Solutions to Extra Homework for Weeks 4 & 5

Problem 1: $ty'' + 2y' + te^t y = 0, \quad t > 0$

$$y'' + \frac{2}{t}y' + e^t y = 0, \quad W = c \cdot e^{-\int \frac{2}{t}dt} = c \cdot e^{-2\ln t}$$

$$= c \cdot e^{\ln t^{-2}}$$

$$\boxed{W = c \cdot t^{-2}}$$

Problem 25: $y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \geq 0$

(a) $r^2 + 2r + 6 = 0, \quad (r+1)^2 + 5 = 0, \quad r + 1 = \pm\sqrt{5}i, \quad r = +\pm\sqrt{5}i$

$$y = e^{-t} \cdot (C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t)$$

$$y' = -e^{-t}(C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t) + e^{-t} \cdot \sqrt{5} \cdot (-C_1 \sin\sqrt{5}t + C_2 \cos\sqrt{5}t)$$

$$y(0) = 2 = C_1$$

$$y'(0) = \alpha = -C_1 + \sqrt{5} \cdot C_2, \quad C_2 = \frac{\alpha + C_1}{\sqrt{5}} = \frac{\alpha + 2}{\sqrt{5}}$$

$$y(t) = e^{-t}\left(2 \cdot \cos\sqrt{5}t + \frac{\alpha+2}{\sqrt{5}} \cdot \sin\sqrt{5}t\right)$$

(b) $y(1) = 0 = e^{-1} \cdot \left(2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}} \sin\sqrt{5}\right)$

$$2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}} \sin\sqrt{5} = 0$$

$$\frac{\cos\sqrt{5}}{\sin\sqrt{5}} = -\frac{\alpha+2}{2\sqrt{5}} \qquad \alpha + 2 = -\cot\sqrt{5} \cdot 2\sqrt{5}$$

$$\boxed{\alpha = -2 - 2\sqrt{5} \cdot \cot\sqrt{5}}$$

(c) $y(t) = 0 = e^{-t} \cdot \left(2\cos\sqrt{5}t + \frac{\alpha+2}{\sqrt{5}} \sin\sqrt{5}t\right)$

because $e^{-t} > 0$ for all real t,

$y(t) = 0$ implies $2\cos\sqrt{5}t + \frac{\alpha+2}{\sqrt{5}} \sin\sqrt{5}t = 0$

$$\frac{2\cos\sqrt{5}t}{\sin\sqrt{5}t} = -\frac{\alpha+2}{2\sqrt{5}}, \quad \tan\sqrt{5}t = -\frac{2\sqrt{5}}{\alpha+2}$$

$$\sqrt{5}\,t = \tan^{-1}\left(-\frac{2\sqrt{5}}{\alpha+2}\right)$$

$$t = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{-2\sqrt{5}}{\alpha+2}\right)$$

$$t = \frac{1}{\sqrt{5}}\left[\text{Principle value of } \tan^{-1}\left(-\frac{2\sqrt{5}}{\alpha+2}\right) \text{ (which is between } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) + n\pi\right]$$

Smallest value of t is when $n=1$ because $t>0$

(C) $\alpha \to \infty$ $\quad t = \frac{1}{\sqrt{5}}\cdot\left(\tan^{-1}\left(-\frac{2\sqrt{5}}{\infty+2}\right) + n\pi\right)$

$n=1, \quad t \to 0 + \frac{\pi}{\sqrt{5}}$

Problem 26 $\quad y'' + 2ay' + (a^2+1)y = 0 \quad y(0)=1, \ y'(0)=0$

$r^2 + 2ar + a^2 + 1 = 0$

$(r+a)^2 + 1 = 0, \quad (r+a)^2 = -1 \qquad r+a = \pm i, \quad y = -a \pm i$

$y = e^{-at}\cdot(C_1\cos t + C_2 \sin t)$

$y' = -ae^{-at}\cdot(C_1\cos t + C_2\sin t) + e^{-at}(-C_1\sin t + C_2\cos t)$

$y(0) = 1 = C_1$

$y'(0) = -a\cdot 1 + C_2 = 0, \qquad C_2 = a$

(a) $y(t) = e^{-at}\cdot(\cos t + a\sin t)$

(b) $a=1, \quad y = e^{-t}\cdot(\cos t + \sin t) = e^{-t}\cdot\sqrt{2}\cdot\cos(t - \theta), \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$

$|y(t)| < 0.1 \qquad \left|e^{-t}\cdot\sqrt{2}\cdot\cos(t-\theta)\right| < 0.1$

$e^{-t}\cdot\sqrt{2}\cdot|\cos(t-\theta)| < 0.1$

$e^{-t}\cdot|\cos(t-\theta)| < \frac{0.1}{\sqrt{2}}$

in general $\quad y(t) = e^{-at}\cdot(\cos t + a\sin t) = e^{-at}\sqrt{1+a^2}\cos(t-\theta), \ \theta = \tan^{-1}(a)$

$|y(t)| < e^{-at}\cdot\sqrt{1+a^2} \le \frac{1}{10}, \quad t \ge \frac{1}{a}\ln\left(10\sqrt{1+a^2}\right)$

$\Rightarrow T \le \frac{1}{a}\ln\left(10\sqrt{1+a^2}\right), \quad a=1, \ T \sim 1.8763$

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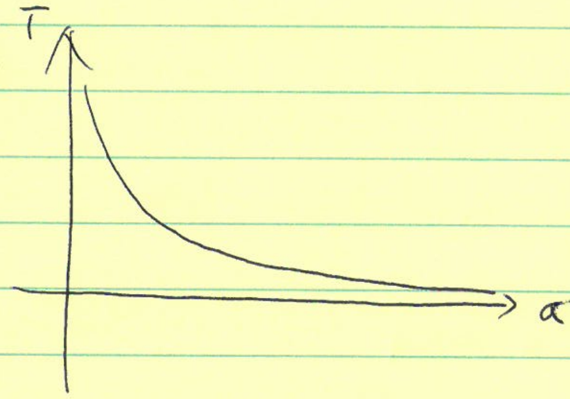
(c) $a = \frac{1}{4}, \quad T \sim 7.4284$

$a = \frac{1}{2}, \quad T \sim 4.3003$

$a = 2, \quad T \sim 1.5116$

$a = 3, \quad T \sim 1.1496$

(d)



Problem 18: $9y'' + 12y' + 4y = 0, \quad y(0) = a > 0, \quad y'(0) = -1$

$9r^2 + 12r + 4 = 0 \qquad (3r+2)^2 = 0, \quad 3r+2=0, \quad r = -\frac{2}{3}, -\frac{2}{3}$

(a) $y(t) = c_1 e^{-\frac{2}{3}t} + c_2 t e^{-\frac{2}{3}t}$

$y' = -\frac{2}{3} c_1 e^{-\frac{2}{3}t} + c_2 e^{-\frac{2}{3}t} + c_2 t \left(-\frac{2}{3}\right) e^{-\frac{2}{3}t}$

$y(0) = c_1 \cdot 1 = a > 0,$

$y'(0) = -\frac{2a}{3} + c_2 = -1, \qquad c_2 = -1 + \frac{2a}{3}$

$\cancel{y(t) = ae^{-\frac{2}{3}t} + \frac{t}{3}e^{-\frac{2}{3}t}}$

(b) $y(t) = ae^{-\frac{2}{3}t} + \left(-1 + \frac{2a}{3}\right) t e^{-\frac{2}{3}t}$

$= e^{-\frac{2}{3}t} \left(a + \left(-1 + \frac{2a}{3}\right) t \right)$

$\frac{2a}{3} - 1 > 0$ solution stays positive because $a > 0$

$\frac{2a}{3} > 1, \quad \boxed{a > \frac{3}{2}}$

Problem 20: (a) $y'' + 2ay' + a^2 y = 0, \qquad r^2 + 2ar + a^2 = 0, \quad (r+a)^2 = 0,$

$r = -a, -a, \qquad y_1 = e^{-at}, \quad y_2 = te^{-at}$

$$W = c \cdot e^{\int -2a\,dt} = c \cdot e^{-2at}$$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = c \cdot e^{-2at}$$

$$\Rightarrow \begin{vmatrix} e^{-at} & y_2 \\ -ae^{-at} & y_2' \end{vmatrix} = c \cdot e^{-2at}$$

$$y_2' e^{-at} + a \cdot e^{-at} \cdot y_2 = c \cdot e^{-2at}$$

$$y_2' + a \cdot y_2 = c \cdot e^{-at}$$

$$\mu = e^{at}, \quad y_2 = \frac{\int c \cdot e^{-at} e^{at}\,dt + d}{e^{at}} = cte^{-at} + de^{-at}$$

$$y_2 = cte^{-at} + \underbrace{de^{-at}}_{\text{repeated from } y_1},$$

$$\Rightarrow \boxed{y_2 = te^{-at}}$$