Math 644, Fall 2012

Homework 1 Due: Friday, 09/21/2012

1. Consider the regression model $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, i = 1, \ldots, n$. Show that $Y_i - \bar{Y} = \beta_1(X_i - \bar{X}) + \epsilon_i$, where $\epsilon_i = \varepsilon_i - \bar{\varepsilon}$ and $\bar{\varepsilon} = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i$.

2. Show that the least squares estimator b_1 of β_1 can be expressed as $b_1 = r_{X,Y}\frac{s_Y}{s_X}$, where

$$\begin{aligned} s_X &= \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}, \quad s_Y = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n-1}} \\ r_{X,Y} &= \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}. \end{aligned}$$

3. Consider the regression model $Y_i = \beta_1 X_i + \varepsilon_i, i = 1, \ldots, n$. Find the least squares estimator of β_1 and show the estimator of β_1 is unbiased.

4. Show that the least squares regression line fitted to data $(5, Y_{1,1})$, $(5, Y_{1,2}), (5, Y_{1,3}), (10, Y_{2,1}), (10, Y_{2,2}), (10, Y_{2,3}), (15, Y_{3,1}), (15, Y_{3,2})$, $(15, Y_{3,3})$, is the same as a model fitted to the three points $(5, \bar{Y}_1), (10, \bar{Y}_2)$ and $(15, \bar{Y}_3)$, where $\bar{Y}_1 = (Y_{1,1} + Y_{1,2} + Y_{1,3})/3, \bar{Y}_2 = (Y_{2,1} + Y_{2,2} + Y_{2,3})/3$, and $\bar{Y}_3 = (Y_{3,1} + Y_{3,2} + Y_{3,3})/3$.

5. (Grade Point average): For graduate students, X is ACT test score, Y is GPA. There are 120 students, and their ACT and GPA are recorded. Suppose we predict Y based on X by a simple linear regression model $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, i = 1, \ldots, n$.

   (a) Obtain the least squares estimators of β_0 and β_1, and state the estimated regression function.

   (b) Plot the estimated regression function and the data. Does the estimated regression appear to fit the data well?

   (c) Obtain a point prediction of GPA for a student with ACT $X = 30$.

   (d) What is the change of the mean response when ACT increases by one point?

   (e) Plot the fitted residuals e_i against X_i.

   (f) Calculate $\sum_{i=1}^{120} e_i^2$.