Math 222 Exam 2, October 26, 2016

Read each problem carefully. Show all your work. Turn off your phones. No calculators!

1. Consider the differential equation $x^2y'' - 2y = \frac{3}{x} + x^2$.

   (a) (5 points) Find r such that $y = x^r$ is a solution of the corresponding homogeneous equation: $x^2y'' - 2y = 0$.

   (b) (10 points) Based on your results from (a), find the general solution for the full equation.

2. Consider the differential equation $(t-1)y'' - ty' + y = F(t)$.

   (a) (10 points) For $F(t) = 0$, given $y_1 = e^t$ is a solution, find another linearly independent solution via reduction of order.

   (b) (5 points) For $F(t) = 0$, use Abel's theorem to find the Wronskian of the two linearly independent solutions y_1 and y_2.

   (c) (5 points) For $F(t) = \ln(t)\tan(t)$ and initial values $y(1/2) = y'(1/2) = 5$, what is the largest interval for t that will guarantee a unique solution for the IVP?

3. Consider the differential equation $y'' - 10y' + 25y = 10e^{5x}$. (7 points) First find the general solution for the equation. (8 points) Then find the solution to the equation with initial values $y(0) = 0$ and $y'(0) = 1$.

4. Use the method of undetermined coefficients for the following problems.

   (a) (10 points) Find the correct form for the particular solution of the differential equation $y'' + 4y = t^2 \sin 2t + (6t + 7)\cos 2t$ (do not compute the coefficients).

   (b) (10 points) Find the solution to the IVP: $y'' + 2y' = 3 + e^{-2t}$, $y(0) = 0$, and $y'(0) = 0$.

5. (15 points) A mass weighing 16 lb stretches a spring 8/17 ft. The mass is attached to a viscous damper with a damping constant of 2 lb s /ft. The mass is set in motion from its equilibrium position with an upward velocity of 6 in/s. Find the position u of the mass at any time t. Plot u versus t. Note that in these units the gravitational acceleration $g = 32$ ft/s^2.

6. (15 points) A mass weighing 4 lb stretches a spring 1.5 in. This undamped spring-mass system is suddenly set in motion from rest at equilibrium at $t = 0$ by an external force of $3\cos 15t$ lb. Determine the position of the mass at any time, and draw a graph of the displacement versus t. Note that in these units the gravitational acceleration $g = 32$ ft/s^2.

Problem 1: $x^2y'' - 2y = \frac{3}{x} + x^2$

(a) $x^2y'' - 2y = 0$, $y = x^r$, $r(r-1) - 2 = 0$, $r^2 - r - 2 = 0$ $\boxed{3\text{pts}}$

$(r-2)(r+1) = 0$, $r = -1, 2$ $\boxed{2\text{pts}}$

$y_1 = x^{-1}$, $y_2 = x^2$

(b) $W[y_1, y_2] = \begin{vmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{vmatrix} = 2 - (-1) = 3$ $\boxed{3\text{pts}}$

$$Y = \left(\int \frac{-x^2 \cdot \left(\frac{3}{x^3} + 1\right)}{3} dx \right) x^{-1} + \left(\int \frac{x^{-1} \cdot \left(\frac{3}{x^3} + 1\right)}{3} dx \right) x^2 \quad \boxed{3\text{pts}}$$

$$= \left(\int -\frac{1}{x} - \frac{x^2}{3} dx \right) \frac{1}{x} + \left(\int \frac{\frac{3}{x^4} + \frac{1}{x}}{3} dx \right) \cdot x^2$$

$$= \left(-\ln x - \frac{1}{3}x^3 \right) \frac{1}{x} + \left(-\frac{1}{3} \frac{1}{x^3} + \frac{\ln x}{3} \right) x^2$$

$$= -\frac{1}{x}\ln x + \frac{x^2}{3}\ln x - \frac{1}{3}x^2 - \frac{1}{3x}$$

general solution $y = \frac{c_1}{x} + c_2 x^2 - \frac{1}{x}\ln x + \frac{x^2}{3}\ln x$ $\boxed{4\text{pts}}$

Problem 2: $(t-1)y'' - ty' + y = F(t)$

(a) $F(t) = 0$, $y_1 = e^t$, $y_2 = ve^t = vy_1$ $\boxed{2\text{ pts}}$

$y_2' = v'y_1 + vy_1'$, $y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$$(t-1)(v''y_1 + 2v'y_1' + vy_1'') - t(v'y_1 + vy_1') + vy_1 = 0 \quad \boxed{3\text{pts}}$$

$y_1 = e^t$,

$$(t-1)y_1 \cdot v'' + \left(2(t-1)y_1' - ty_1\right)v' = 0$$

$$(t-1)v'' + (2t-2-t)v' = 0$$

$$\frac{v''}{v'} = -\frac{t-2}{t-1} = -\frac{t-1-1}{t-1} = -1 + \frac{1}{t-1}$$

$\ln v' = -t + \ln(t-1)$, $v' = (t-1)e^{-t}$, $\boxed{3\text{pts}}$

$$v = \int (t-1)e^{-t} dt = \int te^{-t} dt - \int e^{-t} dt$$

$$= t(-e^{-t}) - \int -e^{-t} dt - \int e^{-t} dt$$

$\boxed{v = -te^{-t}}$ $y_2 = vy_1 = -t$ $\boxed{2\text{pts}}$

p.02

(b) $y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = 0$

$P(t) = -\frac{t}{t-1}$ $\qquad W = c \cdot e^{-\int \frac{-t}{t-1}dt}$ $\boxed{3pts}$

$$\int \frac{t}{t-1}dt = \int \frac{t-1+1}{t-1}dt = \int 1 + \frac{1}{t-1}dt = t + \ln(t-1)$$

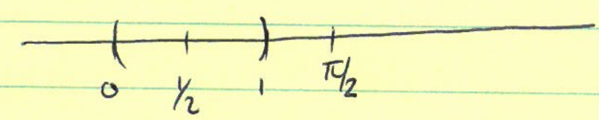
$W = c \cdot e^{t+\ln(t-1)} = c \cdot e^t \cdot (t-1)$ $\boxed{2pts}$

check : $W[e^t, -t] = \begin{vmatrix} e^t & -t \\ e^t & -1 \end{vmatrix} = -e^t + te^t$
$= (-1+t)e^t$

(c) $F(t) = \ln t \cdot \tan t$, $\quad y(1/2) = y'(1/2) = 5$

$$y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = \frac{\ln t \cdot \tan t}{t-1}$$ $\boxed{3pts}$

coefficients diverge at $t=0$
$t=1$
$t=\pi/2 + n\pi$



$\boxed{2pts}$

$1/2$ is in the interval $(0,1)$ $\Rightarrow$ largest interval for t that will guarantee a unique solution.

Problem 3 $\quad y'' - 10y' + 25y = 10e^{5x}$

y_c : $r^2 - 10r + 25 = 0$, $(r-5)^2 = 0$, $r = 5, 5$

$y_c = c_1e^{5x} + c_2xe^{5x}$ $\boxed{4pts}$

$Y = A \cdot x^2e^{5x}$, $\quad Y' = A \cdot 2x \cdot e^{5x} + 5Ax^2e^{5x}$

$= Ae^{5x}(2x + 5x^2)$

$Y'' = 5Ae^{5x}(2x+5x^2) + Ae^{5x}(2+10x)$

$= Ae^{5x}(10x + 25x^2 + 2 + 10x) = Ae^{5x}(2 + 20x + 25x^2)$

$Y'' - 10Y' + 25Y = 10e^{5x}$

$Ae^{5x}(2 + 20x + 25x^2) - 10Ae^{5x}(2x + 5x^2) + 25Ax^2e^{5x} = 10e^{5x}$

x^2: $\quad 25A - 50A + 25A \quad = 0$

x^1: $\quad 20A - 20A \quad = 0$

x^0: $\quad 2A \quad = 10$, $\boxed{A=5}$ $\boxed{3pts}$

general solution $y = C_1 e^{5x} + C_2 x e^{5x} + 5x^2 e^{5x}$ 4pts

$y' = 5C_1 e^{5x} + C_2 e^{5x} + 5C_2 x e^{5x} + 10x e^{5x} + 25x^2 e^{5x}$

$y(0) = 0 = C_1$

$y'(0) = 1 = C_2$ 4pts

$\boxed{y(x) = x e^{5x} + 5x^2 e^{5x}}$

Problem 4: (a) $y'' + 4y = t^2 \sin 2t + (6t+7)\cos 2t$

$y'' + 4y = 0, \quad y_1 = \cos 2t, \quad y_2 = \sin 2t$ 4pts

$\boxed{Y = t(At^2 + Bt + C)\sin 2t + t(Dt^2 + Et + F)\cos 2t}$ 6pts

(b) $y'' + 2y' = 3 + e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0$

$r^2 + 2r = 0, \quad (r+2)r = 0, \quad r = 0, -2$ 2pts

$y_c = C_1 + C_2 e^{-2t}$ 2pts

$Y = t(A + Be^{-2t})$ 3pts

$Y' = A + Be^{-2t} - 2Bte^{-2t}$

$Y'' = -2Be^{-2t} - 2Be^{-2t} + 4Bte^{-2t}$

$(-2Be^{-2t} - 2Be^{-2t} + 4Bte^{-2t}) + 2(A + Be^{-2t} - 2Bte^{-2t}) = 3 + e^{-2t}$

$2A = 3, \quad A = 3/2$

$-2B = 1, \quad B = -1/2$

$y = C_1 + C_2 e^{-2t} + t\left(\frac{3}{2} - \frac{1}{2}e^{-2t}\right)$

$y' = -2C_2 e^{-2t} + \frac{3}{2} - \frac{1}{2}e^{-2t} + te^{-2t}$

$y(0) = 0 = C_1 + C_2$ $\quad C_1 = -\frac{1}{2}$

$y'(0) = -2C_2 + \frac{3}{2} - \frac{1}{2} = 0 \quad -2C_2 + 1 = 0, \quad C_2 = \frac{1}{2}$

$\boxed{y(t) = -\frac{1}{2} + \frac{1}{2}e^{-2t} + \frac{t}{2}\left(3 - e^{-2t}\right)}$ 3pts

P.04

Problem 5

$m = \frac{16}{32} = \frac{1}{2}$ $\boxed{1\text{ pt}}$

$16 = \frac{8}{17} \cdot k, \quad k = 34$ $\boxed{1\text{ pt}}$

$m \cdot u'' + \gamma u' + ku = 0, \quad u(0) = 0, \quad u'(0) = \frac{1}{2}$ ft/s $\boxed{4\text{ pts}}$

$\frac{1}{2}u'' + 2u' + 34u = 0$

$u'' + 4u' + 68u = 0$

$r^2 + 4r + 68 = 0, \quad (r+2)^2 + 68 - 4 = 0, \quad (r+2)^2 = -64 \Rightarrow r = -2 \pm 8i$

$u_1 = e^{-2t}\cos 8t$ $\boxed{4\text{ pts}}$

$u_2 = e^{-2t}\sin 8t$

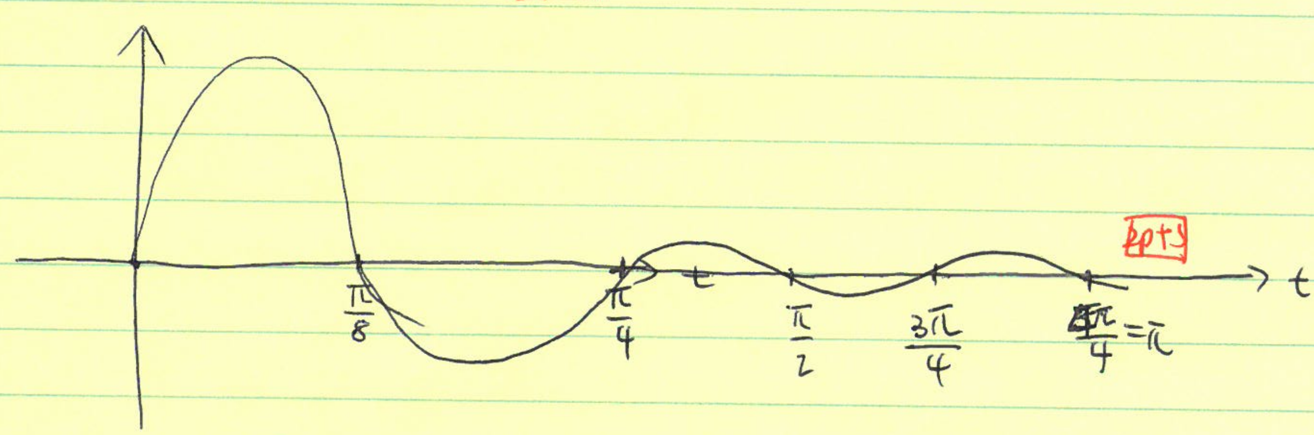
$u = c_1 u_1 + c_2 u_2 = e^{-2t}(c_1\cos 8t + c_2 \sin 8t)$

$u' = -2 \cdot e^{-2t} \cdot (c_1 \cos 8t + c_2 \sin 8t) + e^{-2t}(-8c_1 \sin 8t + 8c_2 \cos 8t)$

$u(0) = 0 = c_1$

$u'(0) = \frac{1}{2} = 8c_2, \quad c_2 = \frac{1}{16}$

$u(t) = \frac{1}{16} e^{-2t} \cdot \sin 8t$ $\boxed{3\text{ pts}}$



Problem 6:

$m = \frac{4}{32} = \frac{1}{8}$ $\boxed{1\text{ pt}}$ $\quad 4\text{ lb} = \frac{1.5}{12} \cdot k = \frac{3/2}{12}k = \frac{3}{24}k = \frac{1}{8}k, \quad k = 32$ $\boxed{1\text{ pt}}$

$\frac{1}{8}u'' + 32u = 3\cos 15t$ $\boxed{4\text{ pts}}$

$u'' + 256u = 24\cos 15t, \quad u(0) = 0, \quad u'(0) = 0$

$r^2 + 256 = 0, \quad r = \pm 16i$ $\boxed{4\text{ pts}}$

$u_c = c_1 \cos 16t + c_2 \sin 16t$

$U = A\cos 15t + B \sin 15t$

$U'' = -15^2(A\cos 15t + B\sin 15t)$

$-15^2 \cdot (A\cos 15t + B\sin 15t) + 256(A\cos 15t + B\sin 15t) = 3\cos 15t$

$A = 3, \quad B = 0$

p.05

$u = c_1 \cos 16t + c_2 \sin 16t + 3 \cos 15t$

$u' = 16(-c_1 \sin 16t + c_2 \cos 16t) + 45 \cancel{\cos 15} \sin 15t$

$u(0) = c_1 + 3 = 0 \quad , \quad c_1 = -3$

$u'(0) = c_2 = 0$

$u = -3 \cos 16t + 3 \cos 15t$

$A + B = 16t \qquad A = \frac{31}{2}t,$

$A - B = 15t \qquad B = \frac{t}{2}$

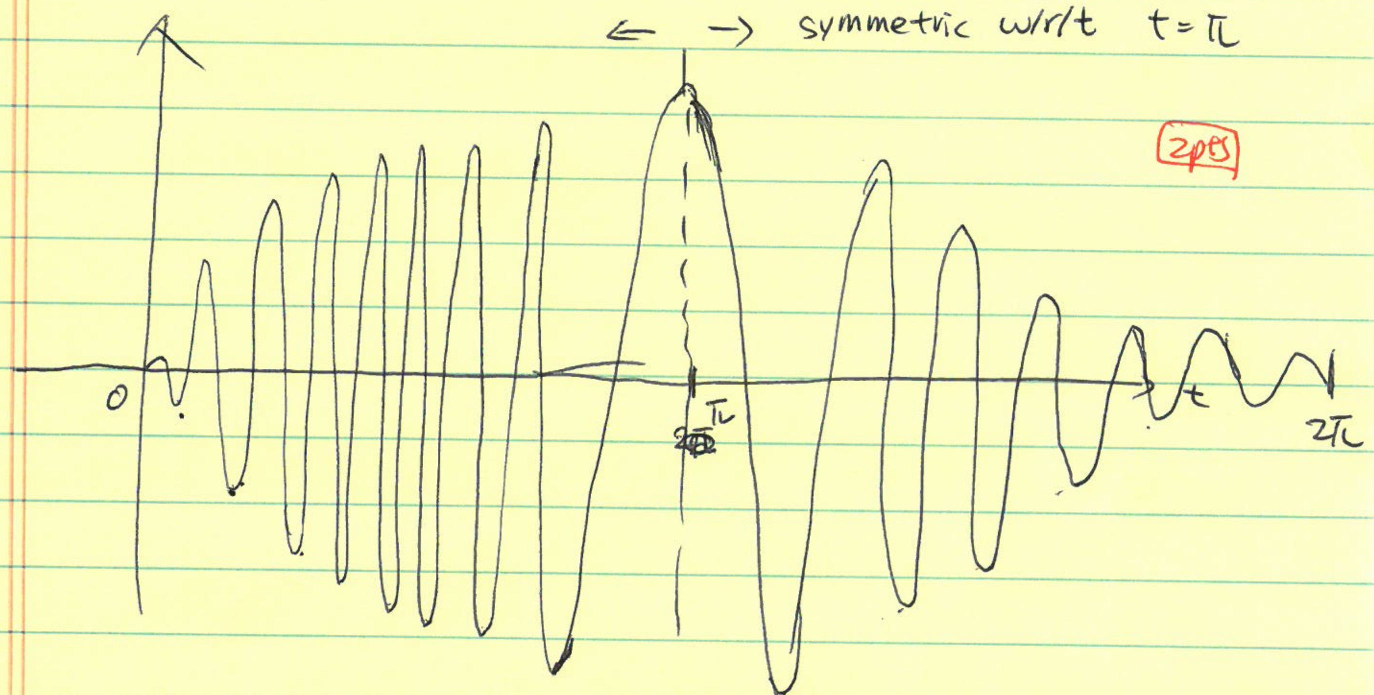
$u = -3(\cos 16t - \cos 15t)$

$= -3 \cdot \left(-2 \cdot \sin \frac{31}{2}t \cdot \sin \frac{t}{2}\right)$

$\boxed{u = 6 \sin \frac{31t}{2} \sin \frac{t}{2}}$

3pts

$\leftarrow \ \rightarrow$ symmetric w/r/t $t = \pi$



2pts