Grading Guidelines for M222 Final Exam May 6, 2016

Problem 1:

$$\det\begin{vmatrix} -2-\lambda & 3 \\ 1 & -4-\lambda \end{vmatrix} = 0, \quad (-2-\lambda)(-4-\lambda)-3=0.$$

$$(2+\lambda)(\lambda+4)-3=0, \quad \lambda^2+6\lambda+8-3=0, \quad \lambda^2+6\lambda+5=0,$$
$$(\lambda+5)(\lambda+1)=0,$$

$\lambda = -5$, $\boxed{2\text{pts}}$

$\lambda = -1$, $\boxed{2\text{pts}}$

$\lambda=-5$ $$\begin{pmatrix} -2+5 & 3 \\ 1 & -4+5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 3x_1+3x_2=0, \quad \frac{x_1}{x_2} = \frac{-1}{1}$$

corresponding eigenvector $\Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\boxed{2\text{pts}}$

$\lambda=-1$ $$\begin{pmatrix} -2+1 & 3 \\ 1 & -4+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad -x_1+3x_2=0, \quad \frac{x_1}{x_2} = \frac{3}{1}$$

Corresponding eigenvector $\Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\boxed{2\text{pts}}$

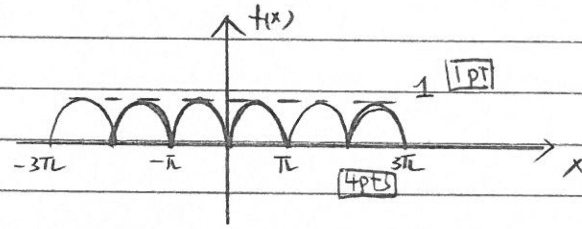
$$X(t) = c_1 e^{-5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{as } t\to\infty, \quad e^{-5t}\to 0, \quad e^{-t}\to 0$$

$$X(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{as } t\to\infty.$$ $\boxed{2\text{pts}}$

Problem 2:

$$f(x) = \begin{cases} -\sin x & -\pi \le x < 0 \\ \sin x & 0 \le x \le \pi \end{cases}$$

(a)



$2L = 2\pi, \quad L = \pi$

(b) $$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos\frac{n\pi x}{\pi} = \frac{a_0}{2} + \sum_{n=1} a_n \cos(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\sin 2x\,dx \qquad n=1$$

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin(1+n)x - \sin(1-n)x}{2}\,dx \qquad n>1$$

$$a_n = \frac{1}{\pi}\left(-\frac{1}{2}\cos 2x \Big|_0^\pi\right) + \frac{1}{\pi}\left(\frac{-1}{1+n}\cos(1+n)x - \frac{-1}{1-n}\cos(1-n)x\right)\Big|_0^\pi$$

$$a_n = -\frac{1}{(n+1)\pi}\left(\cos(n+1)\pi - 1\right) + \frac{1}{(1-n)\pi}\left(\cos(1-n)\pi - 1\right) \quad \boxed{4\text{pts}} \qquad n > 1$$

$$a_1 = \frac{1}{\pi}\left(-\frac{1}{2}\cos 2x\right)\Big|_0^\pi = 0 \quad \boxed{2\text{pts}} \qquad n = 1$$

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}(-\cos x)\Big|_0^\pi = \frac{2}{\pi}(1+1) = \frac{4}{\pi} \quad \boxed{2\text{pts}}$$

Problem 3: $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

$r^2 + \lambda = 0, \quad r = \pm\sqrt{-\lambda}, \qquad \lambda = 0, \quad y = Ax + B, \quad y'(0) = A = 0,$

$y(L) = B = 0$

$\Rightarrow \lambda = 0$ is not an eigenvalue $\boxed{2\text{pts}}$

$\lambda > 0, \quad y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$y' = -\sqrt{\lambda}\,(A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x)$

$y'(0) = 0 = B(-\sqrt{\lambda}), \quad B = 0$

$y(L) = 0 = A\cos\sqrt{\lambda}L = 0, \qquad \sqrt{\lambda}L = n\pi + \frac{\pi}{2}, \quad \lambda = \left(n+\frac{1}{2}\right)^2\left(\frac{\pi}{L}\right)^2 \quad \boxed{3\text{pts}}$

corresponding eigenfunction $\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right) \quad \boxed{3\text{pts}}$

$\lambda < 0 \qquad y = A\cdot e^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$y' = \sqrt{-\lambda}\,(Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x})$

$y'(0) = 0 = (A-B)\sqrt{-\lambda}, \quad A = B$

$\cancel{y(L) = 0 =} \quad A(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}) = 0, \quad A = 0$

$\Rightarrow \lambda < 0$ is not an eigenvalue $\boxed{2\text{pts}}$

Problem 4: (a) $ty' = \frac{1}{y+1}, \quad y(1) = 0,$

$(y+1)y' = \frac{1}{t} \quad \boxed{3\text{pts}} \qquad \frac{y^2}{2} + y = \ln t + C, \quad \frac{0^2}{2} + 0 = \ln 1 + C, \quad C = 0.$

$\frac{y^2}{2} + y = \ln t, \quad \boxed{2\text{pts}} \qquad y = \frac{-1 \pm\sqrt{1 - 4\cdot\frac{1}{2}\ln t}}{2\cdot\frac{1}{2}} = \frac{-1 \pm\sqrt{1-2\ln t}}{1}$

$y(1) = 0, \quad \boxed{y = -1 + \sqrt{1 - 2\ln t}} \quad \boxed{1\text{pt}}$

(b) $y' - y = e^{at}, \quad y(0) = 0$

$(e^{-t}y)' = e^{(a-1)t}$, $\quad e^{-t}y = \int e^{(a-1)t}dt$

$a \neq 1$, $\quad e^{-t}y = \frac{1}{a-1}e^{(a-1)t} + c$, $\quad y = \frac{1}{a-1}e^{at} + ce^{t}$ $\boxed{3pts}$

$y(0) = 0 = \frac{1}{a-1} + c$, $\quad c = -\frac{1}{a-1}$

$$\boxed{y = \frac{1}{a-1}(e^{at} - e^{t})} \quad \boxed{2pts}$$

$a = 1 \quad e^{-t}y = t + c$

$y = te^{t} + ce^{t} \quad y(0) = 0, \quad c = 0, \quad \boxed{y(t) = te^{t}}$ $\boxed{3pts}$

Problem 5: (a) $y'' - 2y' - 3y = e^{-t}$, $\quad r^2 - 2r - 3 = 0$, $\quad (r-3)(r+1) = 0$, $\quad r = -1, 3$ $\boxed{4pts}$

$y = c_1e^{-t} + c_2e^{3t} + Ate^{-t}$

$Y = Ate^{-t}$, $\quad Y' = Ae^{-t} - Ate^{-t}$, $\quad Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

$Ate^{-t} - 2Ae^{-t} - 2(Ae^{-t} - Ate^{-t}) - 3Ate^{-t} = e^{-t}$

$-2A - 2A = 1$, $\quad -4A = 1$, $\quad A = -\frac{1}{4}$ $\boxed{3pts}$

$y = c_1e^{-t} + c_2e^{3t} - \frac{1}{4}te^{-t}$

(b) $y_2 = ve^{t}$, $\quad y_2' = v'e^{t} + ve^{t}$, $\quad y_2'' = v''e^{t} + v'e^{t} + ve^{t} + v'e^{t}$

$\boxed{2pts}$

$(t-1)\cdot(v''e^{t} + 2v'e^{t} + ve^{t}) - t(v'e^{t} + ve^{t}) + ve^{t} = 0$

$(t-1)v'' + (2(t-1) - t)v' = 0$

$(t-1)v'' + (t-2)v' = 0$

$\frac{v''}{v'} = -\frac{t-2}{t-1} = -1 + \frac{1}{t-1}$

$\ln v' = -t + \ln(t-1)$ $\quad \boxed{4pts}$

$v' = (t-1)\cdot e^{-t} = te^{-t} - e^{-t}$,

$v = \int te^{-t}dt + e^{-t}$

$= t(-e^{-t}) - \int -e^{-t} + e^{-t}$

$= -te^{-t} - e^{-t} + e^{-t} = -te^{-t}$

$y_2 = -te^{-t}\cdot e^{t} = -t$

$$\boxed{y = c_1e^{t} + c_2t} \quad \boxed{4pts}$$

Problem 6: $y'' + (x-1)y = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (x-1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} - a_n x^n = 0 \qquad \boxed{4pts}$$

x^0: $2\cdot 1\cdot a_2 - a_0 = 0, \quad a_2 = \frac{1}{2}a_0$

x^1: $3\cdot 2\cdot a_3 + a_0 - a_1 = 0, \quad a_3 = \frac{1}{6}(a_1 - a_0)$

x^2: $4\cdot 3\cdot a_4 + a_1 - a_2 = 0, \quad a_4 = \frac{1}{12}(a_2 - a_1)$

x^n: $(n+2)(n+1)a_{n+2} + a_{n-1} - a_n = 0$

$$a_{n+2} = \frac{1}{(n+2)(n+1)}(a_n - a_{n-1}) \qquad \boxed{2pts}$$

y_1: $a_1 = 0,\ a_0 \neq 0 \quad y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots \qquad \boxed{2pts}$

y_2: $a_1 \neq 0,\ a_0 = 0, \quad y_2 = x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \dots \qquad \boxed{2pts}$

Problem 7: $y'' + \frac{1}{2}y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$

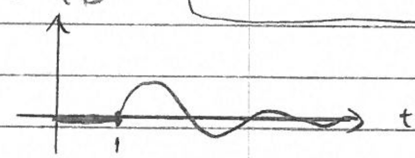
$\mathcal{L}[y'' + \frac{1}{2}y' + y] = e^{-s}$

$(s^2 + \frac{1}{2}s + 1)Y = e^{-s}, \quad Y = \frac{e^{-s}}{s^2 + \frac{1}{2}s + 1} = \frac{e^{-s}}{(s+\frac{1}{4})^2 + 1 - \frac{1}{16}}$ $\boxed{4pts}$ $\boxed{3pts}$

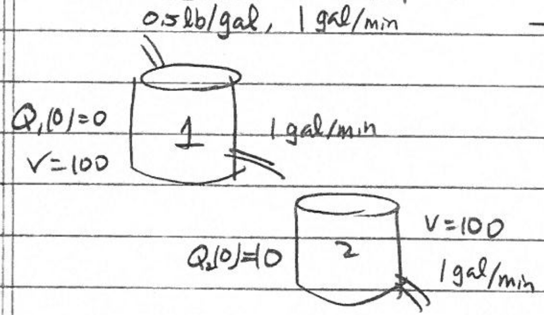
$$Y = \frac{e^{-s}}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}, \quad y(t) = \mathcal{L}^{-1}\left[e^{-s}\cdot \frac{1}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}\right] = \boxed{\frac{4}{\sqrt{15}}\cdot e^{-\frac{1}{4}t}\cdot \sin\left(\frac{\sqrt{15}}{4}t\right) \equiv f(t)} \qquad \boxed{4pts}$$

$$\boxed{y(t) = u_1(t)\cdot f(t-1)} \qquad \boxed{4pts}$$



Problem 8:



$$\frac{dQ_1}{dt} = 0.5\cdot 1 - \frac{Q_1}{100} = \frac{1}{2} - \frac{Q_1}{100} \qquad \boxed{2pts}$$

$$\frac{dQ_2}{dt} = \frac{Q_1}{100} - \frac{Q_2}{100} \qquad \boxed{2pts}$$

$$Q_1' + \frac{Q_1}{100} = \frac{1}{2}, \quad \left(Q_1 e^{\frac{t}{100}}\right)' = \frac{1}{2}e^{\frac{t}{100}}, \quad Q_1 \cdot e^{\frac{t}{100}} = 50\, e^{\frac{t}{100}} + c$$

$$Q_1 = 50 + c\cdot e^{-t/100}, \quad c = -50$$

$$\boxed{Q_1 = 50(1 - e^{-t/100})} \qquad \boxed{3pts}$$

p.5

$$Q_2' + \frac{Q_2}{100} = \frac{1}{2}\left(1 - e^{-t/100}\right),$$

$$\left(Q_2 \cdot e^{\frac{t}{100}}\right)' = \frac{1}{2}\left(1 - e^{-t/100}\right) e^{t/100} = \frac{1}{2}\left(e^{t/100} - 1\right)$$

$$Q_2 \cdot e^{\frac{t}{100}} = 50 e^{\frac{t}{100}} - \frac{t}{2} + c$$

$$Q_2 = 50 - \frac{t}{2} e^{-t/100} + c \cdot e^{-t/100}$$

$$Q_2(0) = 10 = 50 + c, \quad c = -40$$

$$\boxed{Q_2 = 50 - \frac{t}{2} e^{-t/100} - 40 e^{-t/100}}$$ 3pts