributions arising in some standard multiple testing problems which exhibit Property 1 for Assumption 1 to hold and Property 2, with some α_0 for the typically chosen values of α, so that Condition 1 is satisfied in the exchangeable case.

Consider using t-test statistics $T_i = \nu^{1/2} X_i / Y^{1/2}$ $(i = 1, \ldots, n)$ to test $\mu_i = 0$ simultaneously for $i = 1, \ldots, n$. Here the X_i are jointly distributed as multivariate normal with $E(X_i) = \mu_i$, $\text{var}(X_i) = \sigma^2$ for some unknown σ^2, and $\text{corr}(X_i, X_j) = \rho_{ij}$, and Y is distributed independently of the X_i as $\sigma^2 \chi^2_\nu$.

The pair (T_j, T_k) has the central bivariate t distribution with ν degrees of freedom and associated correlation ρ_{jk}, so $(\nu+1)^{1/2}(T_j - \rho_{jk} x)/\{(\nu + x^2)(1 - \rho^2_{jk})\}^{1/2} \sim t_{\nu+1}$, conditional on $T_k = x$ (Kotz & Nadarajah, 2004). Therefore

$$\text{pr}(T_j \geqslant x \mid T_k = x) = 1 - \Psi_{\nu+1}\left[x\left\{\frac{(\nu+1)(1-\rho_{jk})}{(\nu+x^2)(1+\rho_{jk})}\right\}^{1/2}\right], \tag{6}$$

where $\Psi_{\nu+1}$ is the cumulative distribution function of $t_{\nu+1}$, the central t distribution with $\nu+1$ degrees of freedom. The conditional probability in (6) is decreasing in x. Hence, for the multivariate t distribution that arises in testing $\mu_i = 0$ against $\mu_i > 0$ simultaneously for $i = 1, \ldots, n$, Property 1 holds. In the exchangeable case with $\rho_{jk} = \rho$, Property 2 also holds with $\alpha_0 = 1/2$, since here $x = \Psi_\nu^{-1}(1-\alpha) \geqslant 0$, with $0 < \alpha \leqslant 1/2$, making the conditional probability in (6) less than or equal to $1/2$.

For the absolute values, we see that

$$\text{pr}\big(|T_j| \geqslant x \,\big|\, |T_k| = x\big) = 2 - \Psi_{\nu+1}\left[x\left\{\frac{(\nu+1)(1-\rho_{jk})}{(\nu+x^2)(1+\rho_{jk})}\right\}^{1/2}\right] - \Psi_{\nu+1}\left[x\left\{\frac{(\nu+1)(1+\rho_{jk})}{(\nu+x^2)(1-\rho_{jk})}\right\}^{1/2}\right], \tag{7}$$

which is decreasing in $x \geqslant 0$. Hence, for the absolute value multivariate t distribution that arises in the context of testing $\mu_i = 0$ against $\mu_i \neq 0$ simultaneously for $i = 1, \ldots, n$, Property 1 holds. If $x \geqslant 1$, the right-hand side of (7) is less than or equal to

$$2 - \Psi_{\nu+1}\left\{\left(\frac{1-\rho_{jk}}{1+\rho_{jk}}\right)^{1/2}\right\} - \Psi_{\nu+1}\left\{\left(\frac{1+\rho_{jk}}{1-\rho_{jk}}\right)^{1/2}\right\},$$

which is increasing in $|\rho_{jk}|$, as will be shown in the Appendix, and hence it is less than or equal to $2 - \Psi_{\nu+1}(\infty) - \Psi_{\nu+1}(0) = 1/2$, as required for Property 2 to hold in the exchangeable case. Here $x = \Psi_\nu^{-1}(1-\alpha/2)$, and so Property 2 is seen to hold if $\alpha \leqslant 2\{1-\Psi_\nu(1)\}$. Since $1 - \Psi_\nu(1) \geqslant 1 - \Phi(1)$, where Φ is the cumulative distribution function of $N(0, 1)$, a suitable α_0 for Property 2 is $2\{1-\Phi(1)\} \approx 0{\cdot}3173$.

Remark 4. The ranges of α values that guarantee Property 2 for the above distributions contain values of α typically used in practice, but the range can be widened for any fixed ν. Simply by taking $\nu \to \infty$, the above conclusions in terms of Properties 1 and 2 can also be drawn for multivariate and absolute value multivariate normal distributions arising in the contexts of the same testing problems with a known σ^2.

5. Numerical investigations

We conducted simulation studies to investigate how our proposed step-down procedures compare with the method of Seneta & Chen (2005) in the exchangeable case, in terms of both control of the familywise error rate and the average power, i.e., the expected proportion of rejected false null hypotheses, for two-sided tests using multivariate normal test statistics. Since the step-down and step-up methods of Dunnett & Tamhane (1992) would be ideal in this context, as they are designed to fully use the underlying dependence, we have included them as benchmarks. We considered different values of n, but were able to include Dunnett and Tamhane's methods only for $n \leqslant 16$, since beyond this value their critical values become exceedingly difficult to compute, as seen from the R (R Development Core Team, 2016) package DunnettTests. Figure 1, which shows the relative performances of these four methods for $n = 16$, is representative of how our methods perform for relatively few tests.

Figure 1 shows that our method 1, described in Remark 1, always outperforms Seneta and Chen's method, especially when ρ is high. Moreover, its performance tends to that of Dunnett and Tamhane's step-down or step-up method when ρ is close to 1. In Fig. 1 we did not include our method 2 as stated in Theorem 2, because it provides only a marginal improvement upon the results of method 1, unless ρ is high.

The power advantage of our method 1 over Seneta and Chen's method is preserved regardless of n, as observed from our simulations focusing on the comparison between these two methods for some cases of $n > 16$, up to $n = 100$. Simulation results for $n = 100$ are presented in the Supplementary Material.

6. Concluding remarks

The primary goal of this paper is to improve stepwise, step-down or step-up, procedures controlling the familywise error rate based only on marginal p-values under some common distributional models by making use of dependencies among the test statistics. We have achieved this goal for Holm's step-down procedure by further improving upon the work of Seneta & Chen (2005) with little or no additional computation. Hochberg's procedure is the step-up analogue of Holm's procedure and is a valid familywise error rate-controlling procedure under the distributional settings in § 4 (Sarkar & Chang, 1997). Unfortunately, the idea of improving Hochberg's procedure by using the step-up analogues of the proposed step-down procedures does not work, because such analogues no longer control the familywise error rate.

Dunnett and Tamhane's step-down and step-up methods are preferable when they can be implemented, since they are based on exact critical values and hence are more powerful than our methods. However, they may be not implementable, because not all pairwise correlations may be known or estimable, or n may be so large that the critical values are difficult to compute.

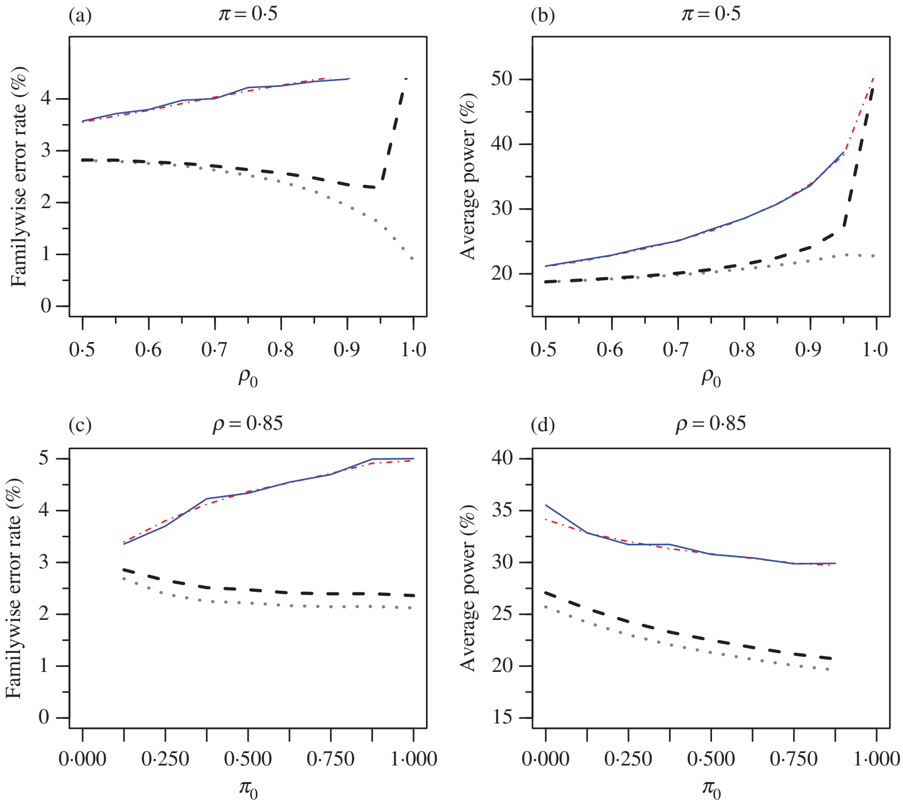

Fig. 1. Comparison of four methods for testing $\mu_i = 0$ against $\mu_i \neq 0$ simultaneously for $i = 1, \ldots, 16$, at level $\alpha = 0{\cdot}05$ based on multivariate normal test statistics with common correlation ρ: our method 1 (dashed), Seneta and Chen's method (dotted), and Dunnett and Tamhane's step-down (dot-dash) and step-up (solid) methods. The mean is chosen to be 2 when a null hypothesis is false, and π_0 is the proportion of true null hypotheses. One million independent replications were used in all simulations.

The examples in § 4 of bivariate distributions having the desired convexity are commonly seen in applications of Holm's procedure. Many other bivariate distributions share the same property, such as certain families of location and scale mixture distributions containing bivariate gamma and F distributions (Sarkar & Chang, 1997) and families of distributions corresponding to Archimedean copulas (Nelsen, 2007). The convexity results in § 4 can be generalized from bivariate to multivariate distributions, which are important in their own right, since they are not available in the literature as far as we know.

Estimating G or H in general is beyond the scope of this paper, but in the Supplementary Material we demonstrate how one could do so in practice, and we check the desired convexity and other properties before calculating the $\tilde{\alpha}_i$ or α_i^*.

Acknowledgement

The authors greatly appreciate valuable comments from two referees and the editor. This work was supported by the U.S. National Science Foundation.

Supplementary material

Supplementary material available at *Biometrika* online includes additional examples of distributions satisfying Assumption 1, further simulation results, and a demonstration of how to estimate H and check its convexity from data.

Appendix

Proofs of some results in §§ 3 and 4

Assertion 1. The $\tilde{\alpha}_i$ in (2) is increasing in i under Assumption 1 if the null p-values are exchangeable.

Proof. The result follows from the fact that $\alpha/\{mG_m(\alpha/m)\}$, where $G_m(u) = mu - (m-1)H(u)$, is decreasing in $m = 1, \ldots, n$. This is because: (i) for fixed u, $G_m(u)$ is increasing in m since $G_{m+1}(u) - G_m(u) = u - H(u) \geqslant 0$; and (ii) for fixed m, $G_m(u)/u$ is decreasing in $u \in (0, 1)$ because $G_m(u)$ is concave in $u \in (0, 1)$ and $G_m(0) = 0$. □

Assertion 2. The α_i^* in (5) is increasing in i if Condition 1 holds.

Proof. The function $\alpha u + [(1-u)H(\alpha u)/\{1-(1-u)h(\alpha u)\}]$ is increasing in $u \in (0, 1)$ since its derivative, $\{1-(1-u)h(\alpha u)\}^{-2}[\alpha\{1-h(\alpha u)\} + \alpha u h(\alpha u) - H(\alpha u) + \alpha(1-u)^2 h'(\alpha u)H(\alpha u)]$, is nonnegative. This is because: (i) $h'(\alpha u) \geqslant 0$ since $H(u)$ is convex in $u \in (0, 1)$; (ii) $H(\alpha u) \leqslant \alpha u h(\alpha u)$ since $H(u) \geqslant 0$ is convex in $u \in (0, 1)$ and $H(0) = 0$; and (iii) $h(\alpha u) \leqslant h(\alpha) \leqslant 1$ since $h(u)$ is increasing in $u \in (0, 1)$. Thus the result follows. □

Assertion 3. The function

$$\Psi_{\nu+1}\left\{\left(\frac{1+\rho}{1-\rho}\right)^{1/2}\right\} + \Psi_{\nu+1}\left\{\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\right\} \tag{A1}$$

is decreasing in $|\rho| \in [0, 1)$.

Proof. Without loss of generality, assume that $\rho \geqslant 0$; then we see that the first derivative of the function in (A1) with respect to ρ is $[\Gamma\{(\nu+2)/2\}/\Gamma\{(\nu+1)/2\}]\{(\nu+1)\pi\}^{-1/2}(1-\rho^2)^{-1}$ times

$$\left(\frac{1+\rho}{1-\rho}\right)^{1/2}\left\{1+\frac{1+\rho}{(\nu+1)(1-\rho)}\right\}^{-(\nu+2)/2} - \left(\frac{1-\rho}{1+\rho}\right)^{1/2}\left\{1+\frac{1-\rho}{(\nu+1)(1+\rho)}\right\}^{-(\nu+2)/2}.$$

This is nonpositive, as can be checked from the result that $(\nu+2)\log\{(\nu+2+\nu\rho)/(\nu+2-\nu\rho)\} - \nu\log\{(1+\rho)/(1-\rho)\}$ is decreasing in $\rho \in [0, 1)$ and hence $\leqslant 0$. Thus the result is proved. □

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[*Received September* 2014. *Revised October* 2015]