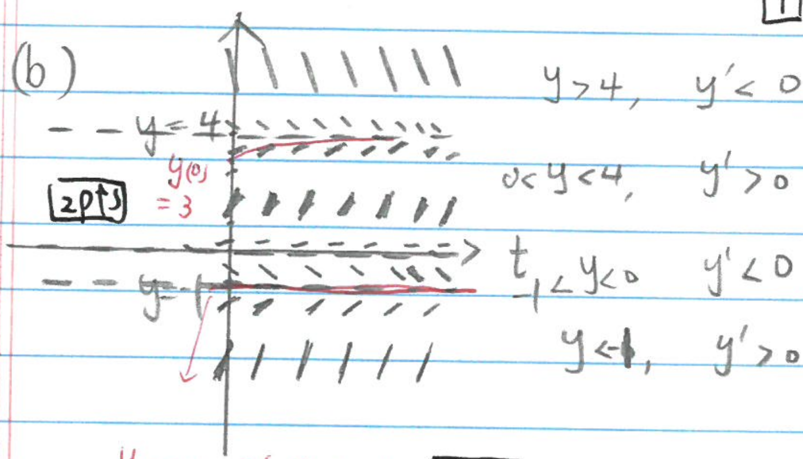
Solutions Exam 1 Math 222 09/28/2016

Problem 1: $y' = -y(y+1)(y-4)$

(a) $y' = 0 = -y(y+1)(y-4)$, $y = -1, 0, 4$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$

(b)



$\boxed{3}$ points for the drawing of direction field

$y = -1$ if $y(0) = -1$ $\boxed{2\text{pts}}$

Problem 2: (a) $\frac{d}{dx}(xy') = \ln(xy)$ 2nd order $\boxed{2\text{pts}}$, nonlinear $\boxed{2\text{pts}}$

(b) $y'' + 4y = 0$, $y_1 = \cos 2t$, $y_1'' = -4\cos 2t$

y_1 is a solution because $y_1'' + 4y_1 = -4\cos 2t + 4\cos 2t = 0$ $\boxed{3\text{pts}}$

$y_2 = \sin 2t$, $y_2'' = -4\sin 2t$

y_2 is a solution because $y_2'' + 4y_2 = -4\sin 2t + 4\sin 2t = 0$ $\boxed{3\text{pts}}$

Problem 3(a): $\frac{dy}{dx} = 2(1+x)(1-y^2)$ $\qquad \frac{dy}{1-y^2} = 2(1+x)\,dx$

$$\frac{1}{2}\int \frac{1}{1+y} + \frac{1}{1-y}\,dy = \int 2(1+x)\,dx \qquad \boxed{5\text{pts}}$$

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = (1+x)^2 + C \qquad \boxed{4\text{pts}}$$

$$\ln\left|\frac{1+y}{1-y}\right| = 2(1+x)^2 + C$$

$$\left|\frac{1+y}{1-y}\right| = A\cdot e^{2(1+x)^2}$$

$y(0)=-2$, $y'|_0 < 0$, $y' = 2(1+x)(1-y^2)$, $1+x>0$ for $x>0$

therefore as x increase above zero, y' stays negative

y decreases below -2 as x increase from zero

$$\begin{matrix} 1+y<0 \\ 1-y>0 \end{matrix} \Rightarrow -\frac{1+y}{1-y} = A\,e^{2(1+x)^2}$$

$y(0)=-2$, $-\frac{-1}{3} = Ae^2$, $\frac{1}{3} = Ae^2$ $\boxed{A=\frac{1}{3e^2}}$

$$\frac{1+y}{1-y} = -\frac{1}{3e^2}e^{2(1+x)^2} = -\frac{1}{3}e^{2((1+x)^2-1)}$$ $\boxed{4\text{pts}}$

$$1+y = -\frac{1}{3}e^{2((1+x)^2-1)}(1-y)$$

$$y\left(1-\frac{1}{3}e^{2((1+x)^2-1)}\right) = -1-\frac{1}{3}e^{2((1+x)^2-1)}$$

$$y = -\frac{1+\frac{1}{3}e^{2(1+x)^2-2}}{1-\frac{1}{3}e^{2(1+x)^2-2}}$$ $\boxed{1\text{ pts}}$

as x increases from zero, the denominator may vanish $\Rightarrow 1-\frac{1}{3}e^{2(1+x)^2-2}=0$

$$1 = \frac{1}{3}e^{2(1+x)^2-2} \qquad 3 = e^{2(1+x)^2-2}$$

$$\ln 3 = 2(1+x)^2-2 \qquad 2+\ln 3 = 2(1+x)^2$$

$$(1+x) = \pm\sqrt{\frac{2+\ln 3}{2}}, \qquad x = \sqrt{\frac{2+\ln 3}{2}}-1 \sim 0.2447$$

because $x>0$

the solution $y = -\dfrac{1+\frac{1}{3}e^{2(1+x)^2-2}}{1-\frac{1}{3}e^{2(1+x)^2-2}} \rightarrow -\infty$ as x increases toward $\sqrt{\frac{2+\ln 3}{2}}-1$

$\boxed{1pt}$

The original problem 3(a) was asking for a sketch of the solution. However, if any student notices that the solution of the IVP does not extend beyond the asymptote $x=\sqrt{\frac{2+\ln 3}{2}}-1$, the student earns one more point.

Problem 3(b) $y'-y=-\dfrac{e^t}{2}$,

$\mu = e^{\int -dt} = e^{-t}$, $\boxed{5 \text{ pts}}$

$$y = \frac{\int e^{-t}\cdot\left(-\frac{e^t}{2}\right)dt + C}{e^{-t}} = \frac{-\frac{t}{2}+c}{e^{-t}} \quad \boxed{7 \text{ pts}}$$

$y = -\frac{t}{2}e^t + ce^t$, $y'=0$ at $t=2$

$y' = y + \left(-\frac{e^t}{2}\right) = -\frac{t}{2}e^t + ce^t - \frac{e^t}{2}$

$y'=0$ at $t=2$, $0 = -e^2 + ce^2 - \frac{e^2}{2}$, $0=-1-\frac{1}{2}+c$,

$c=\frac{3}{2}$ $\quad y = -\frac{t}{2}e^t + \frac{3}{2}e^t$

$y(0)=y_0=\frac{3}{2}$ $\boxed{2 \text{ pts}}$

Problem 4 $V' = 4-2=2$, $V = 2t + V(0) = 2t + 1$ (L)

$\boxed{3 \text{ pts}}$

$Q' = 4\cdot 1 - 2\cdot\dfrac{Q}{2t+1}$ $\boxed{3 \text{ pts}}$

$Q' + \frac{2Q}{2t+1} = 4, \quad Q(0) = 10\ g$

$\mu = e^{\int \frac{2}{2t+1}} = e^{\ln(2t+1)} = 2t+1$ $\boxed{3\text{ pts}}$

$Q = \frac{\int 4\cdot(2t+1)dt + C}{2t+1} = \frac{(2t+1)^2 + C}{2t+1} = 2t+1 + \frac{C}{2t+1}$

$Q(0) = 10 = 1 + \frac{C}{2}, \quad C = 18$ $\boxed{2\text{ pts}}$

$Q = 2t+1 + \frac{18}{2t+1}$

$\boxed{3\text{ pts}}$

When the tank is overflowing, $V = 9 = 2t+1, \ t = 4\ (hr)$

$Q(4) = 9 + \frac{18}{9} = 9+2 = 11$

concentration $\Rightarrow \left.\frac{Q}{V}\right|_{t=4} = \frac{11}{9}\left(\frac{g}{L}\right)$ $\boxed{1\text{ pt}}$

Problem 5: $y' = y^2 - t^2, \quad y(0) = 1$

$\boxed{4\text{pts}}$

$y_{n+1} = y_n + h\cdot(y_n^2 - t_n^2) \qquad y_0 = 1, \ t_0 = 0, \ h = 0.1$

$t_1 = 0.1 \qquad y_1 = y_0 + 0.1\cdot(y_0^2 - 0^2) = 1 + 0.1\cdot(1^2 - 0^2) = 1.1$

$\boxed{2\text{pts}}$

$t_2 = 0.2 \qquad y_2 = 1.1 + 0.1\times(1.1^2 - 0.1^2)$ $\boxed{2\text{pts}}$

$= 1.1 + 0.1\times(1.21 - 0.01) = 1.1 + 0.12 = 1.22$

$t_3 = 0.3 \qquad y_3 = 1.22 + 0.1\times(1.22^2 - 0.2^2)$ $\boxed{2\text{pts}}$ $\quad 1.22\times1.22 =$

$= 1.22 + 0.1\times(1.4884 - 0.04)$

$= 1.22 + 0.14484 = 1.36484$

$$\begin{array}{r} 1.22 \\ \times\ 1.22 \\ \hline 244 \\ 244\ \ \\ 122\ \ \ \ \\ \hline 14884 \end{array}$$

Problem 6(a) $(r+1)(r-3)=0$, $r^2-2r-3=0$, $y''-2y'-3y=0$

or any nonzero constant $c\cdot(y''-2y'-3y)=0$ $\boxed{7\text{ pts}}$

$y=c_1e^{-t}+c_2e^{3t}$ $\boxed{4\text{pts}}$ $y(0)=c_1+c_2=\alpha$

$y'=-c_1e^{-t}+3c_2e^{3t}$ $y'(0)=-c_1+3c_2=\beta$

want $c_2=0$ $c_1=\boxed{\alpha=-\beta}$ $\boxed{4\text{pts}}$

(b) $y''-4y=0$ $y(0)=1$, $y'(0)=0$

$r^2-4=0$

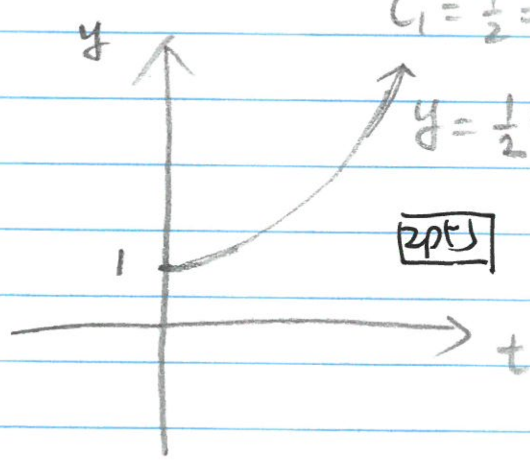
$r^2=4$, $r=\pm 2$, $\boxed{4\text{ pts}}$

$y=c_1e^{-2t}+c_2e^{2t}$ $y(0)=1=c_1+c_2$

$y'=-2c_1e^{-2t}+2c_2e^{2t}$ $y'(0)=0=-2c_1+2c_2$

$c_1=\frac{1}{2}=c_2$ $\boxed{4\text{pts}}$

$y=\frac{1}{2}(e^{2t}+e^{-2t})=\cosh(2t)$



$\boxed{2\text{pts}}$