Math 222 Exam 3, November 30, 2016

Read each problem carefully. Show all your work. Turn off your phones. No calculators!

1. (a) (8 points) Use Laplace transform to solve the IVP:

$$y'' + y' = e^{-t}\cos t, \quad y(0) = 0, \quad y'(0) = 0.$$

(b) (12 points) Find the inverse Laplace transform of the following functions. Compute the integration if you use convolution integral to find the inverse.

$$\text{(i) } F(s) = \frac{e^{-s} - 3s}{s^2 + 4s + 5}, \quad \text{(ii) } G(s) = \frac{3se^{-s}}{(s^2+9)^2}.$$

2. (15 points) Find the solution of the initial value problem with $y(0) = 0$ and $y'(0) = 0$:

$$y'' + 2y' + 2y = g(t) = \begin{cases} 0, & 0 \le t < 5 \\ 1, & 5 \le t < 20, \\ 0, & t \ge 20. \end{cases}$$

3. Consider the IVP

$$y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

(a) (10 points) Express the solution of the given IVP in terms of a convolution integral.

(b) (10 points) If $g(t) = \delta(t - \frac{\pi}{4}) - \delta(t - \frac{\pi}{2})$, carry out the convolution integral in (a) to find the corresponding solution $y(t)$.

4. Consider the IVP:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} \alpha \\ 6 \end{pmatrix}.$$

(a) (8 points) Find the general solution.

(b) (7 points) Find the value of α such that $\vec{x} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

5. Consider the differential equation $y'' - xy = 0$.

(a) (8 points) Seek a power series solution around the point $x_0 = 0$. Find the recurrence relation.

(b) (7 points) Find the first two non-zero terms in each of the two solutions y_1 and y_2.

6. (a) (8 points) Show that $x = 1$ and $x = -1$ are regular singular points of the differential equation $(1 - x^2)y'' - xy' + y = 0$.

(b) (7 points) Find condition(s) on β so that all solutions of the Euler equation $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.

Grading Guidelines for Exam 3 M222 Fall 2016

Problem 1 (a) $y''+y' = e^{-t}\cos t \quad y(0)=0,\ y'(0)=0$

$$\mathcal{L}[y''+y'] = \mathcal{L}[e^{-t}\cos t] = \frac{s+1}{(s+1)^2+1}, \quad \boxed{2pts}$$

$$s^2Y+sY = \frac{s+1}{(s+1)^2+1}, \quad Y = \frac{1}{s((s+1)^2+1)} = \frac{A}{s} + \frac{Cs+D}{s^2+2s+2}$$

$$A(s^2+2s+2) + (Cs^2+Ds) = 1$$

$A + C = 0 \qquad C = -1/2$

$2A + D = 0 \qquad D = -1$

$2A = 1 \qquad A = 1/2$

$$Y = \frac{1}{2}\frac{1}{s} + \frac{-\frac{1}{2}s-1}{(s+1)^2+1} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1} \quad \boxed{3pts}$$

$$y(t) = \mathcal{L}^{-1}[Y] = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \quad \boxed{3pts}$$

(b)(i) $$F(s) = \frac{e^{-s}-3s}{s^2+4s+5} = \frac{e^{-s}-3(s+2)+6}{(s+2)^2+1} \quad \boxed{3\ pts}$$

$$\mathcal{L}^{-1}[F] = \underbrace{-3\cdot e^{-2t}\cos t}_{\boxed{1pt}} + \underbrace{6e^{-2t}\sin t}_{\boxed{1pt}} + \underbrace{u_1(t)e^{-2(t-1)}\sin(t-1)}_{\boxed{2pts}}$$

(ii) $$G(s) = \frac{3se^{-s}}{(s^2+9)^2} \qquad \frac{3s}{(s^2+9)^2} = -\frac{3}{2}\frac{d}{ds}\left(\frac{1}{s^2+9}\right) = -\frac{1}{2}\frac{d}{ds}\left(\frac{3}{s^2+3^2}\right) \quad \boxed{3pts}$$

$$\mathcal{L}^{-1}\left[\frac{3s}{(s^2+9)^2}\right] = -\frac{1}{2}\cdot(-t)\sin 3t = h(t) \quad \boxed{3pts}$$

$h(t) = \frac{t}{2}\sin 3t$

$$\mathcal{L}^{-1}[G] = \mathcal{L}\left[\frac{3s}{(s^2+9)^2}\cdot e^{-s}\right] = u_1(t)\,h(t-1) \quad \boxed{2pts}$$

Problem 2 $\mathcal{L}[y''+2y'+2y] = \mathcal{L}[g], \quad g = u_5 - u_{20}$

$$(s^2+2s+2)Y = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s} \quad \boxed{5pts}$$

$$Y = \left(e^{-5s} - e^{-20s}\right)\frac{1}{s(s^2+2s+2)} \quad \boxed{3pts}$$

From 1(a) $$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+2s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \equiv h(t) \quad \boxed{4pts}$$

$$y(t) = u_5\, h(t-5) - u_{20}\, h(t-20).\quad \boxed{3pts}$$

Problem 3

$$y'' + y = g(t),\quad y(0) = 1,\quad y'(0) = -1$$

(a) $\mathcal{L}[y''+y] = \mathcal{L}[g] = G$

$$s^2 Y - s y(0) - y'(0) + Y = G,\quad (s^2+1)Y = s - 1 + G,\quad \boxed{4pts}$$

$$Y = \frac{s-1}{s^2+1} + \underbrace{\frac{G}{s^2+1}}_{\boxed{2pts}},\qquad y(t) = \cos t - \sin t + \int_0^t \sin(t-\tau)\, g(\tau)\, d\tau \quad \boxed{4pts}$$

(b) $g(t) = \delta(t-\frac{\pi}{4}) - \delta(t-\frac{\pi}{2})$

$$y(t) = \cos t - \sin t + \int_0^t \sin(t-\tau)\, g(\tau)\, d\tau$$

$$= \cos t - \sin t + \int_0^t \sin(t-\tau)\cdot\left(\delta(\tau-\tfrac{\pi}{4}) - \delta(\tau-\tfrac{\pi}{2})\right) d\tau \quad \boxed{4pts}$$

$$= \cos t - \sin t + u_{\frac{\pi}{4}} \sin(t-\tfrac{\pi}{4}) - u_{\frac{\pi}{2}} \sin(t-\tfrac{\pi}{2}) \quad \boxed{6pts}$$

Problem 4

$$x' = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix} x,\quad x(0) = \begin{pmatrix} \alpha \\ 6 \end{pmatrix}$$

(a) $\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 2 \\ -3 & -4-\lambda \end{pmatrix} = 0,\quad (3-\lambda)(-4-\lambda) + 6 = 0$

$\cancel{\lambda^2 + \lambda + 12 + 6} = 0 \quad (\lambda+4)(\lambda-3) + 6 = 0,\quad \lambda^2 + \lambda - 6 = 0\quad (\lambda+3)(\lambda-2) = 0,$

$\lambda = -3, 2\quad \boxed{4pts}$

$\lambda = -3$: $\begin{pmatrix} 3+3 & 2 \\ -3 & -4+3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad 6x_1 + 2x_2 = 0,\quad 3x_1 + x_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad x_1 + 2x_2 = 0 \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$x(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \boxed{4pts}$$

(b) want $\underbrace{C_2 = 0}_{\boxed{3pts}}$ $\quad \begin{pmatrix} C_1 \\ -3C_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 6 \end{pmatrix}\quad C_1 = \alpha,\ -3C_1 = 6,\quad C_1 = -2 = \alpha \quad \boxed{4pts}$

Problem 5: $y'' - xy = 0, \quad y = \sum_{n=0}^{\infty} a_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

(a)

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

x^0: $2 \cdot 1 \cdot a_2 = 0$, $\quad a_2 = 0$ $\boxed{2pts}$

x^1: $3 \cdot 2 \cdot a_3 - a_0 = 0$, $\quad a_3 = \frac{a_0}{3 \cdot 2}$

x^2: $4 \cdot 3 \cdot a_4 - a_1 = 0$, $\quad a_4 = \frac{a_1}{4 \cdot 3}$

x^3: $5 \cdot 4 \cdot a_5 - a_2 = 0$, $\quad a_5 = \frac{a_2}{5 \cdot 4} = 0$

x^4: $6 \cdot 5 \cdot a_6 - a_3 = 0$, $\quad a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$

x^5: $7 \cdot 6 \cdot a_7 - a_4 = 0$, $\quad a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$

$\boxed{6pts}$

x^n: $(n+2)(n+1) a_{n+2} - a_{n-1} = 0$, $\quad \boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad n = 1, 2, \cdots}$

(b) $y_1 = a_0 \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \cdots\right)$ $\boxed{3.5pts}$

$y_2 = a_1 \left(x + \frac{1}{12} x^4 + \frac{1}{504} x^7 + \cdots\right)$ $\boxed{3.5pts}$

Problem 6: (a) $y'' - \frac{x}{1-x^2} y' + \frac{1}{1-x^2} y = 0$

$x = 1$ $\quad \lim_{x \to 1} \frac{-x}{1-x^2}(x-1) = \lim_{x \to 1} \frac{-x}{(1+x)(1-x)}(x-1) = \frac{1}{2}$ $\boxed{2pts}$

$\lim_{x \to 1} \frac{1}{1-x^2}(x-1)^2 = \lim_{x \to 1} \frac{1}{(1+x)(1-x)}(x-1)^2 = \cancel{\frac{1}{2}}\ 0$ $\boxed{2pts}$

$x = 1$ is a regular singular point

$x = -1$ $\quad \lim_{x \to -1} \frac{-x}{1-x^2}(x+1) = \lim_{x \to -1} \frac{-x}{(1+x)(1-x)}(x+1) = \frac{1}{2}$ $\boxed{2pts}$

$\lim_{x \to -1} \frac{1}{1-x^2}(x+1)^2 = \lim_{x \to -1} \frac{1}{(1+x)(1-x)}(x+1)^2 = 0$ $\boxed{2pts}$

$x = -1$ is a regular singular point.

(b) $x^2y'' + \beta y = 0$ $\quad y \sim x^r \quad r(r-1) + \beta = 0, \quad r^2 - r + \beta = 0$

$r = \frac{1 \pm \sqrt{1-4\beta}}{2}$ want β such that $y \to 0$ as $x \to 0$ $\boxed{3pts}$

$\Rightarrow$ real$(r) > 0$

(i) if $1-4\beta < 0$, real$(r) = \frac{1}{2}$

$(\frac{1}{4} < \beta)$

(ii) if $1-4\beta = 0$, $r = \frac{1}{2}, \frac{1}{2}$, $y_1 = x^{\frac{1}{2}}$, $y_2 = x^{\frac{1}{2}} \ln x$

$(\beta = \frac{1}{4})$ $\quad y_1 \to 0$ as $x \to 0$

$$\lim_{x\to 0} x^{1/2}\ln x = \lim_{x\to 0} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0} -2\cdot x^{1/2} = 0$$

(iii) if $1-4\beta > 0$ want $\frac{1-\sqrt{1-4\beta}}{2} > 0$, $\quad 1 > \sqrt{1-4\beta}$

$\frac{1}{4} > \beta$ $\quad 1 > 1-4\beta$, $\underline{\underline{\beta > 0}}$

combining everything $\Rightarrow$ $\boxed{\beta > 0}$ $\boxed{4pts}$

This problem is taken from the textbook (problem 36 on page 280). It's also an extra homework problem (problem 3) for week 8. Consequently, it seems reasonable to expect the students to be able to carry out some of the steps & reasoning to arrive at the correct answer. The textbook provided an answer in the back. Therefore the four points are assigned for derivation of the condition $\beta > 0$. If the student answers $\beta = c$ (such as 1 or 2, etc), less than half of the four points should be considered.