Math 222 Final Exam, December 16, 2016

Trigonometric identities: $\sin(A+B) = \sin(A)\cos(B)+\sin(B)\cos(A)$, $\cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B)$. Closed-book, show all your work, all phones off and no calculator.

1. (8 points) Find the solution of the IVP: $x^2y'' - 3xy' + 4y = 0$, $y(1) = 2$, $y'(1) = 3$.

2. A function $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \sin(x), & 0 \le x < \pi \end{cases}$ is periodically extended as $f(x + 2\pi) = f(x)$.

   (a) (5 points) Sketch three periods of this function.

   (b) (10 points) Find the Fourier series of this function.

3. (15 points) Find the eigenvalues λ and eigenfunctions of the given boundary value problem

$$y'' + \lambda y \;=\; 0, \quad y(0) = y(2), \quad y'(0) = y'(2).$$

4. (a) Consider the ODE $y' = (y^2 - 1)(y - 3)$. (6 points) First find the equilibrium solutions (i.e., $y' = 0$). (4 points) Then sketch the direction field.

   (b) Consider the IVP: $y' + \frac{2}{t}y = \frac{1}{t}$, and the initial value is given at $t = 1$ as $y(1) = 1$. (10 points) Find the solution and describe its behavior as $t \to \infty$.

5. (a) (10 points) Solve the IVP: $y'' + 4y' + 5y = 2e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

   (b) (10 points) Find the general solution for the differential equation $y'' + 4y = \sin(2t)$.

6. (12 points) Use Laplace transform to find the solution of the IVP: $y'' - y = (t-1)u_1(t)$, $y(0) = 0$, $y'(0) = 1$. $u_1(t)$ is a unit step function that is zero when $t < 1$ and unity when $t \ge 1$.

7. (10 points) Consider two tanks, labeled tank A and tank B. Tank A contains 100 gal of water and 20 lb of salt. Tank B contains 200 gal of water and 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A. Solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B, also at a rate of 5 gal/sec.

   (a) Set up an IVP for the amount of salt in each tank by writing down two linear differential equations for the total amount of salt in tank A and tank B. Formulate the IVP as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Find the 2×2 matrix $\mathbf{A}$ and the initial condition $\mathbf{x}(0)$.

   (b) Express the solution in terms of the eigenvectors of $\mathbf{A}$. What is the total amount of salt in both tanks as functions of time?

Grading Guidelines for Final Exam M222 Fall 2016

Problem 1

$x^2y'' - 3xy' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 3$

$y \sim x^r$, $\boxed{1pt}$ $\quad r(r-1) - 3r + 4 = 0, \quad r^2 - 4r + 4 = 0, \quad r = 2, 2$ $\boxed{3pts}$

$y_1 = x^2, \quad y_2 = x^2 \ln x \qquad y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \ln x$

$y' = 2c_1 x + 2c_2 x \ln x + c_2 x$ $\boxed{2pts}$

$y'(1) = 3 = 2c_1 + c_2$

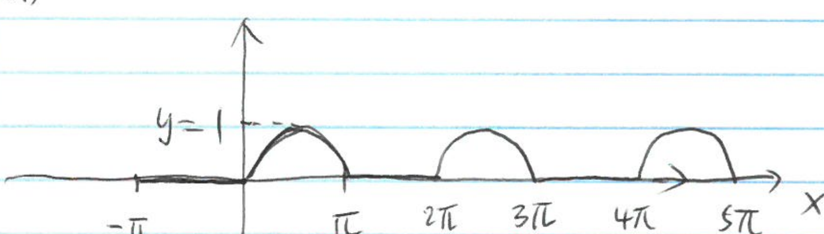
$y(1) = 2 = c_1$ $\quad c_2 = 3 - 4 = -1$

$$\boxed{y = 2x^2 - x^2 \ln x}$$ $\boxed{2pts}$

Problem 2

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \sin x & 0 \le x < \pi \end{cases}$$

(a)



$\boxed{3pts}$ $\quad \boxed{2pts}$ for the extension

(b) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}, \quad L = \pi$ $\boxed{2pts}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x\,dx\right)$$

$$= -\frac{1}{\pi}\cos x \Big|_0^{\pi} = -\frac{1}{\pi}(\cos\pi - \cos 0) = \frac{2}{\pi}$$ $\boxed{2pts}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos\frac{n\pi x}{\pi}\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$\sin x \cdot \cos nx = \frac{1}{2}\left(\sin(1+n)x + \sin(1-n)x\right)$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin(1+n)x + \sin(1-n)x\right)dx$$

$$= \frac{1}{2\pi}\left(-\frac{1}{1+n}\cos(1+n)x - \frac{1}{1-n}\cos(1-n)x\right)\Big|_0^{\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{1}{1+n}\left(\cos(1+n)\pi - 1\right) - \frac{1}{1-n}\left(\cos(1-n)\pi - 1\right)\right)$$

$$\boxed{a_n = \frac{-1}{2\pi}\frac{1}{(1+n)}\left((-1)^{n+1} - 1\right) - \frac{1}{2\pi(1-n)}\left((-1)^{1-n} - 1\right)}$$ [3pts]

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$$\sin x \cdot \sin nx = \frac{1}{2}\left(\cos(1-n)x - \cos((1+n)x)\right)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos(1-n)x - \cos((1+n)x)\right)dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \cos(1-n)x - \cos(1+n)x \, dx$$

$$= \frac{1}{2\pi}\frac{1}{1-n}\sin(1-n)x\Big|_0^{\pi} - \frac{1}{1+n}\sin(1+n)x\Big|_0^{\pi}$$ $n \neq 1$

$b_n = 0$ when $n > 1$

$n = 1$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} 1 - \cos 2x \, dx = \frac{1}{2\pi}\left(x\Big|_0^{\pi} - \frac{\sin 2x}{2}\Big|_0^{\pi}\right) = \frac{1}{2}$$ [3pts]

Problem 3 $\quad y'' + \lambda y = 0 \qquad y(0) = y(2), \quad y'(0) = y'(2)$

$\lambda = 0, \quad y'' = 0, \quad y = ax + b \qquad y(0) = y(2) \Rightarrow \quad b = 2a + b, \quad a = 0$

[5pts] $\quad a = 0 \Rightarrow y = b, \quad y'(0) = 0 = y'(0) \Rightarrow \lambda = 0$ is an eigenvalue and $y = 1$ is the corresponding eigenfunction

$\lambda > 0,\quad y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$$y' = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}\, x + c_2 \cos\sqrt{\lambda}\, x\right)$$

$$y(0) = c_1 = y(2) = c_1 \cos\sqrt{\lambda}\, 2 + c_2 \sin\sqrt{\lambda}\, 2$$

$$\left(1 - \cos 2\sqrt{\lambda}\right) c_1 - \sin 2\sqrt{\lambda}\, c_2 = 0 \quad —— ①$$

$$y'(0) = y(2) \Rightarrow \sqrt{\lambda}\left(-c_1 \cdot \cancel{\sin 0} + c_2 \cos\sqrt{\lambda}\, 0\right) = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}\, 2 + c_2 \cos 2\sqrt{\lambda}\right)$$

$$\sin 2\sqrt{\lambda} \cdot c_1 + \left(1 - \cos 2\sqrt{\lambda}\right) c_2 = 0 \quad —— ②$$

for non-trivial c_1 & c_2 from ① & ② $\begin{vmatrix} 1-\cos 2\sqrt{\lambda} & -\sin 2\sqrt{\lambda} \\ \sin 2\sqrt{\lambda} & 1-\cos 2\sqrt{\lambda} \end{vmatrix} = 0$

$$\left(1 - \cos 2\sqrt{\lambda}\right)^2 + \sin^2 2\sqrt{\lambda} = 0$$

$$1 - 2\cos 2\sqrt{\lambda} + 1 = 0 \qquad \cos 2\sqrt{\lambda} = 1, \quad 2\sqrt{\lambda} = 2n\pi,$$

$$\lambda = (n\pi)^2$$

corresponding eigenfunctions: $\cos n\pi x$ & $\sin n\pi x$ ⇐ 5pts

$\lambda < 0 \quad y = c_1 e^{+\sqrt{|\lambda|}\, x} + c_2 e^{-\sqrt{|\lambda|}\, x}$

$$y' = \sqrt{|\lambda|}\left(c_1 e^{\sqrt{|\lambda|}\, x} - c_2 e^{-\sqrt{|\lambda|}\, x}\right)$$

$$y(0) = c_1 + c_2 = y(2) = c_1 e^{\sqrt{|\lambda|}\, 2} + c_2 e^{-2\sqrt{|\lambda|}}$$

$$y'(0) = \sqrt{|\lambda|}\,(c_1 - c_2) = \sqrt{|\lambda|}\left(c_1 e^{2\sqrt{|\lambda|}} - c_2 e^{-2\sqrt{|\lambda|}}\right)$$

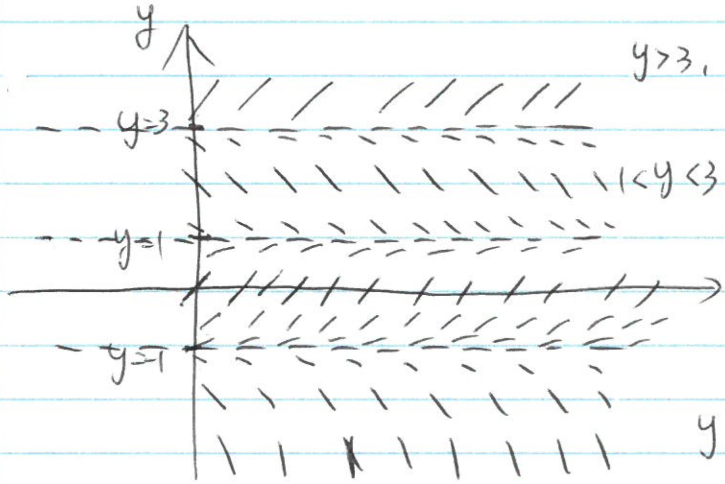
$$\left.\begin{aligned} \left(1 - e^{2\sqrt{|\lambda|}}\right) c_1 + \left(1 - e^{-2\sqrt{|\lambda|}}\right) c_2 = 0 \\ \left(1 - e^{2\sqrt{|\lambda|}}\right) c_1 - \left(1 - e^{-2\sqrt{|\lambda|}}\right) c_2 = 0 \end{aligned}\right)$$

$c_1 = c_2 = 0$ ← 5pts

$\lambda < 0$ is not an eigenvalue

Problem 4 $\quad y' = (y^2-1)(y-3)$

(a) $y' = 0 \quad (y^2-1)(y-3) = 0, \quad y = \pm 1, 3$ $\boxed{6pts}$



$y > 3, \quad y' > 0$ $\boxed{1pt}$

$1 < y < 3 \quad y' < 0$ $\boxed{1pt}$

$-1 < y < 1 \quad y' > 0$ $\boxed{1pt}$

$y < -1 \quad y' < 0$ $\boxed{1pt}$

(b) $y' + \frac{2}{t} y = \frac{1}{t}, \quad y(1) = 1$

$$\mu = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2 \quad \boxed{4pts}$$

$$(t^2 y)' = t, \quad t^2 y = \frac{t^2}{2} + C, \quad y = \frac{1}{2} + \frac{C}{t^2} \quad \boxed{4pts}$$

$$y(1) = 1 = \frac{1}{2} + \frac{C}{1}, \quad C = \frac{1}{2}$$

$$y = \frac{1}{2} + \frac{1}{2t^2}, \quad t \to \infty \quad y \to \frac{1}{2} \quad \boxed{2pts}$$

Problem 5 (a) $y'' + 4y' + 5y = 2e^{-t} \quad y(0) = 0, \; y'(0) = 1$

$$r^2 + 4r + 5 = 0, \quad (r+2)^2 + 1 = 0, \quad (r+2)^2 = -1, \quad r = -2 \pm i \quad \boxed{4pts}$$

$y_1 = e^{-2t} \cos t \qquad Y = Ae^{-t}, \; Y' = -Ae^{-t}, \; Y'' = Ae^{-t}$

$y_2 = e^{-2t} \sin t \qquad (A - 4A + 5A) e^{-t} = 2e^{-t}, \quad 2A = 2, \; A = 1$

$$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t + e^{-t} \quad \boxed{4pts}$$

P.05

$y(0) = C_1 + 1 = 0, \quad C_1 = -1$

$y'(t) = -2e^{-2t}(C_1 \cos t + C_2 \sin t)$
$+ e^{-2t}(-C_1 \sin t + C_2 \cos t) - e^{-t}$

$y'(0) = -2(C_1) + 1 \cdot 1 \cdot C_2 - 1 = 1, \quad C_1 = -1$

$2 + C_2 - 1 = 1 \quad C_2 = 0.$

$y(t) = -e^{-2t}\cos t + e^{-t}$ $\boxed{2pts}$

(b) $y'' + 4y = \sin 2t, \quad y_1 = \cos 2t, \; y_2 = \sin 2t$ $\boxed{4pts}$

$Y = t(A \sin 2t + B \cos 2t)$ $\boxed{2pts}$

$Y' = A \sin 2t + B \cos 2t + t \cdot (2A \cos 2t - 2B \sin 2t)$

$Y'' = 2A \cos 2t - 2B \sin 2t + 2A \cos 2t - 2B \sin 2t + t(-4A \sin 2t - 4B \cos 2t)$

$Y'' + 4Y = 4A \cos 2t - 4B \sin 2t + t(-4A \sin 2t - 4B \cos 2t)$
$+ 4 \cdot t \cdot (A \sin 2t + B \cos 2t) = \sin 2t$

$A = 0, \quad -4B = 1, \quad B = \frac{-1}{4}$

$\boxed{y = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{4} t \cdot \cos 2t}$ $\boxed{4pts}$

Problem 6 $\quad y'' - y = (t-1)u_1(t) \quad y(0) = 0, \; y'(0) = 1$

$\mathcal{L}[y'' - y] = \mathcal{L}[(t-1)u_1] = \frac{e^{-s}}{s^2}$

$s^2 Y - s y(0) - y'(0) - Y = \frac{e^{-s}}{s^2},$ $\boxed{4pts}$

P.06

$$(s^2-1)Y = 1+\frac{e^{-s}}{s^2}, \quad Y = \frac{1}{s^2-1} + \frac{e^{-s}}{s^2(s^2-1)}$$

$$y = \mathcal{L}\left[\frac{1}{s^2-1} + \frac{e^{-s}}{s^2(s^2-1)}\right]$$

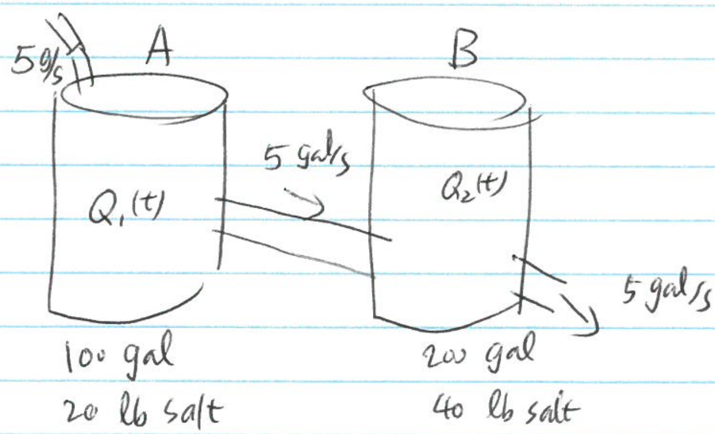
$$= \mathcal{L}\left[\frac{1}{s^2-1}\right] + \mathcal{L}\left[\frac{e^{-s}}{s^2(s^2-1)}\right]$$

$$\mathcal{L}\left[\frac{1}{s^2-1}\right] = \sinh t \qquad \boxed{4\text{pts}}$$

$$\mathcal{L}\left[\frac{1}{s^2(s^2-1)}\right] = \mathcal{L}\left[-\frac{1}{s^2} + \frac{1}{s^2-1}\right] = -t + \sinh t = f(t) \qquad \boxed{4\text{pts}}$$

$$\boxed{y = \sinh t + u_1 \cdot f(t-1)}$$

Problem 7



$$\frac{dV_A}{dt} = 5-5=0 \qquad V_A = \text{const.} = 100 \text{ gal}$$

$$\frac{dV_B}{dt} = 5-5=0, \qquad V_B = \text{const} = 200 \text{ gal}$$

$$Q_1' = \text{in} - \text{out} = 0 - \frac{Q_1}{100}\times 5 = -\frac{Q_1}{20}$$

$$Q_2' = \frac{Q_1}{100}\times 5 - \frac{Q_2}{200}\times 5 = \frac{Q_1}{20} - \frac{Q_2}{40}$$

p.07

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}' = \begin{pmatrix} -\frac{1}{20} & 0 \\ +\frac{1}{20} & -\frac{1}{40} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \quad \boxed{2pt}$$

$\boxed{5pts}$

$$A = \begin{pmatrix} -\frac{1}{20} & 0 \\ \frac{1}{20} & -\frac{1}{40} \end{pmatrix} \qquad \det(A - \lambda I) = 0 \qquad \det \begin{pmatrix} -\frac{1}{20} - \lambda & 0 \\ \frac{1}{20} & -\frac{1}{40} - \lambda \end{pmatrix} = 0 \quad \boxed{2pt}$$

$$\left(-\frac{1}{20} - \lambda\right)\left(-\frac{1}{40} - \lambda\right) = 0, \qquad \lambda = -\frac{1}{20}, -\frac{1}{40} \quad \boxed{1pt}$$

$$\lambda = -\frac{1}{20} \qquad \begin{pmatrix} 0 & 0 \\ \frac{1}{20} & -\frac{1}{40} + \frac{1}{20} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{20} Q_1 + \frac{1}{40} Q_2 = 0 \qquad 2Q_1 + Q_2 = 0 \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \boxed{2pts}$$

$$\lambda = -\frac{1}{40} \qquad \begin{pmatrix} -\frac{1}{20} + \frac{1}{40} & 0 \\ \frac{1}{20} & -\frac{1}{40} + \frac{1}{40} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Q_1 = 0 \Rightarrow Q_2 = 1 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \boxed{2pts}$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = C_1 e^{-\frac{t}{20}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-\frac{t}{40}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} Q_1(0) \\ Q_2(0) \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C_1 = 20$$

$$-2C_1 + C_2 = 40$$

$$C_2 = 40 + 2C_1 = 80$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = 20 e^{-\frac{t}{20}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 80 e^{-\frac{t}{40}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \boxed{1pt}$$