Math 222 Exam 2, October 22, 2014

Read each problem carefully. Show all your work for each problem. No calculators!

1. (a) (8) If the Wronskian of $f(t)$ and $g(t)$ is $3e^{4t}$ and $f(t) = e^{2t}$ find $g(t)$.
   (b) (8) Find the Wronskian of the functions y_1 and y_2 and state whether the functions are linearly dependent or independent

$$\text{(i)}\ \ y_1 = \cos t\,,\quad y_2 = \sin t\,,\qquad \text{(ii)}\ \ y_1 = x\,,\quad y_2 = xe^x\,.$$

2. (16) Find the solution of the initial value problem

$$y'' + 2y' + 5y = 0\,,\quad y(0) = 2\,, y'(0) = 0\,.$$

   Sketch the graph of the solution and describe its behavior.

3. (16) Show that $y_1(t) = t^{-1}$ is a solution of

$$t^2y'' + 3ty' + y = 0\,.$$

   Find a second solution by using the method of reduction of order.

4. (a) (9) Find the general solution of

$$2y'' + 3y' + y = t^2 + 3\sin t\,.$$

   (b) (9) Determine a suitable form (or ansatz) if the method of undetermined coefficients is to be used to find the particular solution y_p of the ODE below. Do NOT attempt to evaluate the coefficients

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t\,.$$

5. (16) Given the ODE

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3\,,$$

   show that $y_1 = t$ and $y_2 = te^t$ satisfy the corresponding homogeneous equation, then use the method of variation of parameters to find a particular solution.

6. A mass weighing 16 lb stretches a spring $\frac{1}{4}$ ft. The mass is attached to a viscous damper with a damping constant of 2 lb s/ft. The mass is set in motion from its equilibrium position $u(0) = 0$ with a downward velocity of $u'(0) = \frac{1}{4}$ ft/s. Recall $g = 32$ ft/s^2.
   (i) (6) Formulate the initial value problem for this system
   (ii) (8) Find the displacement of the mass u at any time t
   (iii) (4) Sketch the graph of u versus t and describe the motion.

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Problem 1 (a) $fg' - f'g = 3e^{4t}$, $f = e^{2t}$, $f' = 2e^{2t}$

$$e^{2t}g' - 2e^{2t}g = 3e^{4t}$$

$$g' - 2g = 3e^{2t} \qquad \mu = e^{\int -2\,dt} = e^{-2t}$$

$$g = \frac{\int 3e^{2t}\cdot e^{-2t}\,dt + c}{e^{-2t}} = \frac{3t + c}{e^{-2t}} = \boxed{3t\,e^{2t} + c\,e^{2t}}$$

(b) $y_1 = \cos t$, $y_2 = \sin t$, $W[y_1, y_2] = \cos t\cdot\cos t - (-\sin t)\sin t = 1 \neq 0$

y_1 & y_2 are linearly independent

$$y_1 = x,\ y_2 = xe^x,\quad W[y_1, y_2] = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x^2e^x \neq 0 \text{ for } x \neq 0$$

y_1 & y_2 are linearly independent.

Problem 2 $y'' + 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = 0$

$r^2 + 2r + 5 = 0$, $(r+1)^2 + 4 = 0$, $r = -1 \pm 2i$

$$y = e^{-t}(c_1\cos 2t + c_2\sin 2t)$$

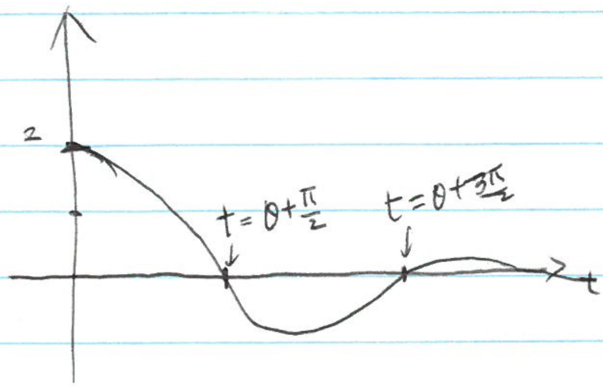
$$y' = -e^{-t}(c_1\cos 2t + c_2\sin 2t) + e^{-t}(-2c_1\sin 2t + 2c_2\cos 2t)$$

$$y(0) = 2 = c_1$$

$$y'(0) = -1\cdot(c_1) + 1\cdot(2c_2) = 0 \qquad 2c_2 = c_1,\ \boxed{c_2 = 1}$$

$$y(t) = e^{-t}\cdot(2\cos 2t + \sin 2t) = \sqrt{5}\,e^{-t}\cdot\left(\cos t\cdot\frac{2}{\sqrt{5}} + \sin 2t\,\frac{1}{\sqrt{5}}\right)$$

$$= \sqrt{5}\,e^{-t}\cdot\cos(t - \theta),\quad \theta = \tan^{-1}\left(\frac{1}{2}\right)$$



$u \to 0$ as $t \to \infty$

Problem 3 :

$y_1 = t^{-1}, \quad t^2 y_1'' + 3t y_1' + y_1 = t^2\left(\frac{2}{t^3}\right) + 3t\left(-\frac{1}{t^2}\right) + \frac{1}{t} = 0$

$y_1' = -\frac{1}{t^2}$ $\quad$ y_1 is a solution.

$y_1'' = \frac{2}{t^3}$

$y_2 = v y_1, \quad y_2' = v' y_1 + v y_1', \quad y_2'' = v'' y_1 + 2v' y_1' + v y_1''$

$t^2(v'' y_1 + 2v' y_1' + v y_1'') + 3t(v' y_1 + v y_1') + v y_1 = 0$

$(t^2 y_1) v'' + (2y_1' t^2 + 3t y_1) v' = 0$

$t v'' + (-2+3) v' = 0, \quad \frac{v''}{v'} = -\frac{1}{t}, \quad \ln v' = -\ln t, \quad v' = \frac{1}{t}$

$v = \ln t, \quad \boxed{y_2 = t \ln t}$

Problem 4 : (a) $2y'' + 3y' + y = t^2 + 3\sin t$

$2r^2 + 3r + 1 = 0 \qquad (2r+1)(r+1) = 0, \quad r = -\frac{1}{2}, -1$

$(2r \quad 1)$
$(r \quad 1)$

$y_c = c_1 e^{-t} + c_2 e^{-\frac{t}{2}}, \quad Y = At^2 + Bt + C + D\cos t + E\sin t$

$Y' = 2At + B - D\sin t + E\cos t$

$Y'' = 2A - D\cos t - E\sin t$

$$2(2A - D\cos t - E\sin t) + 3(2At + B - D\sin t + E\cos t) + At^2 + Bt + C + D\cos t + E\sin t$$
$$= t^2 + 3\sin t$$

t^2: $A = 1$

t^1: $6A + B = 0$, $B = -6$

t^0: $4A + 3B + C = 0$ $\quad 4 - 24 + C = 0$, $C = 20$

$\cos t$: $3E + D = 0$ $\quad 3E + D = 0$

$\sin t$: $-2E - 3D + E = 3$ $\quad -E - 3D = 3$ $\quad -E - 3(-3E) = 9E - E = 8E = 3$,

$E = \frac{3}{8}$, $D = -\frac{9}{8}$

$$\boxed{y = C_1 e^{-t} + C_2 e^{-\frac{t}{2}} + t^2 - 6t + 20 - \frac{9}{8}\cos t + \frac{3}{8}\sin t}$$

(b) $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin 3t$

$r^2 + 3r = 0$, $r = 0, -3$, $y_c = C_1 + C_2 e^{-3t}$

$$Y = t(At^4 + Bt^3 + Ct^2 + Dt + E) + (Ft^2 + Gt + H)e^{-3t}$$
$$+ I\sin 3t + J\cos 3t$$

Problem 5: $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$

$y_1 = t$, $y_1' = 1$, $y_1'' = 0$ $\quad - t(t+2) + (t+2)t = 0$ ✓

$y_2 = te^t$, $y_2' = e^t + te^t$, $y_2'' = e^t + e^t + te^t = 2e^t + te^t$

$$t^2 y_2'' - t(t+2)y_2' + (t+2)y_2$$
$$= t^2(2e^t + te^t) - t(t+2)(1+t)e^t + (t+2)te^t$$
$$= (2t^2 + t^3 - t(t+2)(t+1) + t(t+2))e^t = 0$$ ✓

Problem 6: $m = \frac{16}{32} = \frac{1}{2}$ $\quad 16 = k \cdot \frac{1}{4},\ k = 64 \quad \gamma = 2$

(a) $\frac{1}{2}u'' + 2u' + 64u = 0, \quad u'' + 4u' + 128u = 0 \quad u(0) = 0,\ u'(0) = \frac{1}{4}$

(b) $r^2 + 4r + 128 = 0, \quad (r+2)^2 + 124 = 0, \quad r = -2 \pm \sqrt{124}\, i$

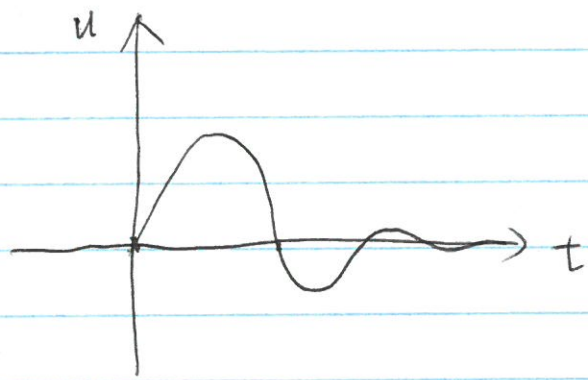
$u = e^{-2t}\left(C_1 \cos\sqrt{124}\,t + C_2 \sin\sqrt{124}\,t\right)$

$u' = -2e^{-2t}\left(C_1 \cos\sqrt{124}\,t + C_2 \sin\sqrt{124}\,t\right) + e^{-2t} \cdot \sqrt{124}\left(-C_1 \sin\sqrt{124}\,t + C_2 \cos\sqrt{124}\,t\right)$

$u(0) = C_1 = 0$

$u'(0) = \sqrt{124}\, C_2 = \frac{1}{4}, \quad C_2 = \frac{1}{4\sqrt{124}}$

$$u(t) = \frac{e^{-2t}}{8\sqrt{31}} \sin\left(2\sqrt{31}\,t\right)$$



$u \to 0$ as $t \to \infty$