1. Suppose we have $n = 10$ observations $(X_i, Y_i)$ and fit the data with model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

with $\varepsilon_i, i = 1, ..., 10$ are IID $N(0, \sigma^2)$. We have the following calculations.

$$\bar{X} = 0.5669, \quad \bar{Y} = 0.9624, \quad \sum_{i=1}^{n} Y_i^2 = 10.2695,$$

$$\sum_{i=1}^{n} X_i^2 = 4.0169, \quad \sum_{i=1}^{n} X_iY_i = 6.2841.$$

(a) Write down the estimated model.

(b) Test $H_0 : \beta_1 = 1$ with $\alpha = 0.05$.

(c) For a new $X = 1$, find the 95% CI for its expected response.

(d) For a new $X = 0.5$, find the 95% prediction interval for its possible response.

2. For the least square estimators $b_0$ and $b_1$ of simple linear regression model, find $\text{Cov}(b_0, b_1)$.

3. Suppose $A : m \times n$ is a constant matrix and $Y : n \times 1$ is a random vector. Then $\text{Var}(AY) = A \text{Var}(Y) A'$. Please give your proof for $m = 2, n = 3$.

4. For multiple linear regression, the normal equations are

$$\sum_{i=1}^{n} e_i = 0$$

$$\sum_{i=1}^{n} e_i X_{i1} = 0$$

$$\vdots$$

$$\sum_{i=1}^{n} e_i X_{ip} = 0.$$
Prove that
\[ \sum_{i=1}^{n} \hat{Y}_i \epsilon_i = 0. \]

5. For each of the following regression models, indicate whether it is a general linear regression model. If not, state whether it can be expressed in the form of a linear regression model after some suitable transformation.

(a) \( Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 \log X_{i2} + \beta_3 X_{i1}^2 + \epsilon_i. \)

(b) \( Y_i = \epsilon_i \exp(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}^2), \) with \( \epsilon_i > 0. \)

(c) \( Y_i = \beta_0 \log(\beta_1 X_{i1}) + \epsilon_i. \)

(d) \( Y_i = \log(\beta_1 X_{i1}) + \beta_2 \log(X_{i2}) + \epsilon_i. \)

(e) \( Y_i = [1 + \exp(\beta_0 + \beta_1 X_{i1} + \epsilon_i)]^{-1}. \)

6. Consider the multiple linear regression models
\[ Y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \ldots, n, \]
where \( \epsilon_i \) are uncorrelated with \( E \epsilon_i = 0 \) and \( E \epsilon_i^2 = \sigma^2. \) State the least square criterion and derive the least squares estimators for \( \beta_1 \) and \( \beta_2. \)