Solutions for HWK week 12

Problem 1 (problem 19 p 405)

$$tx' = Ax \qquad x = \xi \cdot t^r \qquad x' = r\xi t^{r-1} \qquad tx' = r\,\xi t^r$$

$$r\xi t^r = A\xi t^r, \qquad (A\xi - r\xi)\cdot t^r = 0$$

$$(A - rI)\xi = 0 \implies \det(A - rI) = 0 \text{ for a non-trivial solution}$$

(Problem 20 P 406)

$$tx' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x, \qquad x = \xi t^r,$$

$$\det\left[\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0, \qquad \det\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} = 0$$

$$(2-r)(-2-r) + 3 = 0 \qquad (r+2)(r-2) = -3 \qquad r^2 - 4 = -3$$

$$r^2 = 1, \qquad r = \pm 1$$

$$r = 1, \qquad \begin{pmatrix} 2-1 & -1 \\ 3 & -2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 = 0, \qquad x_1 = 1, \; x_2 = 1 \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r = -1 \qquad \begin{pmatrix} 2+1 & -1 \\ 3 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 - x_2 = 0 \qquad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\text{solution } x = c_1 \cdot t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 t^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Problem 2 (Problem 31 on page 407)

$$x' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} x$$

(a) $\alpha = \frac{1}{2}$, $\det\begin{pmatrix} -1-\lambda & -1 \\ -0.5 & -1-\lambda \end{pmatrix} = 0$

$$(-1-\lambda)(-1-\lambda) - \tfrac{1}{2} = 0$$

$$(\lambda+1)^2 = \tfrac{1}{2}, \qquad \lambda = -1 \pm \sqrt{\tfrac{1}{2}}$$

$$\lambda = -1 + \sqrt{\tfrac{1}{2}} \qquad \begin{pmatrix} -1-(-1+\sqrt{\frac{1}{2}}) & -1 \\ -\frac{1}{2} & -1-(-1+\sqrt{\frac{1}{2}}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\sqrt{\frac{1}{2}} & -1 \\ -\frac{1}{2} & -\sqrt{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{x_1}{x_2} = -\frac{1}{\sqrt{\frac{1}{2}}} = -\sqrt{2}, \quad \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\lambda = -1-\sqrt{\frac{1}{2}}$

$$\begin{pmatrix} \sqrt{\frac{1}{2}} & -1 \\ -\frac{1}{2} & \sqrt{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{x_1}{x_2} = \sqrt{2} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

$$x = C_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\left(-1+\sqrt{\frac{1}{2}}\right)t} + C_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\left(-1-\sqrt{\frac{1}{2}}\right)t}$$

$\Rightarrow$ The equilibrium is a stable node.

(b) $\alpha = 2$

$$\det\begin{pmatrix} -1-\lambda & -1 \\ -2 & -1-\lambda \end{pmatrix} = 0, \quad (\lambda+1)^2 - 2 = 0, \quad \lambda = -1 \pm \sqrt{2}$$

$\lambda = -1+\sqrt{2}$

$$\begin{pmatrix} -1-(-1+\sqrt{2}) & -1 \\ -2 & -1-(-1+\sqrt{2}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \quad -\sqrt{2}x_1 - x_2 = 0, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

$\lambda = -1-\sqrt{2}$

$$\begin{pmatrix} -1-(-1-\sqrt{2}) & -1 \\ -2 & -1-(-1-\sqrt{2}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \quad \sqrt{2}x_1 - x_2 = 0, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$x = C_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + C_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{(-1-\sqrt{2})t}$$

$\Rightarrow$ The equilibrium is a saddle point.

(c) $\det\begin{pmatrix} -1-\lambda & -1 \\ -\alpha & -1-\lambda \end{pmatrix} = 0 \quad (1+\lambda)^2 - \alpha = 0, \quad (1+\lambda)^2 = \alpha,$

if $\alpha > 0$ $\quad 1+\lambda = \pm\sqrt{\alpha} \quad \lambda = -1 \pm \sqrt{a}$

λ can be positive when $\underline{\alpha > 1}$ $\Rightarrow$ saddle point

both λ is negative when $\alpha < 1 \Rightarrow$ stable point

when $\alpha = 1$, $\lambda = 0, -2$ $\Rightarrow$ one mode is neutrally stable the other is stable.