29

The Hat Problem And Some Variations

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Abstract: The hat problem arose in the context of computational complexity. What started as a puzzle, the problem is found to have connections with coding theory and has reached the research frontier of Mathematics, Statistics and Computer Science. In this article, some variations of the hat problem are presented along with their solutions. An application is indicated.

Keywords and phrases: Computational complexity, optimization, strategy, winning probability

29.1 Introduction

The 'Hat Problem' has been making rounds in Mathematics, Statistics and Computer Science departments for quite some time. The problem straddles all these disciplines. For a technical description of the problem, see Buhler (2002). For a popular article on the problem, see Robinson (2001). The original hat problem appeared in Todd Ebert's thesis in Computer Science in connection with complexity theory. A version of the problem can be found in Ebert and Vollmer (2000). It is interesting to note that how this purely recreational problem has come to the research frontier with many problems yet unsolved. A simple version of the problem involves three participants and two colors. Three friends (Brenda, Glenda, Miranda say) are planning to participate in a game-show in which a big prize can be won collectively. The host of the game-show places a hat on each of the participants. The hat is either black or red. The choice of the colors is random and the placements are independent. What this means is that all the eight configurations of hats, listed in Table 29.1, on the heads of the participants are equally likely. Each participant can see the colors of the hats of her team mates but has no idea what the color of her hat is. The host asks each of the team mates separately what the color of her hat is. A team mate can guess the color of her hat, red (R) or black (B), or pass (P).

Table 29.1: List of all configurations (3 people and 2 colors)

Configuration	Brenda	Glenda	Miranda	Probability
1	Red	Red	Red	1/8
2	Red	Red	Black	1/8
3	Red	Black	Red	1/8
4	Red	Black	Black	1/8
5	Black	Red	Red	1/8
6	Black	Red	Black	1/8
7	Black	Black	Red	1/8
8	Black	Black	Black	1/8

The other members of the team will not know what her response is. They can win the prize collectively if at least one of the team mates guesses the color and whoever guesses must be right. For example, if every one passes, they can not win the prize. If only one guesses the color and the others pass, the one who guesses must be right in order to win the prize. If two guess the color and the other passes, both the guesses must be right in order to win the prize. If all three guess, all guesses must be right in order to win the prize. Before participating in the game-show, the team mates can get into a huddle and formulate a strategy of responses. The basic question is: what is the best strategy of responses so as to maximize the chances of winning the prize.

Let us analyze a couple of strategies. One simple strategy is that every one guesses. If this is the case, the chances of winning the prize are 1/8. Another strategy is that one elects to guess and the others decide to pass. Winning the prize now solely depends on the one who elects to guess. The chances of winning the prize are then 50 per cent. Is there a strategy which will improve the chances of winning the prize to more than 50 per cent? It is not obvious. In order to improve the chances of winning, it seems that only one of the team mates should guess the color and the others to pass, but who guesses and who passes should be based on what actually they see on the stage. Consider the following strategy.

Instructions to Brenda

a. If the colors of hats of your team mates are both red, say that the color of your hat is black.

b. If the colors of hats of your team mates are both black, say that the color of your hat is red.

c. If the colors of hats of your team mates are different, pass.

The same instructions are given to Glenda and Miranda.

Table 29.2: Actual configurations along with responses and outcomes (3 people and 2 colors)

Actual Configuration			Responses			Outcome
Brenda	Glenda	Miranda	Brenda	Glenda	Miranda	
R	R	R	B	B	B	Loss
R	R	B	P	P	B	Win
R	B	R	P	B	P	Win
R	B	B	R	P	P	Win
B	R	R	B	P	P	Win
B	R	B	P	R	P	Win
B	B	R	P	P	R	Win
B	B	B	R	R	R	Loss

Under this strategy, let us evaluate the chances of winning the prize. The details are provided in Table 29.2.

It is now clear that the chances of winning the prize under this strategy are 75 per cent. One can also show that there is no way one can improve the chances of winning to more than 75 per cent. For future reference, let us call this strategy as *Strategy O*.

There are two main objectives we want to pursue in this article. One is to extend the hat problem to the case of three colors and three team mates. We will present an optimal strategy the team mates can pursue which will maximize the chances of winning the prize collectively. The other is to stay within the environment of two colors and three team mates but the eight configurations that are possible are not equally likely. More precisely, we will be given a probability distribution on the set of all hat configurations and the task is to determine an optimal strategy which will maximize the probability of winning the prize. We will also present some other variations of the hat problem. Finally, we will end the paper with a number of open questions.

29.2 Hamming Codes

The hat problem has a close connection with 'Covering Codes'. In this section, the connection is explained in a rudimentary fashion.

Covering and packing are two of the most intriguing problems in Mathematics useful in Engineering. A *packing problem* in the traditional Euclidean space is to ask for the maximal number of identical non-intersecting spheres in a large volume. As an example, suppose we have a box with dimensions 1 meter $\times$ 1 meter $\times$ 1 meter. We want to pack the box with identical balls of radius 10 centimeters. In what way should we pack the box so as to accommodate maximum number of balls? On the other hand, a *covering problem* in an Euclidean space asks for the minimal number of identical spheres to cover a specified volume.

A discrete analogue of the covering problem involves the so-called *Hamming Space*. For a fixed positive integer n, it is the set of all n-tuples where each component in any n-tuple is either zero or one. The elements of the Hamming space are called *points*. Any non-empty subset of the Hamming space is called a *code* and its elements are called *codewords*. The *Hamming distance* between any two points is the number of components at which the points differ. The Hamming distance is a non-negative integer from zero to n. The *minimum distance* of a code is the smallest of the pairwise distances between its codewords. Let x be a point in the Hamming space and $r > 0$. A sphere of radius r with center at x in the Hamming space consists of all points within distance r from the center x.

Covering problem: Given n and r, what is the smallest number of spheres of radius r so that every point in the Hamming space belong to at least one of the spheres?

Example. Suppose $n = 3$ and $r = 1$.

Hamming Space: 000, 001, 010, 011, 100, 101, 110, 111

S(000, 1) = Sphere with center at 000 and radius one = 000, 001, 010, 100

S(111, 1) = Sphere with center at 111 and radius one = 111, 110, 101, 011

Every point in the Hamming space belongs to one of these two spheres. In other words, these two spheres cover the whole space. This covering is minimal.

Any such minimal covering gives rise to an optimal strategy in the hat problem. Identify $0 = R$ and $1 = B$. Let L be the set of centers of the spheres and W its complement. In this example, $L = 000, 111$ and $W = 001, 010, 100, 110, 101, 011$. A strategy S now can be developed such that for this strategy the set of losing configurations is L and the set of winning configurations is W. We begin with instructions to the team mates that make up the strategy S. The team mates Brenda, Glenda, and Miranda are ordered in the order they are mentioned and instructions to them proceed in that order. To begin with, they should be appraised with the notation $0 = R$ and $1 = B$, and also with the sets L and W.

Instructions to Brenda

Suppose you see 00. (This means that Brenda sees red hats on both Glenda and Miranda.) If there is a unique $u \in \{0, 1\}$ such that $u00 \in W$, say that the color of your hat is u. Otherwise, pass. Here, u is unique and in fact, $u = 1$.

Suppose you see 01. (This means that Brenda sees red hat on Glenda and black hat on Miranda.) If there is a unique $u \in \{0, 1\}$ such that $u01 \in W$, then say that the color of your hat is u. Otherwise, pass. Here, u is not unique. As a matter of fact, 001 and 101 both belong to W. In this case, you should pass.

Suppose you see 10. (This means that Brenda sees black hat on Glenda and red hat on Miranda.) If there is a unique $u \in \{0, 1\}$ such that $u10 \in W$, then say that the color of your hat is u. Otherwise, pass. Here, u is not unique. As a matter of fact, 010 and 110 both belong to W. In this case, you should pass.

Suppose you see 11. (This means that Brenda sees black hats on both Glenda and Miranda.) If there is a unique $u \in \{0, 1\}$ such that $u11 \in W$, then say that the color of your hat is u. Otherwise, pass. Here u is unique. As a matter of fact, $u = 0$.

Instructions to Glenda

Suppose you see 00. (This means that Glenda sees red hats on both Brenda and Miranda.) If there is a unique $u \in \{0, 1\}$ such that $0u0 \in W$, say that the color of your hat is u. Otherwise, pass. Here, u is unique and in fact, $u = 1$.

Suppose you see 01. (This means that Glenda sees red hat on Brenda and black hat on Miranda.) If there is a unique $u \in \{0, 1\}$ such that $0u1 \in W$, then say that the color of your hat is u. Otherwise, pass. Here, u is not unique. As a matter of fact, 001 and 011 both belong to W. In this case, you should pass.

Suppose you see 10. (This means that Glenda sees black hat on Brenda and red hat on Miranda.) If there is a unique $u \in \{0, 1\}$ such that $1u0 \in W$, then say that the color of your hat is u. Otherwise, pass. Here, u is not unique. As a matter of fact, 100 and 110 both belong to W. In this case, you should pass.

Suppose you see 11. (This means that Glenda sees black hats on both Brenda and Miranda.) If there is a unique $u \in \{0, 1\}$ such that $1u1 \in W$, then say that the color of your hat is u. Otherwise, pass. Here, u is unique. As a matter of fact, $u = 0$.

By now, the tone of instructions should be clear. Instructions to Miranda follow in the same tone.

In the general case of 2 colors and n participants, we look at the corresponding Hamming space and a minimal cover. An optimal strategy is built based on the minimal cover in the same way as outlined above. For a connection between the hat problem and minimal covers, see Lenstra and Seroussi (2004). For a comprehensive discussion of Hamming space and covers, see Cohen *et al.* (1997).

29.3 Three Team Mates and Three Colors

We now consider the case of three team mates and three colors. Each of the team mates is fitted with a hat, which is red (R), black (B), or green (G), by the host. All the 27 configurations of hats are equally likely. Each participant can see the color of the hat each of her team mates has but can not see the color of her own hat. Each participant is required to guess the color of her hat or pass. In order to win the prize collectively, at least one team mate should guess the color of her hat and whoever guesses must be right. What is the best strategy that will maximize the probability of winning the prize?

Let us formulate the problem mathematically. Brenda can see the colors of the hats of her team mates. What she sees is: RR, RB, BR, BB, RG, GR, GG, BG, or GB on Glenda and Miranda, respectively. She needs to respond: R, B, G, or P (Pass). Formally, we can introduce a map from the set of all possible hat configurations she sees on her team mates to the set of all possible responses. Thus, an *instruction* is a map f described by,

$$\{RR, RB, BR, BB, RG, GR, GG, BG, GB\} \xrightarrow{f} \{R, B, G, P\}.$$

Let $\mathbf{F}$ be the collection of all instructions. The cardinality of the set $\mathbf{F}$ is $4^9 =$ 262,144. A *strategy* is a triplet $S = (f_1, f_2, f_3)$, where each f_i is a member of $\mathbf{F}$. Using the strategy S means that Brenda follows the instruction f_1, Glenda f_2, and Miranda f_3. Let S be the collection of all strategies. The cardinality of S is $4^{27} \approx 1.8 * 10^{16}$. For any given strategy, one can work out the probability of winning the prize. A complete enumeration of all strategies along with winning probability using a computer in order to find an optimal strategy is not feasible.

We restrict ourselves to symmetric strategies. A strategy $S = (f_1, f_2, f_3)$ is said to be symmetric if $f_1 = f_2 = f_3$. This means that all participants follow the same instructions. The total number of symmetric strategies is $262,144$. This number is manageable by a computer. We have written a program which enumerates all symmetric strategies and computes the corresponding winning probabilities. We have identified optimal strategies from the list. There are several. A careful scrutiny of the optimal strategies led us to synthesize verbally what the instructions should be.

Designate one of the colors as 'primary' and another color as 'secondary.' For example, we may take red as primary and black as secondary. The instructions to the participants are centered on these designations.

Instructions to Brenda

1. If both the colors you see are primary, say that the color of your hat is the secondary color.

Table 29.3: List of all configurations along with responses under the symmetric strategy S (next page) and outcomes (3 people and 3 colors)

Actual Configuration			Responses			Outcome
Brenda	Glenda	Miranda	Brenda	Glenda	Miranda	
R	R	R	B	B	B	Loss
R	R	B	P	P	B	Win
R	B	R	P	B	P	Win
R	B	B	R	P	P	Win
B	R	R	B	P	P	Win
B	R	B	P	R	P	Win
B	B	R	P	P	R	Win
B	B	B	R	R	R	Loss
R	R	G	P	P	B	Loss
R	G	R	P	B	P	Loss
R	G	G	R	P	P	Win
G	R	R	B	P	P	Loss
G	R	G	P	R	P	Win
G	G	R	P	P	R	Win
G	G	G	R	R	R	Loss
B	B	G	R	R	R	Loss
B	G	B	R	R	R	Loss
B	G	G	R	R	R	Loss
G	G	B	R	R	R	Loss
G	B	G	R	R	R	Loss
G	B	B	R	R	R	Loss
R	B	G	R	P	P	Win
R	G	B	R	P	P	Win
B	R	G	P	R	P	Win
B	G	R	P	P	R	Win
G	R	B	P	R	P	Win
G	B	R	P	P	R	Win

2. If only one of the colors you see is primary, pass.

3. If none of the colors you see is primary, say that the color of your hat is the primary color.

If the primary color is red and the secondary is black, mathematically, instructions to Brenda can be spelled out as follows.

$$\begin{gathered}\text{f(RR)} = \text{B;}\\ \text{f(RB)} = \text{P; f(BR)} = \text{P; f(RG)} = \text{P; f(GR)} = \text{P; f(BB)} = \text{R;}\\ \text{f(BG)} = \text{R; f(GB)} = \text{R; f(GG)} = \text{R.}\end{gathered}$$

The same instructions are given to Glenda and Miranda. If they adopt this symmetric strategy $S = (f, f, f)$, the chances of winning the prize are 15/27. In Table 29.3 we outline all possible hat configurations and responses following the optimal symmetric strategy described above. In 15 cases out of 27, the team mates can win the prize. This is an optimal strategy among all symmetric strategies. In Section 29.5, we will show that this symmetric strategy is indeed optimal among all strategies.

29.4 Three Team Mates and m Colors

The problem outlined in Section 29.3 can be generalized to the case of $m(\geq 3)$ colors. The number of participants remains the same. Each participant is fitted with a hat whose color is one of the m colors given. Let $C_1, C_2, ..., C_m$ be the colors that are used in the game. The total number of configurations of hats is m^3. As in Section 29.3, we confine our attention to symmetric strategies $S = (f, f, f)$, where f is any *instruction*, i.e., f is a map from the set

$$\{(x, y)\,;x, y \in \{C_1, C_2, ..., C_m\}\}$$

into the set

$$\{C_1, C_2, ..., C_m, P\},$$

where the symbol P stands for 'Pass.' The vector (x, y) stands for the colors of the hats any participant will see on her team mates. When the host asks a participant about the color of her hat, she needs to respond $C_1, C_2, ..., C_m$, or P. An optimal strategy uses the following instruction f for each participant.

To begin with, declare one of the colors as 'primary' and one of the remaining colors as 'secondary.'

Table 29.4: List of all configurations of hats and winning ones (3 people and m colors)

Configurations	Cardinality	No. Winning Configurations
(C_1, C_1, C_1)	1	0
(C_1, C_1, C_j), $j = 2, 3, ..., m$	$m-1$	1
(C_1, C_j, C_1), $j = 2, 3, ..., m$	$m-1$	1
(C_j, C_1, C_1), $j = 2, 3, ..., m$	$m-1$	1
(C_1, C_i, C_j), $i, j = 2, 3, ..., m$	$(m-1)^2$	$(m-1)^2$
(C_i, C_1, C_j), $i, j = 2, 3, ..., m$	$(m-1)^2$	$(m-1)^2$
(C_i, C_j, C_1), $i, j = 2, 3, ..., m$	$(m-1)^2$	$(m-1)^2$
(C_i, C_j, C_k), $i, j, k = 2, 3, ..., m$	$(m-1)^3$	0

Instructions (f) to any participant

1. If the colors of the hats of your team mates are both primary, you should say that the color of your hat is secondary color.

2. If only one of the colors of the hats of your team mates is primary, you should pass.

3. If none of the colors of the hats of your team mates is primary, you should say that the color your hat is the primary color.

Let us calculate the probability of winning the prize under the strategy $S = (f, f, f)$, where f is the instruction described above. For simplicity, let us declare that C_1 is the primary color and C_2 the secondary. We will make a complete list of all configurations of hats and then count how many of these configurations lead to winning the prize. To facilitate the calculations, form eight subsets of the set of all hat configurations based on the number of times the primary color C_1 is present in the configurations. The entire set of configurations is given in Table 29.4.

An explanation is in order on the above table. As an example, look at the hat configuration (C_1, C_i, C_j) for some $i, j = 2, 3, \ldots, m$. Under the instructions f outlined above, Brenda's response would be C_1, in which case she is right, and Glenda and Miranda would pass. Thus (C_1, C_i, C_j) would be a winning configuration under the strategy $S = (f, f, f)$. The total number of such hat configurations is $(m-1)^2$, and as we have just observed, each one of them is a winning configuration. In totality, the team will win the prize in $3(m-1)^2 + 3$ cases out of m^3 possible configurations. Hence the probability of winning the prize under the strategy S is given by

$$\frac{3(m-1)^2 + 3}{m^3}.$$

Let us contrast this strategy with the simple strategy, in which one of the participants chooses to guess the color of her hat while others choose to pass. Under this simple strategy, the probability of winning the prize is $\frac{1}{m}$. This probability is certainly less than $\frac{3(m-1)^2+3}{m^3}$.

We do not know that the strategy S, which is optimal in the set of all symmetric strategies, is optimal in the set of all strategies. However, the winning probability for S is very close to the upper bound, which will be discussed in the next section.

29.5 An Upper Bound for the Winning Probability

Let us consider the hat problem with q colors $C_1, C_2, \ldots, C_q$ and $n(\geq 3)$ participants. The participants are numbered serially from 1 to n. The *modus operandi* is similar to the basic hat problem. Each participant will be seeing the hats of the remaining $(n-1)$ participants. Her response is $C_1, C_2, \ldots, C_q$, or P(Pass). The set of all hat configurations is $\Sigma = \{C_1, C_2, \ldots, C_q\}^n$. Let $\mathbf{C} = \{C_1, C_2, \ldots, C_q, P\}$. An *instruction* to Participant No. i is a map f_i from the set

$$\{x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n : x_i \in \{C_1, C_2, \ldots, C_q\} \text{ for all } i\}$$

into the set $\mathbf{C}$. The entity $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$ stands as a generic symbol for the colors of the hats Participant No.i would see on her team mates and $f_i(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n)$ is the response to the query what the color of her hat is. A *strategy* $S = (f_1, f_2, \ldots, f_n)$ is an n-tuple, where f_i is the instruction that Participant No. i follows, $i = 1, 2, \ldots, n$. For a given strategy S, we can check whether or not a configuration of hats is winning. Let W_S denote the set of all winning configurations under the strategy S and L_S losing configurations. Obviously, $\#W_S + \#L_S = q^n$. The objective is to find a strategy S for which $\#W_S$ is maximum, or equivalently, $\#L_S$ is minimum.

We will now work out an upper bound for $\#W_S$. For each $i = 1, 2, \ldots, n$, let

$$Q_i = \{x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_n \in \Sigma; f_i\left(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n\right) \neq P\}.$$

Note that $Q_i = \phi$, the null set, if and only if $f_i \equiv P$, i.e., as per the instruction f_i, Participant No. i passes all the time. It is now clear that Q_i is a multiple of q. Let $\#Q_i = q * t_i$, where t_i is a non-negative integer. Take any $x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_n$ in Q_i. Then, $x_1 x_2 \cdots x_{i-1} C_j x_{i+1} \cdots x_n \in Q_i$ for all $j = 1, 2, \ldots, q$. Of these q configurations, in only one configuration the guess by Participant No. i will be correct. Consequently, in t_i configurations from Q_i, the guesses by Participant No. i will be correct and in the remaining $(q-1)t_i$ configurations the guesses will be incorrect.

Let us interpret and understand all these entities in the context of the hat problem with two colors (R and B) and 3 participants. Suppose the instructions f_1, f_2 and f_3 to Brenda, Glenda and Miranda, respectively, are:

Brenda	Glenda	Miranda
$f_1(RR) =$ P	$f_2(RR) =$ B	$f_3(RR) =$ R
$f_1(RB) =$ P	$f_2(RB) =$ P	$f_3(RB) =$ R
$f_1(BR) =$ P	$f_2(BR) =$ P	$f_3(BR) =$ R
$f_1(BB) =$ B	$f_2(BB) =$ R	$f_3(BB) =$ R

Then $Q_1 = \{$RBB, BBB$\}$, $Q_2 = \{$RRR, RBR, BRB, BBB$\}$, $Q_3 = \{$RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB$\}$. Further, $t_1 = 1$, $t_2 = 2$ and $t_3 = 4$. Of the two configurations in Q_1, if BBB is the configuration of hats, Brenda's guess will be correct. Of the four configurations in Q_2, Glenda's guess will be correct for each of the configurations RBR and BRB. Finally, of the eight configurations in Q_3, Miranda's guess will be correct for each of the configurations RRR, RBR, BRR and BBR.

In the general case, the configurations in Q_i can be partitioned into two sets, one set Q_{i1} containing configurations in each of which Participant No. i's guess will be correct as per her instruction f_i and the other set Q_{i2} containing configurations in each of which Participant No. i's guess will be incorrect, with cardinalities t_i and $(q-1)t_i$, respectively. Now take any configuration from Σ. Let us determine whether or not it is a winning configuration as per the strategy $S = (f_1, f_2, ..., f_n)$. It is a winning configuration if at least one participant guessed correctly. Consequently,

$$W_S \subset Q_{i1} \cup Q_{i2} \cup \cdots \cup Q_{in}.$$

Therefore,

$$\#W_S \leq t_1 + t_2 + \cdots + t_n.$$

On the other hand, if at least one participant guesses wrongly under a given configuration, then it is a losing configuration. Therefore,

$$\#L_S \geq \frac{(q-1)(t_1 + t_2 + \cdots + t_n)}{n}.$$

Since $\#W_S + \#L_S = q^n$, it follows that

$$\#W_S \leq q^n - \frac{(q-1)(t_1 + t_2 + \cdots + t_n)}{n}.$$

Thus an upper bound for $\#W_S$ over all strategies S reduces to the following optimization problem:

$$Maximize \quad min\{t_1 + t_2 + \cdots + t_n, q^n - \frac{(q-1)(t_1 + t_2 + \cdots + t_n)}{n}\}$$

$$subject\ to\ the\ constraints \quad 0 \le t_i \le q^{n-1},\ i = 1, 2, ..., n.$$

Let us review the optimization problem vis-a-vis the hat problem with 2 colors, 3 participants, the strategy S spelled out above. Note that $W_S = \{$RBR, BRR, BBR$\}$; $\#W_S = 3$; $Q_{11} = \{$BBB$\}$; $Q_{21} = \{$RBR, BRB$\}$; $Q_{31} = \{$RRR, RBR, BRR, BBR$\}$; $\#W_S < t_1 + t_2 + t_3$; and $\#L_S = 5 > \frac{(t_1+t_2+t_3)}{3}$.

Let us now tackle the general optimization problem.

$$\begin{aligned}
&Maximize\ min\left\{t_1 + t_2 + \cdots + t_n, q^n - \frac{(q-1)(t_1 + t_2 + \cdots + t_n)}{n}\right\} \\
&\quad \le min\left\{max\{t_1 + t_2 + \cdots + t_n\}, q^n - max\left\{\frac{(q-1)(t_1 + t_2 + \cdots + t_n)}{n}\right\}\right\} \\
&\quad = min\left\{z, q^n - \frac{q-1}{n}z\right\} \\
&\quad = \frac{n}{n+q-1}q^n,
\end{aligned}$$

where $z = max\{t_1 + t_2 + \cdots + t_n\}$ and all the maximums are taken over all $t_1, t_2, ..., t_n$ subject to the constraints spelled out above. Consequently, an upper bound for the winning probability is given by $\frac{n}{n+q-1}$.

In the case of the hat problem with 2 colors and n participants, an upper bound for the winning probability is $\frac{n}{n+1}$. In particular, for the problem with 2 colors and 3 participants, an upper bound for the winning probability is 3/4. The strategy presented in Section 29.1 has the winning probability 3/4 and hence it is indeed optimal. In the case of the hat problem with 2 colors and 4 participants, an upper bound for the winning probability is 4/5. However, there is no strategy for which the winning probability is 4/5. This can be shown as follows. First of all, we show that there is a strategy S^* with winning probability 3/4. Suppose the four participants are: Brenda, Glenda, Miranda and Yolanda. We instruct Brenda to pass. We instruct Glenda, Miranda and Yolanda to ignore Brenda and play the game as though they are the only participants, and follow the three player optimal strategy. Under this strategy S^*, the probability of winning the prize is 3/4. Now, let S be any strategy. Its winning probability must be of the form m/16. Note that $3/4 = 12/16 < 4/5$ but $13/16 > 4/5$. Consequently, the winning probability under S has to be $\le 3/4$. Hence S^* is optimal for the game with four players and 2 colors.

For the hat problem with 2 colors and n participants, one can always find a strategy with winning probability 3/4. Instruct $(n-3)$ participants to pass all the time and the three remaining participants play the game as though they are the only participants.

For the hat problem with 2 colors and 5 participants, an upper bound with winning probability is 5/6. However, there is no strategy which achieves this winning probability. An optimal strategy in this case has a winning probability 3/4 only.

For the hat problem with 2 colors and n participants with n of the form $2^k - 1$, there is always an optimal strategy with winning probability $n/(n+1)$, the upper bound. For a description of an optimal strategy, see Buhler (2002).

If q (number of colors) $= 3$ and $n = 3$, the upper bound is 3/5. The strategy described in Section 29.3 has the winning probability 15/27. For any strategy S in this context, the winning probability must be of the form $m/27$. Note that $16/27 < 3/5$ but $17/27 > 3/5$. The question arises whether or not there is a strategy S with winning probability 16/27. Using Coding Theory argument, which is not present here, we have shown that there is no strategy with winning probability 16/27. Consequently, the strategy presented in Section 29.3 is indeed optimal for the game with 3 colors and 3 participants.

In the case of the hat problem with m colors and 3 participants, the upper bound for winning probability is $3/(m+2)$. The symmetric strategy we have described in Section 29.4 has the winning probability $\frac{3(m-1)^2+3}{m^3}$. The difference between the upper bound and $\frac{3(m-1)^2+3}{m^3}$ is very small. As a matter of fact,

$$\frac{3}{(m+2)} - \frac{3(m-1)^2+3}{m^3} = \frac{6}{m^3},$$

which is close to zero even for moderate values of m. Consequently, we can say that the strategy presented in Section 29.4 is almost optimal.

29.6 General Distribution

We now work in the environment of 2 colors and 3 participants. The eight possible configurations of hats need not be equally likely. Let the distribution on the set of all configurations be given by

Configuration	Brenda	Glenda	Miranda	Probability
1	Red	Red	Red	p_1
2	Red	Red	Black	p_2
3	Red	Black	Red	p_3
4	Red	Black	Black	p_4
5	Black	Red	Red	p_5
6	Black	Red	Black	p_6
7	Black	Black	Red	p_7
8	Black	Black	Black	p_8

Given the distribution p_i's, the objective is to find an optimal strategy which maximizes the probability of winning the prize. For example, if $p_1 = 0 = p_8$, then there is a strategy which gives the probability of winning as unity no matter what the values of the other probabilities are. If $p_1 = 0.47 = p_8$ and $p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0.01$, the strategy described in Section 29.1 is no longer optimal.

For a given distribution, one way to find an optimal strategy is to calculate the probability of winning the prize for each of the possible 531,441 strategies. From this collection of all strategies, we are able to identify 12 strategies and it is enough to calculate the probability of winning for each of these 12 strategies in order to determine an optimal strategy. The reasoning now follows.

Recall that an instruction to a participant is a map

$$f : \{RR, RB, BR, BB\} \to \{R, B, P\}.$$

A strategy is a triplet $S = (f_1, f_2, f_3)$, where f_1 is an instruction to Brenda, f_2 to Glenda, and f_3 to Miranda. Note that the total number of strategies is $81^3 = 531,441$. Given any strategy S, one can determine the set W_S of all winning configurations of hats. For example, if $f_1 \equiv R, f_2 \equiv R, and f_3 \equiv B$, then the only configuration that leads to the prize is RRB if the participants adopt the strategy $S = (f_1, f_2, f_3)$. Thus, $W_S = \{RRB\}$. We can now introduce a relation in the set of all strategies. Say that the strategy $S = (f_1, f_2, f_3)$ is at least as good as the strategy $T = (g_1, g_2, g_3)$ if $W_T \subseteq W_S$. Denote this relation by $T \leq S$. Given a choice between S and T, we would adopt the strategy S. The relation $\leq$ is transitive and reflexive. Consequently, it is a partial order.

We have written a computer program to make a complete list of all strategies along with their sets of winning configurations. A careful scrutiny of the list yields 12 maximal strategies. What this means in terms of the stipulated partial order is that given any strategy T one can find one of the maximal strategies S such that $W_T \subseteq W_S$. It is now transparent that for a given distribution on the set of all configurations, an optimal strategy is one of these 12 strategies. We will now give a list of all these 12 maximal strategies.

Maximal Strategy 1

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = B$	$f_2 = f_1$	$f_3 = f_1$
$f_1(RB) = P$		
$f_1(BR) = P$		
$f_1(BB) = R$		

W_S = Winning set of configurations
= {RRB, RBR, RBB, BRR, BRB, BBR}

Note: This strategy is the same as the one described in Section 1. **Maximal**

Strategy 2

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = R$	$f_2(RR) = P$	$f_3(RR) = P$
$f_1(RB) = P$	$f_2(RB) = R$	$f_3(RB) = R$
$f_1(BR) = P$	$f_2(BR) = B$	$f_3(BR) = B$
$f_1(BB) = B$	$f_2(BB) = P$	$f_3(BB) = P$

W_S = Winning set of configurations
= {RRR, RRB, RBR, BRB, BBR, BBB}

Maximal Strategy 3

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = R$	$f_3(RR) = P$
$f_1(RB) = R$	$f_2(RB) = P$	$f_3(RB) = B$
$f_1(BR) = B$	$f_2(BR) = P$	$f_3(BR) = R$
$f_1(BB) = P$	$f_2(BB) = B$	$f_3(BB) = P$

W_S = Winning set of configurations
= {RRR, RRB, RBB, BRR, BBR, BBB}

Maximal Strategy 4

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = P$	$f_3(RR) = R$
$f_1(RB) = B$	$f_2(RB) = B$	$f_3(RB) = P$
$f_1(BR) = R$	$f_2(BR) = R$	$f_3(BR) = P$
$f_1(BB) = P$	$f_2(BB) = B$	$f_3(BB) = B$

W_S = Winning set of configurations
= {RRR, RBR, RBB, BRR, BRB, BBB}

Maximal Strategy 5

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = R$	$f_2(RR) = P$	$f_3(RR) = P$
$f_1(RB) = R$	$f_2(RB) = P$	$f_3(RB) = P$
$f_1(BR) = R$	$f_2(BR) = P$	$f_3(BR) = P$
$f_1(BB) = R$	$f_2(BB) = P$	$f_3(BB) = P$

W_S = Winning set of configurations
= {RRR, RRB, RBR, RBB}

Maximal Strategy 6

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = B$	$f_2(RR) = P$	$f_3(RR) = P$
$f_1(RB) = B$	$f_2(RB) = P$	$f_3(RB) = P$
$f_1(BR) = B$	$f_2(BR) = P$	$f_3(BR) = P$
$f_1(BB) = B$	$f_2(BB) = P$	$f_3(BB) = P$

W_S = Winning set of configurations
= {BRR, BRB, BBR, BBB}

Maximal Strategy 7

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = R$	$f_3(RR) = P$
$f_1(RB) = P$	$f_2(RB) = R$	$f_3(RB) = P$
$f_1(BR) = P$	$f_2(BR) = R$	$f_3(BR) = P$
$f_1(BB) = P$	$f_2(BB) = R$	$f_3(BB) = P$

W_S = Winning set of configurations
= {RRR, RRB, BRR, BRB}

Maximal Strategy 8

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = B$	$f_3(RR) = P$
$f_1(RB) = P$	$f_2(RB) = B$	$f_3(RB) = P$
$f_1(BR) = P$	$f_2(BR) = B$	$f_3(BR) = P$
$f_1(BB) = P$	$f_2(BB) = B$	$f_3(BB) = P$

W_S = Winning set of configurations
= {RBR, RBB, BBR, BBB}

Maximal Strategy 9

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = P$	$f_3(RR) = R$
$f_1(RB) = P$	$f_2(RB) = P$	$f_3(RB) = R$
$f_1(BR) = P$	$f_2(BR) = P$	$f_3(BR) = R$
$f_1(BB) = P$	$f_2(BB) = P$	$f_3(BB) = R$

W_S = Winning set of configurations
= {RRR, RBR, BRR, BBR}

Maximal Strategy 10

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = P$	$f_2(RR) = P$	$f_3(RR) = B$
$f_1(RB) = P$	$f_2(RB) = P$	$f_3(RB) = B$
$f_1(BR) = P$	$f_2(BR) = P$	$f_3(BR) = B$
$f_1(BB) = P$	$f_2(BB) = P$	$f_3(BB) = B$

W_S = Winning set of configurations
= {RRB, RBB, BRB, BBB}

Maximal Strategy 11

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = R$	$f_2(RR) = R$	$f_3(RR) = R$
$f_1(RB) = B$	$f_2(RB) = B$	$f_3(RB) = B$
$f_1(BR) = B$	$f_2(BR) = B$	$f_3(BR) = B$
$f_1(BB) = R$	$f_2(BB) = R$	$f_3(BB) = R$

W_S = Winning set of configurations
= {RRR, RBB, BRB, BBR}

Maximal Strategy 12

Instruction to Brenda	Instruction to Glenda	Instruction to Miranda
$f_1(RR) = B$	$f_2(RR) = B$	$f_3(RR) = B$
$f_1(RB) = R$	$f_2(RB) = R$	$f_3(RB) = R$
$f_1(BR) = R$	$f_2(BR) = R$	$f_3(BR) = R$
$f_1(BB) = B$	$f_2(BB) = B$	$f_3(BB) = B$

W_S = Winning set of configurations
= {RRB, RBR, BRR, BBB}

A summary of these strategies along with their winning and losing configurations is given in the following table.

Max. Str.	Config. RRR	RRB	RBR	RBB	BRR	BRB	BBR	BBB
1	L	W	W	W	W	W	W	L
2	W	W	W	L	L	W	W	W
3	W	W	L	W	W	L	W	W
4	W	L	W	W	W	W	L	W
5	W	W	W	W	L	L	L	L
6	L	L	L	L	W	W	W	W
7	W	W	L	L	W	W	L	L
8	L	L	W	W	L	L	W	W
9	W	L	W	L	W	L	W	L
10	L	W	L	W	L	W	L	W
11	W	L	L	W	L	W	W	L
12	L	W	W	L	W	L	L	W

Some comments are in order on these maximal strategies. No two of these strategies are comparable. This means that the winning set of configurations of any of these strategies is neither contained in nor contains the winning set of configurations of any one of the other strategies. In addition, an examination of the entire set of strategies yields other valuable information. There are no strategies with exactly three winning configurations or five winning configurations.

For any given distribution p_i's, the above table can be used to determine an optimal strategy which maximizes the probability of winning the prize. One simply calculates the probability of winning under each of these 12 strategies. Pick the one with maximum probability then. Let us look at the distribution $p_1 = p_8 = 0.47$ and $p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0.01$. There are three optimal strategies available: Strategies 2, 3 and 4. The winning probability is 0.98.

These maximal strategies have certain symmetric or anti-symmetric properties with respect to the configurations. For any strategy S, let WL_S (Win-Loss Map) denote the map from the set {RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB} of all configurations into the set $\{W, L\}$ defined by

$$\begin{aligned} WL_S(\text{Configuration}) &= W \text{ if the configuration is a winning one,} \\ &= L \text{ if the configuration is a losing one.} \end{aligned}$$

A strategy S is symmetric if

$$\begin{aligned} WL_S(RRR) &= WL_S(BBB), \\ WL_S(RRB) &= WL_S(BBR), \\ WL_S(RBR) &= WL_S(BRB), \end{aligned}$$

and

$$WL_S(RBB) = WL_S(BRR).$$

If we flip R and B in the arguments of the map WL_S, the map remains invariant. We can now check that the maximal strategies 1, 2, 3, and 4 are symmetric. In addition, for each of these strategies, the number of winning configurations is six. The total number of symmetric strategies each with six winning configurations is four. We have exhausted all these strategies and they are indeed the first four strategies listed above.

A strategy S is anti-symmetric if

$$\begin{aligned} WL_S(RRR) &\neq WL_S(BBB), \\ WL_S(RRB) &\neq WL_S(BBR), \\ WL_S(RBR) &\neq WL_S(BRB), \end{aligned}$$

and

$$WL_S(RBB) \neq WL_S(BRR).$$

The next eight strategies in the list are all anti-symmetric. Each of these strategies has exactly four winning configurations. These strategies can be enumerated systematically by defining WL on the configurations RRR, RRB, RBR and RBB only.

Configurations	Win-Loss Maps							
	WL_1	WL_2	WL_3	WL_4	WL_5	WL_6	WL_7	WL_8
RRR	W	L	W	L	W	L	W	L
RRB	W	L	W	L	L	W	L	W
RBR	W	L	L	W	W	L	L	W
RBB	W	L	L	W	L	W	W	L

This is a complete enumeration of all anti-symmetric strategies each with four winning configurations.

The idea expounded so far can be extended to hat problems with n players and two colors. Consider, for example, four players and two colors. The total number of hat configurations is 16. Let $p_1, p_2, \ldots, p_{16}$ be any given probability distribution on the set of all configurations. The objective is to determine an optimal strategy which maximizes the winning probability. The total number of strategy is 3^{24}. A complete enumeration of all these strategies is outside the scope of any computer. However, one can write down maximal strategies for this problem. For example, the total number of maximal strategies each with 12 winning configurations is 28. (Note that no strategy will give more than 12 winning configurations.) All these strategies will have to be symmetric! The win-loss maps of all these strategies are obtained by selecting the configurations from {RRRR, RRRB, RRBR, RRBB, RBRR, RBRB, RBBR, RBBB} and assigning them the letter W. The win-loss maps can be completed by symmetry.

There are maximal strategies each with 10 wining configurations and also each with 8 winning configurations.

29.7 Other Variations

There are a number of variations of the hat problem considered in the literature. See Buhler (2002) for some of these. We would like to mention one new variation. Consider the hat problem with n colors and n participants. Each participants is fitted with a hat whose color is randomly picked from the given set of colors. Every participants can see the colors of hats of her team mates but does not know the colors of her hat. Each participant is asked separately to guess the color of her hat. No one is allowed to "Pass." They can win collectively the prize provided at least one of the guesses is correct. Is there a strategy of responses which will guarantee 100% chances of winning the prize? Yes, there is one. The reader may try to find one.

29.8 Some Open Problems

There are many open problems in the environment of traditional hat problem. Take the case of 2 colors and n participants. Optimal strategies are known for $n = 3, 4, 5, 6, 7,$ and 8. Optimal strategy is known if $n = 2^k - 1$ for some positive integer $k \geq 2$. For all other cases, optimal strategies are not known. Take the case of 3 colors and n participants. Except for the case $n = 3$, which has been dealt in this paper, optimal strategies are not known. For the general cases of q colors and n participants, virtually nothing is known.

29.9 The Yeast Genome Problem

One of the most important problems in cell biology is to understand functionality of each and every gene of any living organism. A mammoth project, called Deletion Project, is underway to study the DNA of the yeast organism. The genome of yeast organism has been completely mapped out. It has about 6,000 genes. Experiments on yeast cells, under the Project, have the following basic ingredients:

1. Remove a gene from the cell.
2. Place the cell in a chamber at a set temperature.
3. Examine every one of the remaining cells whether or not it is active.

The data vector generated is of order 1×6000. Every entry in the vector, except one, is 0 (inactive) or 1 (active). The missing entry corresponds to the deleted gene. Repeat the Steps 1, 2 and 3 with respect to every gene. At the set temperature, we will thus have 6,000 binary data vectors each vector having exactly one blank space. The whole cell is also placed in the chamber without removing any of its genes. The data vector generated will not have any blanks. Using all these data vectors, one has to guess what would have been the role of the deleted gene had it been present in the cell. It is hoped that hat problem might provide some pointers.

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