P.01

Problem 1 (problem 13 page 612)

Solutions for HWK Week 14

$y'' + \omega^2 y = \sin nt \quad y(0)=0, \quad y'(0)=0$

$y = c_1 \cos\omega t + c_2 \sin\omega t$

$Y = A \sin nt + B\cos nt$ when $n \neq \omega$

$Y'' = -n^2 (A \sin nt + B\cos nt)$

$(\omega^2 - n^2)(A \cdot \sin nt + B \cos nt) = \sin nt$

$B = 0, \quad A = \frac{1}{\omega^2 - n^2} \qquad y = c_1 \cos\omega t + c_2 \sin\omega t + \frac{1}{\omega^2-n^2}\sin nt$

when $n = \omega$,

$Y = t(A \cdot \sin nt + B\cos nt), \quad Y' = (A\sin nt + B\cos nt) + nt(+A\cos nt - B\sin nt)$

$Y'' = nA\cos nt - nB\sin nt + n(A\cos nt - B\sin nt) + n^2 t(-A\sin nt - B\cos nt)$

$Y'' + \omega^2 Y = -n^2 t(A\sin nt + B\cos nt) + 2n(A\cos nt - B\sin nt) + n^2 t \cdot (A\sin nt + B\cos nt)$

$= \sin nt$

$A = 0, \quad B = -\frac{1}{2n}$

$\therefore \; y = c_1 \cos nt + c_2 \sin nt - \frac{t\cos nt}{2n}$

when $n \neq \omega$, $\quad y = c_1 \cos\omega t + c_2 \sin\omega t + \frac{1}{\omega^2-n^2}\sin nt$

$y' = -\omega c_1 \sin\omega t + \omega c_2 \cos\omega t + \frac{n}{\omega^2-n^2}\cos nt$

$y(0) = 0 = c_1$

$y'(0) = \omega c_2 + \frac{n}{\omega^2-n^2} = 0, \qquad c_2 = -\frac{n}{\omega(\omega^2-n^2)}$

$$\boxed{y = -\frac{n}{\omega(\omega^2-n^2)}\sin\omega t + \frac{1}{\omega^2-n^2}\sin nt}$$

when $n = \omega$, $\quad y = c_1\cos nt + c_2 \sin nt - \frac{t}{2n}\cos nt$

$y' = -nc_1 \sin nt + nc_2 \cos nt - \frac{1}{2n}\cos nt + \frac{t}{2}\sin nt$

$y(0) = 0 = c_1$

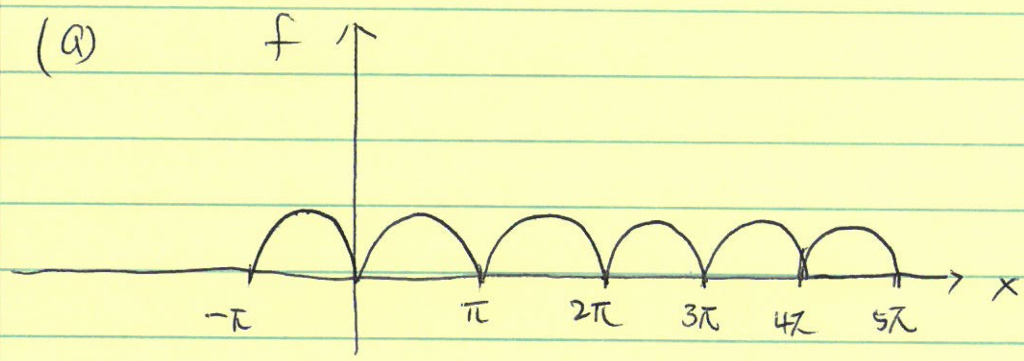
$y'(0) = c_2 - \frac{1}{2n} = 0, \qquad c_2 = \frac{1}{2n}$

$$\boxed{y = +\frac{1}{2n}\sin nt - \frac{t}{2n}\cos nt}$$

Problem 2

$$f(x) = \begin{cases} -\sin x & -\pi \le x < 0 \\ \sin x & 0 \le x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

(a)



(b) $f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad L = \frac{2\pi}{2} = \pi$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos \frac{0\pi x}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot 1 \cdot dx = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx$$

$$= \frac{2}{\pi}(-\cos x)\Big|_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}(1-(-1)) = \frac{4}{\pi}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cdot \cos nx \, dx \Rightarrow \qquad \sin x \cdot \cos nx = \frac{1}{2}(\sin(n+1)x + \sin(1-n)x)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}\left(\sin(n+1)x + \sin(1-n)x\right) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+1)x + \sin(1-n)x \, dx = \frac{1}{\pi}\left(-\frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi} - \frac{1}{1-n}\cos(1-n)x\Big|_0^{\pi}\right)$$

$$\boxed{a_n = \frac{1}{\pi}\left(-\frac{1}{n+1}(\cos(n+1)\pi - 1) - \frac{1}{1-n}(\cos(1-n)\pi - 1)\right)}$$