Math 222 Exam 3, April 22, 2015

Read each problem carefully. Show all your work for each problem. No calculators!

1. Find power series solutions about $x = 0$ of the equation $2y'' + xy' + 3y = 0$:
(a) (8 points) Find the recurrence relation.
(b) (8 points) Find the first four terms in each of two solutions y_1 and y_2 unless the series terminates sooner.

2. (a) (8 points) Find the general solution of the Euler equation and find the solution of the initial value problem
$$x^2y'' - 5xy' + 9y = 0, \quad y(1) = 2, \quad y'(1) = 5.$$
(b) (8 points) Find all singular points of the given equation and determine whether each is regular or irregular
$$x^2(x-1)^2y'' + 3(x-1)y' + 4y = 0.$$

3. (a) (6 points) Sketch the function, then express it in terms of the unit step function $u_c(t)$, and hence find its Laplace transform
$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t-1 & 1 \le t < 3 \\ 0 & 3 \le t. \end{cases}$$
(b) (6 points) Find the inverse Laplace transform of $F(s) = \frac{s-2}{s^2+2s+5}$.
(c) (6 points) Find the inverse Laplace transform in terms of a convolution for
$$F(s) = \frac{s}{(s+2)^2(s^2+9)}.$$

4. (16 points) Use Laplace transforms to find the solution of the initial value problem, then sketch the forcing function and the solution versus t
$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1.$$

5. (16 points) Solve the initial value problem by Laplace transforms, where a is a parameter. Then find the value of a such that $y = 0$ for all $t > \frac{3\pi}{2}$. Sketch the solution for $0 \le t < 2\pi$. (Note that $\sin(t - \frac{3\pi}{2}) = \cos t$)
$$y'' + 4y' + 5y = a\,\delta(t - \frac{3\pi}{2}), \quad y(0) = 1, \quad y'(0) = -2.$$

6. (a) (8 points) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.
(b) (10 points) Find the general solution and the solution of the initial value problem for the ODE system $\boldsymbol{x}' = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. How does the solution behave as $t \to \infty$?

Solutions for Exam 3 April 2015

Problem 1

$2y'' + xy' + 3y = 0$

(a)

$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

x^0: $\quad 2\cdot 2\cdot 1\cdot a_2 + 3a_0 = 0, \qquad a_2 = -\frac{3}{4} a_0$

x^1: $\quad 2\cdot 3\cdot 2\cdot a_3 + a_1 + 3a_1 = 0, \qquad a_3 = -\frac{4}{12} a_1 = -\frac{1}{3} a_1$

x^2: $\quad 2\cdot 4\cdot 3\cdot a_4 + 2a_2 + 3a_2 = 0, \qquad a_4 = -\frac{5}{24}\cdot a_2$

x^n: $\quad 2\cdot(n+2)(n+1) a_{n+2} + n a_n + 3a_n = 0, \qquad \boxed{a_{n+2} = -\frac{(n+3)}{2(n+2)(n+1)} a_n}$ for $n=0,1,2,\dots$

(b)

$$y_1 = a_0\left(1 - \frac{3}{4}x^2 + \left(-\frac{5}{24}\right)\left(-\frac{3}{4}\right)x^4 + \left(-\frac{7}{60}\right)\left(-\frac{5}{24}\right)\left(-\frac{3}{4}\right)x^6 + \cdots\right)$$

$$y_2 = a_1\left(x - \frac{1}{3}x^3 + \left(-\frac{6}{40}\right)\left(-\frac{1}{3}\right)x^5 + \left(-\frac{8}{84}\right)\left(-\frac{6}{40}\right)\left(-\frac{1}{3}\right)x^7 + \cdots\right)$$

Problem 2

(a) $x^2y'' - 5xy' + 9y = 0, \qquad y(1) = 2, \quad y'(1) = 5$

$y \sim x^r$

$r(r-1) - 5r + 9 = 0, \qquad r^2 - 6r + 9 = 0, \quad (r-3)^2 = 0, \quad r = 3, 3,$

$y_1 = x^3, \quad y_2 = x^3 \ln x, \qquad y = c_1 x^3 + c_2 x^3 \ln x$

$\qquad\qquad\qquad\qquad\qquad y' = 3c_1 x^2 + 3c_2 x^2 \ln x + c_2 x^2$

$y(1) = c_1 = 2$

$y'(1) = 3c_1 + c_2 = 5 \quad c_2 = -1$

$y = 2x^3 - x^3 \ln x$

(b) $x^2(x-1)^2 y'' + 3(x-1)y' + 4y = 0$

$x^2(x-1)^2 = 0, \quad x = 0, 0, 1, 1$ are singular points

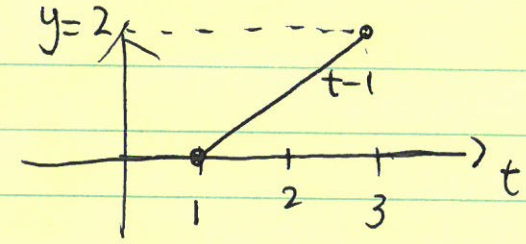
$$y'' + \frac{3}{x^2(x-1)} y' + \frac{4}{x^2(x-1)^2} y = 0$$

$\lim_{x\to 0} \frac{3}{x^2(x-1)} x = \infty \qquad x = 0$ is an irregular singular point

$\lim_{x\to 1} \frac{3}{x^2(x-1)}(x-1) = 3, \quad \lim_{x\to 1} \frac{4}{x^2(x-1)^2}(x-1)^2 = 4 \qquad x = 1$ is a regular singular point

Problem 3

(a) $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t-1 & 1 \le t < 3 \\ 0 & t \ge 3 \end{cases}$



$$f(t) = 0 + (t-1)u_1 + (1-t)u_3$$

$$\mathcal{L}[f] = \mathcal{L}[(t-1)u_1 + u_3(-t+3-2)]$$
$$= e^{-s}\frac{1}{s^2} + e^{-3s}\cdot\left(-\frac{1}{s^2} - \frac{2}{s}\right)$$

(b) $F(s) = \dfrac{s-2}{s^2+2s+5} = \dfrac{s+1-3}{(s+1)^2+2^2}$, $\mathcal{L}^{-1}[F] = e^{-t}\cos 2t - \frac{3}{2}e^{-t}\cdot\sin 2t$

(c) $F(s) = \dfrac{s}{(s+2)^2(s^2+9)} = \dfrac{1}{(s+2)^2}\cdot\dfrac{s}{s^2+9}$

$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$, $\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$,

$$\mathcal{L}^{-1}[F] = \int_0^t (t-\tau)e^{-2(t-\tau)}\cdot\cos 3\tau\, d\tau$$

Problem 4

$y'' + 3y' + 2y = u_2(t)$, $y(0) = 0$, $y'(0) = 1$

$s^2Y - \cancel{sy(0)} - y'(0) + 3(sY - \cancel{y(0)}) + 2Y = \dfrac{e^{-2s}}{s}$

$(s^2+3s+2)Y = 1 + \dfrac{e^{-2s}}{s}$, $Y = \dfrac{1}{(s+2)(s+1)} + \dfrac{e^{-2s}}{s(s+1)(s+2)}$

$$= \frac{1}{s+1} - \frac{1}{s+2} + \left(\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right)e^{-2s}$$

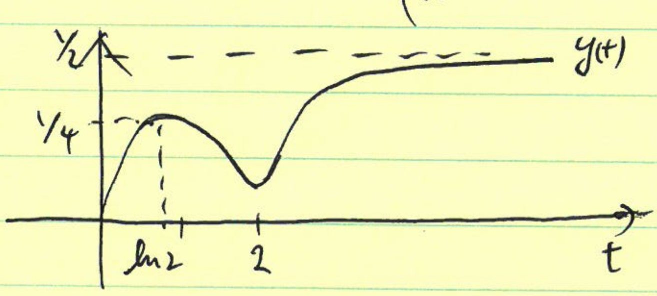
$$A(s+1)(s+2) + B\cdot s(s+2) + Cs\cdot(s+1) = 1$$

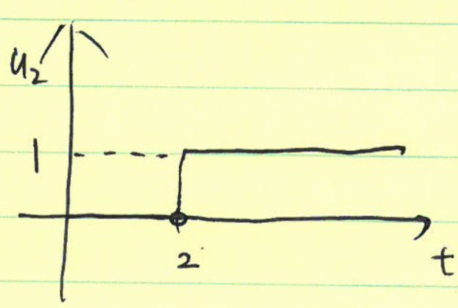
$s=0$, $2A = 1$, $A = 1/2$

$s=-1$, $-B = 1$, $B = -1$

$s=-2$, $-2\cdot(-1)\cdot C = 1$, $C = 1/2$

$$y = e^{-t} - e^{-2t} + \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)u_2$$





Problem 5 $y''+4y'+5y = a\,\delta(t-\frac{3\pi}{2}),\ y(0)=1,\ y'(0)=-2$

$$\mathcal{L}[y''+4y'+5y] = \mathcal{L}[a\delta(t-\tfrac{3\pi}{2})] = a\cdot e^{-\frac{3\pi}{2}s}$$

$$s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 5Y = a e^{-\frac{3\pi}{2}s}$$

$$(s^2+4s+5)Y - s+2-4 = a e^{-\frac{3\pi}{2}s}$$

$$(s^2+4s+5)Y = s+2 + a e^{-\frac{3\pi}{2}s}, \quad Y = \frac{s+2}{(s+2)^2+1} + \frac{a e^{-\frac{3\pi}{2}s}}{(s+2)^2+1}$$

$$y = e^{-2t}\cos t + a\cdot u_{\frac{3\pi}{2}}\cdot e^{-2(t-\frac{3\pi}{2})}\cdot \sin(t-\tfrac{3\pi}{2})$$

$$t > \tfrac{3\pi}{2},\ y = 0 = e^{-2t}\cos t + a\cdot e^{-2t}\cdot\cos t\cdot e^{3\pi}, \quad a\cdot e^{3\pi} = -1,$$

$$\boxed{a = -e^{-3\pi}}$$

Problem 6 (a) $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ $\det\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = 0$

$$(1-\lambda)(4-\lambda)+2=0, \quad (\lambda-1)(\lambda-4)+2=0,$$

$$\lambda^2-5\lambda+6=0, \quad (\lambda-3)(\lambda-2)=0, \quad \lambda=2,3$$

$\lambda=2$ $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $x_1-x_2=2x_1,\ x_1=-x_2 \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda=3$ $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $x_1-x_2=3x_2,\ x_1=4x_2 \Rightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$ $\det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} = 0$

$$(3-\lambda)(-2-\lambda)+4=0$$

$$(\lambda-3)(\lambda+2)+4=0, \quad \lambda^2-\lambda-2=0, \quad (\lambda-2)(\lambda+1)=0,$$

$$\lambda=-1,2$$

$\lambda=-1,$ $\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$ $3x_1-x_2=-x_1,\ 4x_1=x_2 \Rightarrow \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\lambda=2$ $\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$ $3x_1-x_2=2x_1,\ x_1=x_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x = c_1 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} c_1+c_2 \\ 4c_1+c_2 \end{pmatrix} \quad c_1=1,\ c_2=2$$

as $t\to\infty$ $x \to 2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \,/\!/\, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$