Math 222 EXAM II, October 24, 2007

Read each problem carefully. Show all your work for each problem. No Calculators!

1. (a) (8) Determine if the functions y_1 , y_2 are linearly dependent or independent:

   (i) $y_1 = |t-1|$, $y_2 = 2(t-1)$, (ii) $y_1 = 3t+1$, $y_2 = t+3$

   (b) (8) Find a function $g(x)$ which satisfies the conditions: $W(f,g) = x$, $f(x) = x$.

2. (a) (12) Use the method of undetermined coefficients to find a particular solution of the differential equation

$$y'' - y' = 2e^t - 1 - t$$

   (b) (6) Determine the general solution of the above equation

3. (a) (12) Given that $y_1 = e^{-x}$ is a solution of the differential equation

$$xy'' + (x-1)y' - y = 0, \quad x > 0,$$

   use the method of reduction of order to find the second linerly independent solution y_2.

   (b) (6) Determine the homogeneous ODE whose general solution is

$$y = c_1e^t + c_2te^t + e^{-t}(c_3cos2t + c_4sin2t)$$

4. (16) Use the method of variation of parameter to find a particular solution of the differential equation

$$2y'' + 4y' + 2y = \tfrac{1}{t}e^{-t}, \; t > 0$$

5. (16) Determine the form of particular solution of the following ODE, using the method of undetermined coefficients. Do NOT evaluate the constants.

$$y^{(4)} + 2y^{(3)} + 2y'' = 4e^t - 2e^{-t}cos(t) + te^{-t}$$

6. (16) Solve the initial value problem

$$y^{(3)} - y'' - y' + y = 0, \; y(0) = 2, \; y'(0) = -1, \; y''(0) = 0$$

Problem1 (a) $y_1 = |t-1|$, $y_2 = 2(t-1)$

$c_1 y_1 + c_2 y_2 = 0$

$(c_1, c_2) = (0, 0)$ is the only solution if y_1 & y_2 are linearly independent

$c_1 |t-1| + c_2 \cdot 2 \cdot (t-1) = 0$ $\qquad t > 1$

$c_1 (t-1) + c_2 \cdot 2(t-1) = 0$ $\qquad c_1 + 2c_2 = 0$

$-c_1 (t-1) + c_2 \cdot 2(t-1) = 0$ $\qquad t < 1$

$-c_1 + 2c_2 = 0$

$\Rightarrow c_1 = 0, \ c_2 = 0$

$\Rightarrow |t-1|$ & $2(t-1)$ are linearly independent

$y_1 = 3t+1, \quad y_2 = t+3$

$c_1 y_1 + c_2 y_2 = 0$

$c_1 (3t+1) + c_2 (t+3) = 0$

$3c_1 + c_2 = 0$

$c_1 + 3c_2 = 0$ $\qquad c_1 = 0 = c_2 \Rightarrow 3t+1$ & $t+3$ are linearly independent

(b) $fg' - f'g = x, \quad f(x) = x$

$xg' - g = x$

$g' - \frac{1}{x} g = 1$

$\mu = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

$g = \frac{\int 1 \cdot \frac{1}{x} dx + c}{\frac{1}{x}} = \frac{\ln x + c}{\frac{1}{x}} = x \ln x + cx$

Problem 2 (a) $y'' - y' = 2e^t - 1 - t$

$r^2 - r = 0, \quad r(r-1) = 0, \quad r = 0, 1 \quad y_1 = 1, \ y_2 = e^t$

$Y = t(At + B) + tCe^t$

$Y' = 2At + B + Ce^t + Cte^t$

$Y'' = 2A + Ce^t + Ce^t + Cte^t$

P.02

$$Y'' - Y' = 2e^t - 1 - t$$

$$2A + 2ce^t + cte^t - (2At + B + ce^t + cte^t) = 2e^t - 1 - t$$

$$2A - B - 2At + ce^t = 2e^t - 1 - t$$

$$-2A = -1, \quad A = \tfrac{1}{2}$$

$$2A - B = -1, \quad 1 - B = -1, \quad B = 2, \quad \boxed{Y = t\left(\tfrac{t}{2} + 2\right) + 2te^t}$$

(b)

$$c = 2$$

$$\boxed{y = c_1 + c_2 e^t + t\left(\tfrac{t}{2} + 2\right) + 2te^t}$$

Problem 3 (a) $y_2 = vy_1, \quad y_2' = v'y_1 + vy_1', \quad y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$$x \cdot (v''y_1 + 2v'y_1' + vy_1'') + (x-1)(v'y_1 + vy_1') - vy_1 = 0$$

$$(xy_1)v'' + (2xy_1' + (x-1)y_1)v' = 0$$

$$y_1 = e^{-x}$$

$$xe^{-x} \cdot v'' + (-2xe^{-x} + (x-1)e^{-x})v' = 0$$

$$xv'' + (-x-1)v' = 0$$

$$\frac{v''}{v'} = \frac{x+1}{x} = 1 + \frac{1}{x}$$

$$\ln v' = x + \ln x, \quad v' = x \cdot e^x$$

$$v = \int xe^x dx = xe^x - e^x$$

$$\boxed{y_2 = ve^{-x} = x - 1}$$

(b) $y = c_1 e^t + c_2 te^t + e^{-t} \cdot (c_3 \cos 2t + c_4 \sin 2t)$

$$r = 1 \quad r = 1 \quad r = -1 \pm 2i$$

$$(r+1)(r+1)[(r+1)^2 + 4] = 0$$

$$(r^2 + 2r + 1)(r^2 + 2r + 5) = 0$$

$$r^4 + 2r^3 + r^2 + 2r^3 + 4r^2 + 2r + 5r^2 + 10r + 5 = 0$$

$$r^4 + 4r^3 + 10r^2 + 12r + 5 = 0$$

$$\boxed{y^{(4)} + 4y^{(3)} + 10y^{(2)} + 12y' + 5y = 0}$$

P.03

Problem 4: $2y''+4y'+2y=\frac{1}{t}e^{-t},\ t>0$

$y''+2y'+y=\frac{1}{2t}e^{-t}$

$r^2+2r+1=0,\quad r=\frac{-2\pm\sqrt{2^2-4}}{2}=-1,-1$

$y_1=e^{-t},\ y_2=te^{-t},\quad W=\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}-te^{-t} \end{vmatrix}=e^{-2t}$

$$Y=\left(\int\frac{-te^{-t}\cdot\frac{1}{2t}e^{-t}}{e^{-2t}}dt\right)e^{-t}+\left(\int\frac{e^{-t}\cdot\frac{1}{2t}e^{-t}}{e^{-2t}}dt\right)te^{-t}$$

$$=-\frac{1}{2}\int\frac{t}{t}dt\cdot e^{-t}+\frac{1}{2}\int\frac{1}{t}dt\cdot te^{-t}$$

$$Y=-\frac{t}{2}e^{-t}+\frac{1}{2}t\ln t\, e^{-t}$$

Problem 5: $y^{(4)}+2y^{(3)}+2y''=4e^{t}-2e^{-t}\cos t+te^{-t}$

$r^4+2r^3+2r^2=0$

$r^2(r^2+2r+2)=0$

$r^2((r+1)^2+1)=0$

$r^2=0,\ r=0,0$

$(r+1)^2+1=0,\ (r+1)^2=-1,\quad r=-1\pm i$

$y_1=1$

$y_2=t$

$y_3=e^{-t}\cos t$

$y_4=e^{-t}\sin t$

$$Y=Ae^{t}+tBe^{-t}\cos t+tCe^{-t}\cdot\sin t+(Dt+E)e^{-t}$$

$$\boxed{Y=Ae^{t}+t\cdot(Be^{-t}\cos t+Ce^{-t}\sin t)+(Dt+E)e^{-t}}$$

P.04

Problem 6

$y^{(3)} - y'' - y' + y = 0, \quad y(0)=2, \ y'(0)=-1, \ y''(0)=0$

$r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - r + 1 = 0$

$(r^2-1)(r-1) = 0$

$(r+1)(r-1)(r-1) = 0$

$r = -1, 1, 1$

$y = c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}$ $\qquad y(0) = 2 = c_1 + c_2$

$y' = -c_1 e^{-t} + c_2 e^{t} + c_3 e^{t} + c_3 t e^{t}$ $\qquad y'(0) = -1 = -c_1 + c_2 + c_3$

$y'' = c_1 e^{-t} + c_2 e^{t} + c_3 e^{t} + c_3 e^{t} + c_3 t e^{t}$ $\qquad y''(0) = c_1 + c_2 + 2c_3 = 0$

$c_1 + c_2 = 2$

$-c_1 + c_2 + c_3 = -1$

$c_1 + c_2 + 2c_3 = 0$

$2 + 2c_3 = 0, \quad c_3 = -1$

$-c_1 + c_2 = 0$

$c_1 + c_2 = 2$ $\qquad c_2 = 1, \ c_1 = 1$

$$\boxed{y(t) = e^{-t} + e^{t} + (-1) t e^{t}}$$