# Nonparametric Regression and Bonferroni joint confidence intervals

# Simultaneous Inferences

- In chapter 2, we know how to construct confidence interval for $\beta_0$ and $\beta_1$.
- If we want a confidence level of 95% of both $\beta_0$ and $\beta_1$
- One could construct a separate confidence interval for $\beta_0$ and $\beta_1$. BUT, then the probability of both happening is below 95%.
- How to create a joint confidence interval?

# Bonferroni Joint Confidence Intervals

- Calculation of Bonferroni joint confidence intervals is a general technique
- We highlight its application in the regression setting
  - Joint confidence intervals for $\beta_0$ and $\beta_1$
- Intuition
  - Set each statement confidence level to larger than $1 - \alpha$ so that the family coefficient is at least $1 - \alpha$
  - BUT how much larger?

# Ordinary Confidence Intervals

- Start with ordinary confidence intervals for $\beta_0$ and $\beta_1$

$$b_0 \pm t(1-\alpha/2; n-2)s\{b_0\}$$
$$b_1 \pm t(1-\alpha/2; n-2)s\{b_1\}$$

- And ask what the probability that one or both of these intervals is incorrect

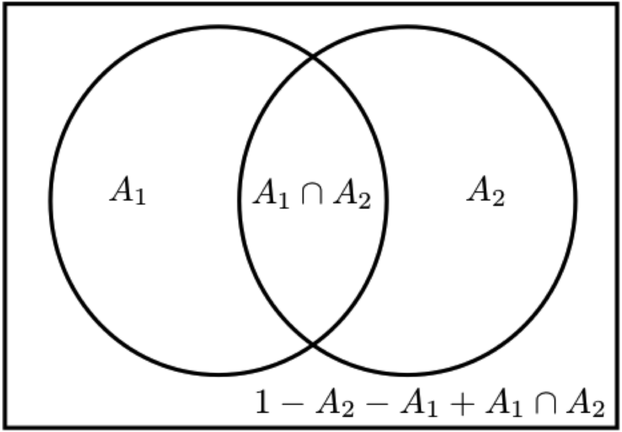
Remember

$$s^2\{b_0\} = MSE\left[\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i-\bar{X})^2}\right]$$
$$s^2\{b_1\} = \frac{MSE}{\sum(X_i-\bar{X})^2}$$

# General Procedure

- Let $A_1$ denote the event that the first confidence interval does not cover $\beta_0$, i.e. $P(A_1) = \alpha$
- Let $A_2$ denote the event that the second confidence interval does not cover $\beta_1$, i.e. $P(A_2) = \alpha$
- We want to know the probability that both estimates fall in their respective confidence intervals, i.e. $P(\bar{A}_1 \cap \bar{A}_2)$
- How do we get there from what we know?

# Venn Diagram



# Bonferroni inequality

- We can see that $P(\bar{A}_1 \cap \bar{A}_2) = 1 - P(A_2) - P(A_1) + P(A_1 \cap A_2)$
  - Size of set is equal to area is equal to probability in a Venn diagram.
- It also is clear that $P(A_1 \cap A_2) \geq 0$
- So, $P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - P(A_2) - P(A_1)$ which is the Bonferroni inequality.
- In words, in our example
  - $P(A_1) = \alpha$ is the probability that $\beta_0$ is *not* in $A_1$
  - $P(A_2) = \alpha$ is the probability that $\beta_1$ is *not* in $A_2$
  - $P(\bar{A}_1 \cap \bar{A}_2)$ is the probability that $\beta_0$ is in $A_1$ *and* $\beta_1$ is in $A_2$
  - So $P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - 2\alpha$

# Using the Bonferroni inequality

- Forward (less interesting) :
  - If we know that $\beta_0$ and $\beta_1$ are lie within intervals with 95% confidence, the Bonferroni inequality guarantees us a family confidence coefficient (i.e. the probability that *both* random variables lie within their intervals simultaneously) of at least 90% (if both intervals are correct). This is

$$P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - 2\alpha$$

- Backward (more useful):
  - If we know what to *specify* a family confidence interval of 90%, the Bonferroni procedure instructs us how to adjust the value of $\alpha$ for each interval to achieve the overall family confidence desired

# Using the Bonferroni inequality cont.

- To achieve a $1-\alpha$ *family* confidence interval for $\beta_0$ and $\beta_1$ (for example) using the Bonferroni procedure we know that both individual intervals must shrink.
- Returning to our confidence intervals for $\beta_0$ and $\beta_1$ from before

$$b_0 \pm t(1-\alpha/2; n-2)s\{b_0\}$$
$$b_1 \pm t(1-\alpha/2; n-2)s\{b_1\}$$

- To achieve a $1-\alpha$ *family* confidence interval these intervals must *widen* to

$$b_0 \pm t(1-\alpha/4; n-2)s\{b_0\}$$
$$b_1 \pm t(1-\alpha/4; n-2)s\{b_1\}$$

- Then
$P(\bar{A}_1 \cap \bar{A}_2) \geq 1 - P(A_2) - P(A_1) = 1 - \alpha/4 - \alpha/4 = 1 - \alpha/2$

# Using the Bonferroni inequality cont.

- The Bonferroni procedure is very general. To make joint confidence statements about multiple simultaneous predictions remember that

$$\begin{aligned} \hat{Y}_h &\pm t(1-\alpha/2; n-2)s\{pred\} \\ s^2\{pred\} &= MSE\left[1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum_i (X_i - \bar{X})^2}\right] \end{aligned}$$

- If one is interested in a $1-\alpha$ confidence statement about $g$ predictions then Bonferroni says that the confidence interval for each individual prediction must get wider (for each $h$ in the $g$ predictions)

$$\hat{Y}_h \pm t(1-\alpha/2g; n-2)s\{pred\}$$

Note: if a sufficiently large number of simultaneous predictions are made, the width of the individual confidence intervals may become so wide that they are no longer useful.

# The Toluca Example

- Say, we want to get a 90 percent confidence interval for $\beta_0$ and $\beta_1$ simultaneously.
- Then we require $B = t(1 - .1/4; 23) = t(.975, 23) = 2.069$
- Then we have the joint confidence interval:

$$b_0 \pm B * s(b_0)$$

and

$$b_1 \pm B * s(b_1)$$

# Confidence Band for Regression Line

- Remember in Chapter 2.5, we get the confidence interval for $E\{Y_h\}$ to be

$$\hat{Y}_h \pm t(1-\alpha/2; n-2)s\{\hat{Y}_h\}$$

- Now, we want to get a confidence band for the entire regression line $E\{Y\} = \beta_0 + \beta_1 X$.
- So called Working-Hotelling $1-\alpha$ confidence band is

$$\hat{Y}_h \pm W \times s\{\hat{Y}_h\}$$

  here $W^2 = 2F(1-\alpha; 2, n-2)$.
- Same form as before, except the $t$ multiple is replaced with the $W$ multiple.

# Example: toluca company

- Say we want to estimate the boundary value for the band at $X_h = 30, 65, 100$.
- We have

| $X_h$ | $\hat{Y}_h$ | $s\{\hat{Y}_h\}$ |
|---|---|---|
| 30 | 169.5 | 16.97 |
| 65 | 294.4 | 9.918 |
| 100 | 419.4 | 14.27 |

- Looking up the table,
  $W^2 = 2F(1-\alpha; 2, n-2) = 2F(.9; 2, 23) = 5.098$.
  R code:

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w2 = 2 * qf(1-0.1,2,23)
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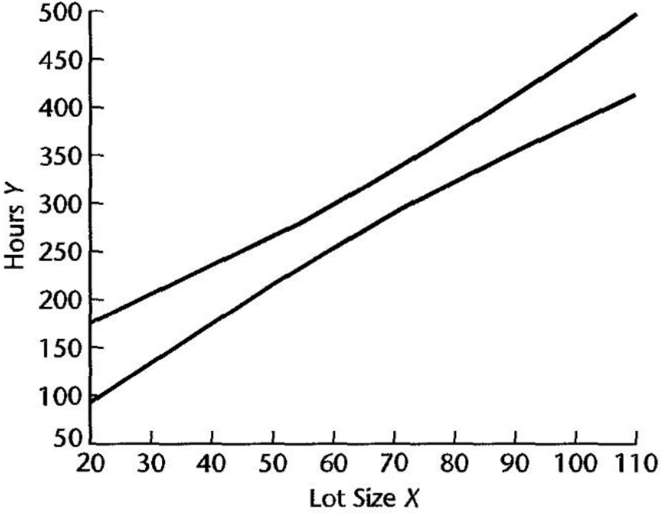
Now we have the confidence band for the three points are

$$131.2 = 169.5 - 2.258(16.97) \leq E\{Y_h\} \leq 169.5 + 2.258(16.97) = 207.8$$
$$272.0 = 294.4 - 2.258(9.918) \leq E\{Y_h\} \leq 294.4 + 2.258(9.918) = 316.8$$
$$387.2 = 419.4 - 2.258(14.27) \leq E\{Y_h\} \leq 419.4 + 2.258(14.27) = 451.6$$

# Example confidence band



# Compare with Bonferroni Procedure

- Say we want to simultaneously estimate response for $X_h = 30, 65, 100$.
- Then the simultaneous confidence intervals are

$$\hat{Y}_h \pm t(1-\alpha/(2g); n-2)s\{\hat{Y}_h\}$$

- We have $B = t(1-\alpha/(2g); n-2) = t(1-.1/(2*3), 23) = 2.263$, the confidence intervals are

$$131.1 = 169.5 - 2.263(16.97) \leq E\{Y_h\} \leq 169.5 + 2.263(16.97) = 207.9$$
$$272.0 = 294.4 - 2.263(9.918) \leq E\{Y_h\} \leq 294.4 + 2.263(9.918) = 316.8$$
$$387.1 = 419.4 - 2.263(14.27) \leq E\{Y_h\} \leq 419.4 + 2.263(14.27) = 451.7$$

# Bonferroni v.s. Working-Hotelling

- This instance, working-hotelling confidence limits are slighter tighter(better) than bonferroni limits
- However, in larger families (more $X$) to be considered simultaneously, working-hotelling is always tighter, since $W$ stays the same for any number of statements but $B$ becomres larger.
- The levels of predictor variables are sometimes not known in advance. In such cases, it is better to use Working-Hotelling procedure since the family encompasses all possible levels of $X$.

$$131.1 = 169.5 - 2.263(16.97) \leq E\{Y_h\} \leq 169.5 + 2.263(16.97) = 207.9$$
$$272.0 = 294.4 - 2.263(9.918) \leq E\{Y_h\} \leq 294.4 + 2.263(9.918) = 316.8$$
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# Regression through the origin

Model

$$Y_i = \beta_1 X_i + \epsilon_i$$

- Sometimes it is known that the regression function is linear and that it *must* go through the origin.
- $\beta_1$ is parameter
- $X_i$ are known constants
- $\epsilon_i$ are i.i.d $N(0, \sigma^2)$.
- The least squares and maximum likelihood estimators for $\beta_1$ coincide as before, the estimator is $b_1 = \frac{\sum X_i Y_i}{\sum X_i^2}$

# Regression through the origin, Cont

- In regression through the origin there is only one free parameter ($\beta_1$) so the number of degrees of freedom of the MSE

$$s^2 = MSE = \frac{\sum e_i^2}{n-1} = \frac{\sum(Y_i - \hat{Y}_i)^2}{n-1}$$

is increased by one.
- This is because this is a "reduced" model in the general linear test sense and because the number of parameters estimated from the data is less by one.

| Estimate of | Estimated Variance | Confidence Limits | |
|---|---|---|---|
| $\beta_1$ | $s^2\{b_1\} = \frac{MSE}{\sum X_i^2}$ | $b_1 \pm ts\{b_1\}$ | (4.18) |
| $E\{Y_h\}$ | $s^2\{\hat{Y}_h\} = \frac{X_h^2 MSE}{\sum X_i^2}$ | $\hat{Y}_h \pm ts\{\hat{Y}_h\}$ | (4.19) |
| $Y_{h(new)}$ | $s^2\{pred\} = MSE\left(1 + \frac{X_h^2}{\sum X_i^2}\right)$ | $\hat{Y}_h \pm ts\{pred\}$ | (4.20) |
| | | where: $t = t(1 - \alpha/2; n-1)$ | |

# A few notes on regression through the origin

- $\sum e_i \neq 0$ in general now. Only constraint is $\sum X_i e_i = 0$.
- SSE may exceed the total sum of squares SSTO. In the case of a curvilinear pattern or linear pattern with a intercept away from the origin.
- Therefore, $R^2 = 1 - SSE/SSTO$ may be negative!
- Generally, it is safer to use the original model opposed with regression-through-the-origin model.
- Otherwise, it is the wrong model to start with!